**2009**

**TRIAL**

# HIGHER SCHOOL CERTIFICATE

## Mathematics Extension 1

**General Instructions:**

- Reading Time - 5 minutes
- Working time - 2 hours
- Write using black or blue pen.
- Board - approved calculators may be used
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown in every question.

**Total marks - 84**

Attempt Questions 1 - 7

All questions are of equal value

**Question 1 ( 12 marks ).** Start on a **SEPARATE** page. **Marks**

**(a)** If P(x) = $x^4 - 3x^3 + ax^2 - 12$ is divisible by (x-3), find the value of $a$. **2**

**(b) i)** Find the gradients at the point P( 1,1) of the tangents to the curves $y = x^3$ and $y = 1 - \ln x$. **2**

**ii)** Hence find the acute angle between these tangents, giving the answer correct to the nearest degree. **2**

**(c)** A (1, - 3) and B ( 6, 7)) are two points. Find the coordinates of the point P( x,y) which divides the interval AB internally in the ratio 2:3 **2**

**(d)** Find $\int \cos^2 4x dx$. **2**

**(e)** Differentiate $\cos^{-1}(3x)$ with respect to $x$. **2**

**Question 2. ( 12 marks )** start on a **SEPARATE** page.

**(a)** Use the substitution $u = x - 1$ to find $\int 5x\sqrt{x-1}\,dx$ **3**

**(b)** T ( $2t, t^2$ ) is a point on the parabola $x^2 = 4y$.

**i)** Show that the tangent to the parabola at T has equation

$tx - y - t^2 = 0$. **2**

**ii)** Hence find the values of $t$ such that the tangent to the parabola at T passes through the point P ( 1, -2). **2**

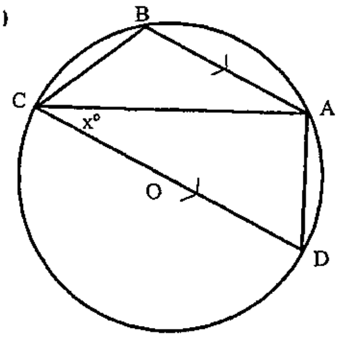
**c ) i)** Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$

where R > 0 and $0 < \alpha < \frac{\pi}{2}$ **3**

**ii)** Hence solve $\cos x - \sqrt{3}\sin x = -2$ for $0 \le x \le 2\pi$. **2**

**Question 3.** ( **12 marks** ) .Start on a **SEPARATE** page.

**(a)** The points A, B, C and D lie on the circumference of a circle centred at 0.
CD is a diameter of the circle and AB is parallel to CD. $\angle$ACD = $x^0$.




Find an expression for $\angle$ ACB in terms of x **3**

**(b)** Use the method of mathematical induction to show that the expression $9^n - 8n - 1$ is divisible by 64 for all integers $n \geq 2$. **3**

**(c)** For the expansion of the expression $(x - \frac{3}{x})^8$, find the term independent of $x$. **3**

**(d)** **i)** Sketch the graph of $y = 2\sin^{-1} 3x$. **2**

**ii)** State the domain and range of the function **1**

**Question 4. ( 12 marks )** .Start on a **SEPARATE** page.

**(a)** How many groups of 2 men and 2 women can be formed from 6 men and 8 women. **1**

**(b)** Six letter words are formed from the letters of the word CYCLIC.

i) How many different 6-letter words can be formed? **1**

**ii)** How many 6 letter words can be formed, if no "C' s are together.? **2**

**(c)**

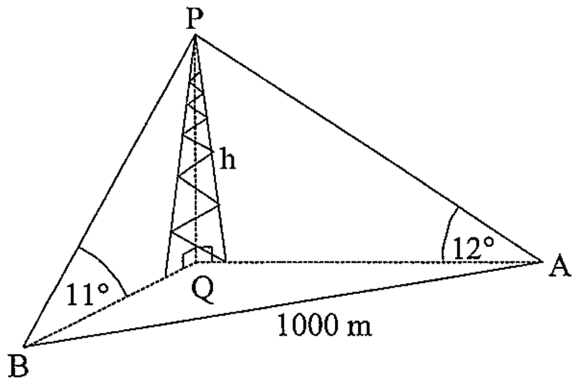



FIGURE NOT TO SCALE

The angle of elevation of a tower PQ of height h metres at a point A due east of it is 12°. From another point B, the bearing of the tower is 051°T and the angle of elevation is 11°. The points A and B are 1000 metres apart and on the same level as the base Q of the tower.

