New South Wales

Higher School Certificate

4 Unit Mathematics

Examinations 1990-1994

Typeset with $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

NSW HSC 4 Unit Mathematics Examination 1990

1. **(a)** Let $z = a + ib$, where a and b are real numbers and $a \neq 0$.

(i) Express $|z|$ and $\tan(\arg z)$ in terms of a and b.

(ii) Express $\frac{z}{3+5i}$ in the form $x + iy$, where x and y are real.

(b) **(i)** If $w = \frac{1+i\sqrt{3}}{2}$, show that $w^3 = -1$.

(ii) Hence calculate w^{10}.

(c) If $z = 5 - 5i$, write z, z^2 and $\frac{1}{z}$ in modulus-argument form.

(d) Let u and v be two complex numbers, where $u = -2 + i$, <u>and</u> v is defined by $|v| = 3$ and $\arg v = \frac{\pi}{3}$.

(i) On an Argand diagram plot the points A and B representing the complex numbers u and v respectively.

(ii) Plot the points C and D represented by the complex numbers $u - v$ and iu, respectively.

Indicate any geometric relationships between the four point A, B, C, and D.

2. **(a)** Find the exact values of: **(i)** $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\, dx$ **(ii)** $\int_0^{\sqrt{2}} \sqrt{4 - x^2}\, dx$.

(b) Find: **(i)** $\int \frac{dx}{(x+1)(x^2+2)}$ **(ii)** $\int \cos^3 x\, dx$, by writing $\cos^3 x = (1 - \sin^2 x) \cos x$, or otherwise.

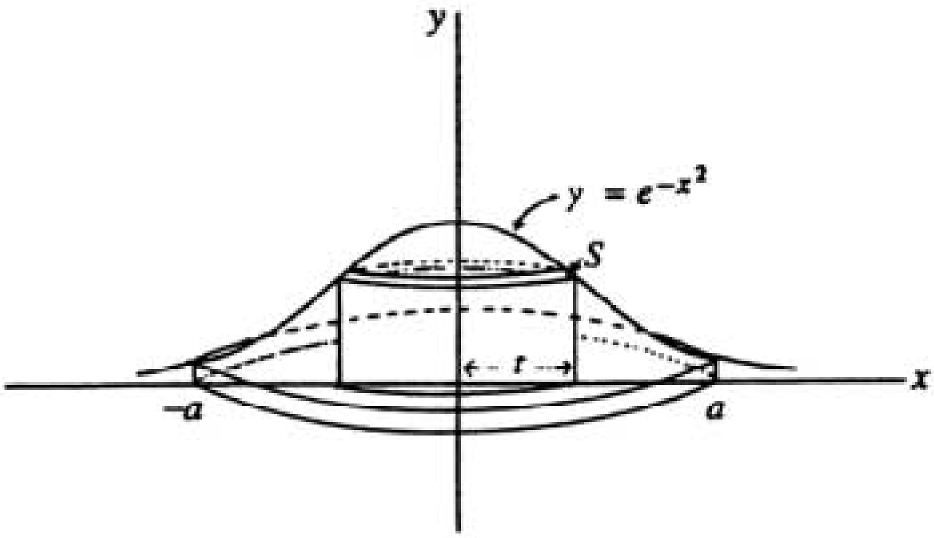
(c) Let $I_n = \int_0^x (1 + t^2)^n\, dt, n = 1, 2, 3, \ldots$.

Use integration by parts to show that $I_n = \frac{1}{2n+1}(1 + x^2)^n x + \frac{2n}{2n+1} I_{n-1}$.

<u>Hint</u> : Observe that $(1 + t^2)^{n-1} + t^2(1 + t^2)^{n-1} = (1 + t^2)^n$.

3. The region under the curve $y = e^{-x^2}$ and above the x axis for $-a \leqq x \leqq a$ is rotated about the y axis to form a solid.

(i) Divide the resulting solid into cylindrical shells S of radius t as in the diagram.



Show that each shell S has approximate volume $\delta V = 2\pi t e^{-t^2}\,\delta t$ where δt is the thickness of the shell.

(ii) Hence calculate the volume of the solid.

(iii) What is the limiting value of the volume of the solid as a approaches infinity?

(b) Consider the functions f, g defined by $f(x) = \frac{x-1}{x+2}$ for $x \neq -2$,

$$g(x) = [f(x)]^2$$

(i) Sketch the hyperbola $y = f(x)$, clearly labelling the horizontal and vertical asymptotes and the points of intersection with the x and y axes.

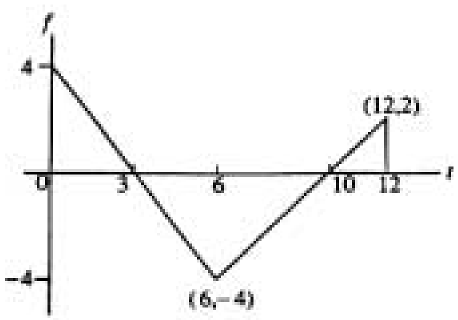
(ii) Find all turning points of $y = g(x)$.

(iii) Using the same diagram as used in **(i)**, sketch the curve $y = g(x)$ clearly labelling it.

(iv) On a separate diagram sketch the curve given by $y = g(-x)$.

4. (a) The diagram shows the graph of the function f,

$$\text{where } f(t) = \begin{cases} 4 - \frac{4}{3}t, & \text{for } 0 \leqq t \leqq 6 \\ t - 10, & \text{for } 6 \leqq t \leqq 12. \end{cases}$$



The function F is defined for $0 \leqq x \leqq 12$ by $F(x) = \int_0^x f(t)\ dt$.