**i.** Show that $\angle AQB = 141°$. **1**

**ii.** Consider the triangle APQ and show that $AQ = h \tan 78°$. **1**

**iii.** Find a similar expression for BQ. **1**

**iv.** Use the cosine rule in the triangle AQB to calculate h to the nearest metre. **2**

**Question 4 continues on the next page.**

**(d) i)** Show that : $\dfrac{2\tan x}{1+\tan^2 x}=\sin 2x$ **1**

**ii)** Hence, or otherwise, find $\displaystyle\int_0^{\frac{\pi}{4}}\frac{\tan x}{1+\tan^2 x}dx$ **2**

**Question 5. ( 12 marks )** .Start on a **SEPARATE** page.

**(a)** The rate at which a body cools in air is proportional to the difference between the temperature, T, of the body and the constant surrounding temperature, S. This can be expressed as

$\dfrac{dT}{dt}=k(T-S)$ where $t$ is time in minutes and $k$ is a constant.

**i)** Show $T=S+Be^{kt}$ where $B$ is a constant, is a solution of the above equation. **1**

**ii)** If a particular body cools from $100^0$ to $80^0$ in 30 minutes, find the temperature of the body after a further 30 minutes, given the surrounding temperature remains constant at $25^0$. Give your answer to the nearest degree. **4**

**(b)** A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity $v\ ms^{-1}$ given by $v=2-x$ and acceleration $a\ ms^{-2}$.

Initially the particle is 4 metres to the left of O.

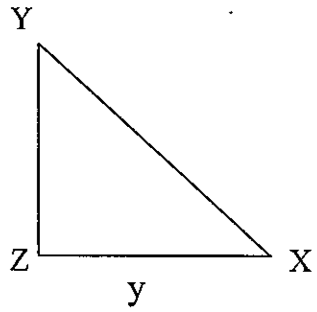
**(i)** Find an expression for $a$ a in terms of $x$. **2**

**(ii)** Use integration to show that $x=2-6e^{-t}$ **3**

**(iii)** Find the exact time taken by the particle to travel 4 metres from its starting point. **2**

**Question 6. ( 12 marks )** .Start on a **SEPARATE** page.

**(a)**




In $\Delta XYZ$, $ZX = y$ and $\angle YZX = 90^0$

**i)** Show that the area A and perimeter P of the triangle are given by

$A = \frac{1}{2}y^2 \tan X$ and $P = y(1 + \tan X + \sec X)$ respectively. **2**

**ii)** If $X = \frac{\pi}{4}$ radians and y is increasing at a constant rate of $0.1 \text{cms}^{-1}$

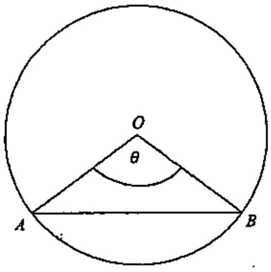
find the rate at which the area of the triangle is increasing at the instant when $y = 20$ cm. **2**

**iii)** If y =10cm and X is increasing at a constant rate of 0.2 radians per second, find the rate at which the perimeter of the triangle is increasing when $X = \frac{\pi}{6}$ radians. **2**

**Question 6 continues on the next page**

**Question 6**

**(b)**




AB is a chord of a circle of radius 1 metre that subtends an angle $\theta$ at the centre of the circle, where $0 < \theta < \pi$. The perimeter of the minor segment cut off by AB is equal to the diameter of the circle.

**(i)** Show that $\theta + 2\sin\frac{1}{2}\theta - 2 = 0.$ **2**

**(ii)** Show that the value of $\theta$ lies between 1 and 2 **2**

**(iii)** Use one application of Newton's method with an initial approximation of $\theta_0 = 1$ to find the next approximation to the value of $\theta$, giving your answer correct to 1 decimal place. **2**

**Question 7. ( 12 marks )** .Start on a **SEPARATE** page.

**(a)** A ball is projected from a point O on horizontal ground in a room of length R metres with an initial speed of $U$ ms $^{-1}$ at an angle of projection of $\alpha$ . There is no air resistance and the acceleration due to gravity is g ms$^{-2}$

**(i)** Assuming after t seconds the ball's horizontal distance x metres,

is given by: $x = U t \cos\alpha$ , and

the vertical component of motion is $\ddot{y} = -g$ ,

show that the vertical displacement y of the ball is given by:

$$y = U t \sin\alpha - \frac{1}{2}gt^2$$ **2**

**(ii)** Hence show that the range R metres for this ball is given by:

$$R = \frac{U^2 Sin2\alpha}{g}$$ **2**

**(iii)** Suppose that the room has a height of 3. 5 metres and the angle of projection is fixed for $0 < \alpha < \frac{\pi}{2}$ but the speed of projection U varies.

Prove that: the maximum height will occur when $U^2 = 7g\,\text{cosec}^2\alpha$ and the maximum range would be 14 cot $\alpha$ . **4**

**(b)**

**i)** Write down the binomial expansion of $(1-x)^{2n}$ in ascending powers of $x$ **1**

**ii)** Hence show that :

$$\binom{2n}{1} + 3\binom{2n}{3} + \ldots + (2n-1)\binom{2n}{2n-1} = 2\binom{2n}{2} + 4\binom{2n}{4} + \ldots + 2n\binom{2n}{2n}.$$

**3**

**End of paper**

# TRIAL HSC Ext. 1 SOLUTION. 2009.

## Question 1.

a) $P(x) = x^4 - 3x^3 + ax^2 - 12$

$P(3) = 3^4 - 3.3^3 + a.3^2 - 12 = 0$ ①

$\therefore 81 - 81 + 9a - 12 = 0$

$9a = 12$

$a = \frac{12}{9} = \frac{4}{3}$ ①

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b) $y = x^3$ $\qquad$ $y = 1 - \ln x$

$y' = 3x^2$ $\qquad$ $y' = \frac{-1}{x}$ ②

at $(1,1)$, $y' = 3.1 = 3$ $\qquad$ at $(1,1)$, $y' = -1$

$\therefore m_1 = 3$ $\qquad$ $\therefore m_2 = -1$

$\text{Tan}\,\theta = \left|\frac{3 - -1}{1 - 3}\right| = \left|\frac{4}{-2}\right| = 2$

$\therefore \theta = \text{Tan}^{-1}(2) = 63^\circ$ ②

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c) $\int \cos^2 4x\, dx$

$\cos^2 4x = \frac{\cos 8x}{2} + \frac{1}{2}$

$\therefore \int \cos^2 4x\, dx = \frac{1}{2}\int(\cos 8x + 1)\, dx$

$= \frac{1}{2}\left[\frac{\sin 8x}{8} + x\right] + c$

$= \frac{\sin 8x}{16} + \frac{x}{2} + C.$ ②

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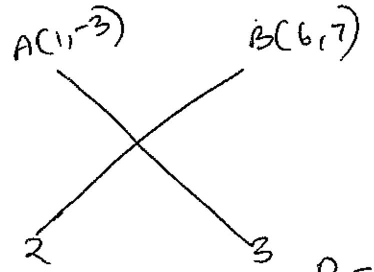
d) 


$P = \left(\frac{3+12}{5}, \frac{-9+14}{5}\right)$

$= (3, 1)$

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e) $\frac{d}{dx}(\cos^{-1} 3x) = \frac{-1}{\sqrt{1-9x^2}} \times 3$

$= \frac{-3}{\sqrt{1-9x^2}}$ ②

## Question 2

a) $u = x - 1$ $\quad \therefore \frac{du}{dx} = 1 \Rightarrow du = dx$

$\int 5x\sqrt{x-1}\, dx = \int 5.(u+1)\sqrt{u}\, du$

$= 5\int(u^{\frac{3}{2}} + u^{\frac{1}{2}})\, du$

$= 5\left[\frac{2}{5}.u^{\frac{5}{2}} + \frac{2}{3}.u^{\frac{3}{2}}\right] + c$

$= 2u^{\frac{5}{2}} + \frac{10}{3}u^{\frac{3}{2}} + c$

$= 2(x-1)^{\frac{5}{2}} + \frac{10}{3}(x-1)^{\frac{3}{2}} + c$ ③

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b) (i) $4y = x^2$

$\therefore y = \frac{x^2}{4} \Rightarrow y' = \frac{2x}{4} = \frac{x}{2}$

at $(2t, t^2)$, $y' = \frac{2t}{2} = t$

$\therefore$ The equation is

$y - t^2 = t(x - 2t)$

$y = tx - t^2 \Rightarrow$

$tx - y - t^2 = 0$ ②

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(ii) Sub. $P(1,-2)$ in the equation

$tx - y - t^2 = 0$

$\therefore t + 2 - t^2 = 0 \Rightarrow t^2 - t - 2 = 0$

$(t-2)(t+1) = 0$

$\therefore t = 2$ or $-1$ ②

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c) (i) $\cos x - \sqrt{3}\sin x = R\cos(x + \alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore R\cos\alpha = 1$ and $R\sin\alpha = \sqrt{3}$

$\therefore \text{Tan}\,\alpha = \sqrt{3}$ $\qquad$ $R^2 = 4$

$\therefore \alpha = \frac{\pi}{3}$ $\qquad$ $R = 2$

$\therefore \cos x - \sqrt{3}\sin x = 2\cos(x + \frac{\pi}{3})$ (

(ii) $2\cos(x + \frac{\pi}{3}) = -2$

$\therefore \cos(x + \frac{\pi}{3}) = -1 \Rightarrow x + \frac{\pi}{3} = \cos^{-1}(-1)$ $\quad (0 \le x \le 2\pi$

$\therefore x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ ②

a) $\angle CAD = 90°$ (angle at circumference in semicircle).

$\angle BAC = x°$ (alternate angles are equal, $AB \parallel CD$)

$\therefore \angle BAD = 90° + x°$ ①

ABCD is a cyclic quadrilateral

$\therefore$ opposite angles are supplementary ①

$\therefore \angle BCD = 180 - (90+x) = 90 - x$

$\therefore \angle ACB = \angle BCD - \angle ACD = 90 - x - x$

$= 90 - 2x°$ ①

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b) When $n=2$, $T_2 = 9^2 - 16 - 1 = 64$ which is divisible by 64

$\therefore$ True for $n=2$ ①

Assume that it is true for $n=k$, $k \geq 2$

$\therefore 9^k - 8k - 1 = 64A$, and $A \in \mathbb{Z}$.

Then $9^{k+1} - 8(k+1) - 1 = 9 \cdot 9^k - 8k - 9$

$= 9(9^k - 1) - 8k$

$= 9(9^k - 8k - 1) + 64k$

$= 9 \times 64A + 64k$

$= 64(9A + 64)$ ②

$\therefore$ true for $n = k+1$

$\therefore 9^n - 8n - 1$ is divisible by 64, $n \geq 2$

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c) $\left(x - \frac{3}{x}\right)^8 = \sum_{r=0}^{8} {}^8C_r \, x^{8-r} \left(-\frac{3}{x}\right)^r$

$= \sum_{r=0}^{8} {}^8C_r (-3)^r \cdot x^{8-2r}$ ①

$8 - 2r = 0 \Rightarrow r = 4$ ①

$\therefore$ The term is ${}^8C_4(-3)^4 = 5670$ ①

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a) ${}^6C_2 \times {}^8C_2 = 420$ ①

b) (i) Number of words $= \frac{6!}{3!}$

$= 120$ ①

(ii)
C _ C _ C _
_ C _ C _ C
C _ _ C _ C
C _ C _ _ C

Total, if no C's are together

$= 4 \times 3! = 24$ ①

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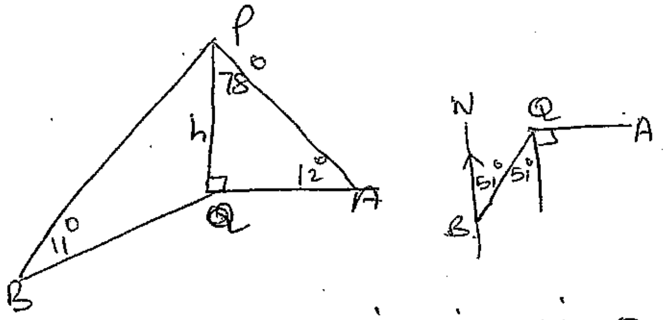
c)




(i) $\angle AQB = 90 + 51 = 141$ ①

(ii) In $\triangle APQ$, $\angle APQ = 78°$

$\therefore \frac{AQ}{h} = \tan 78° \Rightarrow AQ = h\tan 78$ ①

(iii) In $\triangle PQB$, $\frac{BQ}{h} = \tan 79°$

$\therefore BQ = h \tan 79°$ ①

(iv) In $\triangle ABQ$,

$AB^2 = AQ^2 + BQ^2 - 2AQ \cdot BQ \cos 141$

$\therefore 1000000 = h^2\tan^2 78 + h^2 \tan^2 79° - 2h^2 \tan 78 \tan 79 \cos$

$\therefore h^2 = \frac{1000000}{86.2188} \Rightarrow h = 107.69$

$= 108\text{m}$ ②

Question 4 (cont)

d) (i) $\frac{2\tan x}{1+\tan^2 x} = \frac{2\frac{\sin x}{\cos x}}{\sec^2 x}$

$= 2\frac{\sin x}{\cos x} \times \cos^2 x$

$= 2\sin x . \cos x$

$= \sin 2x$ (1)

(ii) $\int_0^{\frac{\pi}{4}} \frac{\tan x}{1+\tan^2 x}\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 2x\,dx$

$= -\frac{1}{4}\left[\cos 2x\right]_0^{\frac{\pi}{4}}$

$= -\frac{1}{4}\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= -\frac{1}{4}\left[0-1\right] = \frac{1}{4}$ (2)

Question 5

a) (i) $\frac{dT}{dt} = k.Be^{kt}$

$= k(T-S)$ (1)

(ii) $\left.\begin{array}{l}T=100\\ t=0\end{array}\right\}$ $100 = 25 + Be^{k\times 0} \Rightarrow B = 75$ (1)

$\left.\begin{array}{l}T=80\\ t=30\end{array}\right\}$ $80 = 25 + 75e^{30k}$

$\frac{55}{75} = e^{30k}$

$\frac{11}{15} = e^{30k}$

$\log_e\left(\frac{11}{15}\right) = 30k$

$\therefore k = \frac{1}{30}\log_e\left(\frac{11}{15}\right)$ (2)

$t=60, T=?$

$T = 25 + 75e^{60\times k}$

$= 65.33^{\circ}$ (1)

$= 65^{\circ}$

b) (i) $v = 2-x$

$a = v\frac{dv}{dx} = (2-x)(-1)$

$= x-2$ (2)

(ii) $v = \frac{dx}{dt} = 2-x$

$\therefore \frac{dt}{dx} = \frac{1}{2-x}$

$dt = \frac{dx}{2-x}$ (1)

$\int dt = \int\frac{dx}{2-x}$

$t = -\ln(2-x) + c$

$t=0, x=-4$

$\therefore 0 = -\ln 6 + c$

$\Rightarrow c = \ln 6$

$\therefore t = -\ln(2-x) + \ln 6$

$\therefore t = \ln\frac{6}{2-x}$ (1)

$\therefore e^t = \frac{6}{2-x}$

$2e^t - xe^t = 6$

$\therefore xe^t = 2e^t - 6$

$x = 2 - 6e^{-t}$ (1)

(iii) When $t=0$, $x=-4$

After the particle has travelled 4m from its starting position,

$x=0 \Rightarrow 0 = 2-6e^{-t}$

$\therefore 6e^{-t} = 2$

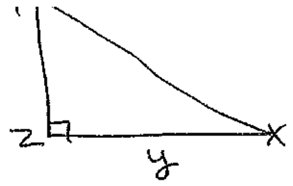
$e^{-t} = \frac{2}{6} = \frac{1}{3}$

$\therefore t = \ln 3.$

(2)

Questions .

a)(i)




$\tan x = \frac{ZY}{y} \Rightarrow ZY = y\tan x$

$\therefore \text{Area} = \frac{1}{2} \times y\tan x \times y$

$= \frac{1}{2} y^2 \tan x$ (1)

$\frac{y}{XY} = \cos x \Rightarrow XY = y\sec x$

$\therefore \text{Perimeter } P = y + y\tan x + y\sec x$

$= y[1 + \tan x + \sec x]$ (1)

(ii) $\frac{dy}{dt} = 0.1 \text{cm s}^{-1}$.

$\frac{dA}{dt} = \frac{dA}{dy} \times \frac{dy}{dt}$

$A = \frac{1}{2} y \tan^2 x$

$\therefore \frac{dA}{dy} = y\tan x$.

$= y\tan\frac{\pi}{4}$ when $x = \frac{\pi}{4}$

$= y$ (1)

$\therefore \frac{dA}{dt} = y \times 0.1 = 20 \times 0.1$

$= 2 \text{cm}^2/\text{sec}$. (1)

(iii) $\frac{dx}{dt} = 0.2$ radians/sec.

$\frac{dP}{dt} = \frac{dP}{dx} \times \frac{dx}{dt}$

$P = y(1 + \tan x + \sec^2 x)$

$\therefore \frac{dP}{dx} = y(\sec^2 x + \sec x \tan x)$ (1)

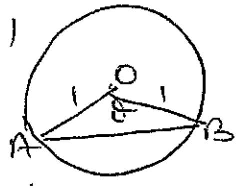
$= 10(\sec^2\frac{\pi}{6} + \sec\frac{\pi}{6} \cdot \tan\frac{\pi}{6})$

when $y = 10$, $x = \frac{\pi}{6}$

$= 10\left[\frac{4}{3} + \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right] = 10\left[\frac{4}{3} + \frac{2}{3}\right]$

$\therefore \frac{dP}{dt} = 20 \times 0.2 \ldots$ /sec. $= 20$ . (1)

)(i)




Using cosine rule,

$AB^2 = 1^2 + 1^2 - 2\cos\theta$

$= 2(1 - \cos\theta)$

$= 4 \cdot \sin^2\frac{1}{2}\theta$

$\therefore AB = 2 \cdot \sin\frac{1}{2}\theta$

Arc $AB = r\theta = \theta$ . $(r = 1)$

$\therefore$ Perimeter = diameter $\Rightarrow$

$\theta + 2\sin\frac{1}{2}\theta = 2$

$\therefore \theta + 2\sin\frac{1}{2}\theta - 2 = 0.$ (2)

(ii) $f(\theta) = \theta + 2\sin\frac{1}{2}\theta - 2$

$f(1) = 1 + 2\sin\frac{1}{2} - 2 \approx -0.04$

$f(2) = 2 + 2\sin 1 - 2 \approx 1.68 > 0$

Since $f(\theta)$ is continuous,

$f(\theta) = 0$ for some $1 < \theta < 2$ (2)

(iii) $f(\theta) = \theta + 2\sin\frac{\theta}{2} - 2$

$f'(\theta) = 1 + \cos\frac{\theta}{2}$

$\therefore \theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$

$= 1 - \frac{-1 + 2\sin\frac{1}{2}}{1 + \cos\frac{1}{2}}$

$\approx 1.0$ (to one dec plac

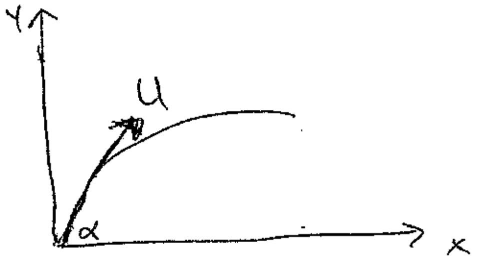
(2).

Question ...

(i) $t=0$, $x=0$ and $y=0$


$\dot{y} = u\sin\alpha$

$\dot{x} = u\cos\alpha$




$\ddot{y} = -g$

$\dot{y} = -\int g\,dt = -gt + c$

when $t=0$, $\dot{y} = u\sin\alpha \Rightarrow C = u\sin\alpha$

$\therefore \dot{y} = -gt + u\sin\alpha$

$\therefore y = \int(-gt + u\sin\alpha)\,dt$

$= ut\sin\alpha - \frac{gt^2}{2} + D$ (2)

$t=0$, $y=0 \Rightarrow D = 0$

$\therefore y = ut\sin\alpha - \frac{gt^2}{2}$ ....(i)

(ii) For the range, $y=0$

$\therefore t\left(u\sin\alpha - \frac{gt}{2}\right) = 0$

$\therefore t=0$ or $t = \frac{2u\sin\alpha}{g}$

$\therefore$ Range $= x = ut\cos\alpha$

$= \frac{u \cdot 2u\sin\alpha}{g} \cdot \cos\alpha$

$= \frac{u^2\sin 2\alpha}{g}$ (2)

(iii) At maximum height, $\dot{y} = 0$

$\therefore u\sin\alpha - gt = 0$

$\therefore t = \frac{u\sin\alpha}{g}$

Sub. $t = \frac{u\sin\alpha}{g}$ in (i)

maximum height $= 3.5 = \frac{u \cdot u\sin\alpha}{g} - \frac{g}{2}\cdot\frac{u^2\sin^2\alpha}{g}$

$3.5 = \frac{u^2\sin^2\alpha}{2g} \Rightarrow u^2 = \frac{7g}{\sin^2\alpha} = 7g\,\mathrm{cosec}^2\alpha$ (2)

(iv) Maximum range $R = \frac{u^2\sin 2\alpha}{g}$

$= \frac{7g}{\sin^2\alpha} \times \frac{2\sin\alpha\cos\alpha}{g}$

$= \frac{14\cos\alpha}{\sin\alpha}$

$= 14\cot\alpha$ (2)

b) (i)

$(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1 x + {}^{2n}C_2 x^2 - {}^{2n}C_3 x^3 + \cdots + {}^{2n}C_{2n-1}(-x)^{2n-1} + {}^{2n}C_{2n} x^{2n}$ (1)

(ii) By differentiating both sides w.r.t. $x$

$-2n(1-x)^{2n-1} = -{}^{2n}C_1 + 2\cdot{}^{2n}C_2 - 3\cdot{}^{2n}C_3 + \cdots + 2n\cdot{}^{2n}C_{2n}\cdot x^{2n-1}$ (1)

Sub. $x=1$ both sides,

$0 = -{}^{2n}C_1 + 2\cdot{}^{2n}C_2 - 3\cdot{}^{2n}C_3 + \cdots - (2n-1)\,{}^{2n}C_{2n-1} + 2n\cdot{}^{2n}C_{2n}$

$\therefore {}^{2n}C_1 + 3\cdot{}^{2n}C_3 + \cdots + (2n-1)\,{}^{2n}C_{2n-1}$

$= 2\cdot{}^{2n}C_2 + 4\,{}^{2n}C_4 + \cdots + 2n\cdot{}^{2n}C_{2n}$ (1)