(i) Calculate $F(6)$ and $F(12)$.

(ii) Calculate those values of x for which $F(x) = 0$.

(iii) Find all turning points of F.

(b) Let $L_1 \equiv 4x - 5y + 1$ and $L_2 \equiv 2x + 3y - 5$.

(i) Find the point P of intersection of the two straight lines $L_1 = 0$ and $L_2 = 0$.

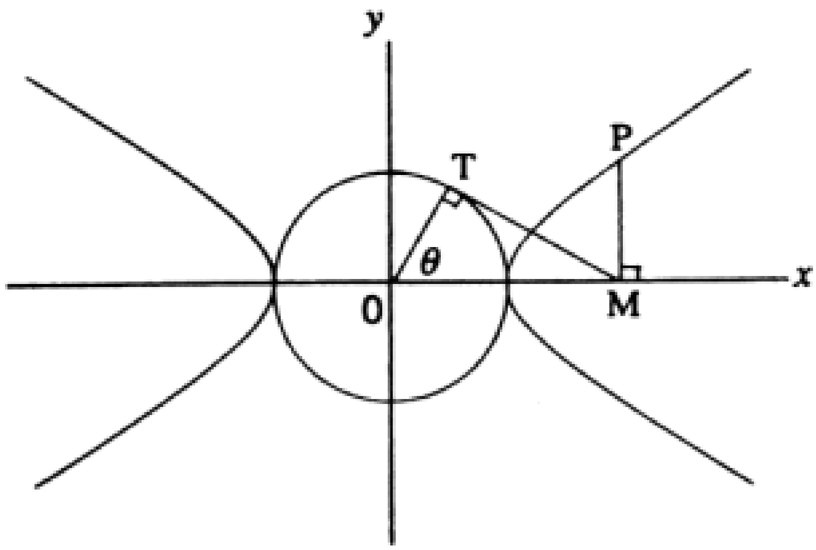
(ii) In the Cartesian plane draw the lines $L_1 = 0$ and $L_2 = 0, L_1 = 6, L_2 = 7$, marking the point P.

Explain why these four lines define a parallelogram.

(iii) If a and b are constants, not both zero, explain why $aL_1 + bL_2 = 0$ defines a straight line through P.

(iv) Using part **(iii)**, or otherwise, prove that the diagonal through P of the parallelogram defined in part **(ii)** has equation: $7L_1 - 6L_2 = 0$.

5. (a)



The figure shows the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$, where $a, b > 0$. The point T lies on the circle with $\angle TOx = \theta$, where $0 < \theta < \frac{\pi}{2}$.

The tangents to the circle at T meets the x axis at M; MP is perpendicular to Ox and P is a point on the hyperbola in the first quadrant.

(i) Show that P has coordinates $(a \sec\theta, b\tan\theta)$.

(ii) Suppose that Q is a point on the hyperbola with coordinates $(a\sec\phi, b\tan\phi)$.

If $\theta + \phi = \frac{\pi}{2}$, and $\theta \neq \frac{\pi}{4}$, show that the chord PQ has equation $ay = b(\cos\theta + \sin\theta)x - ab$.

(iii) Show that every such chord PQ passes through a fixed point and find its coordinates.

(iv) Show that as θ approaches $\frac{\pi}{2}$ the chord PQ approaches a line parallel to an asymptote.

(b) **(i)** How many different five figure numbers can be formed from the digits 1,2,3,4,5 without repetition?

(ii) How many of these numbers are greater than 45321?

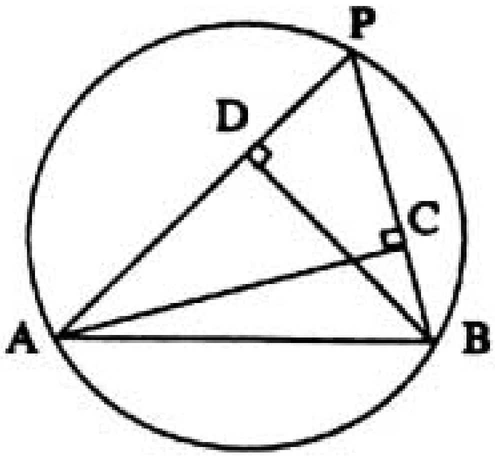
(iii) How many of these numbers are less than 45321?

6. **(a)** **(i)** Write down the relations which hold between the roots α, β, γ of the equation $ax^3 + bx^2 + cx + d = 0, (a \neq 0)$, and the coefficients a, b, c, d.

(ii) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$. You are given that the roots α, β, γ of this equation satisfy $\alpha = \beta + \gamma$. Use part **(i)** to find α.

(iii) Suppose that the equation $x^3 + px^2 + qx + r = 0$ has roots λ, μ, ν which satisfy $\lambda = \mu + \nu$. Show that $p^3 - 4pq + 8r = 0$.

(b) In the diagram, AB is a fixed chord of a circle, P a variable point in the circle and AC and BD are perpendicular to BP and AP respectively. Copy this diagram.



(i) Show that $ABCD$ is a cyclic quadrilateral on a circle with AB as diameter.

(ii) Show that the triangles PCD and APB are similar.

(iii) Show that as P varies, the segment CD has constant length.

(iv) Find the locus of the midpoint of CD.

7. **(a)** A mass of m kilograms falls from a stationary balloon at height h metres above the ground. It experiences air resistance during its fall equal to mkv^2, where v is its speed in metres per second and k is a positive constant.

Let x be the distance in metres of the mass from the balloon, measured positively as it falls.

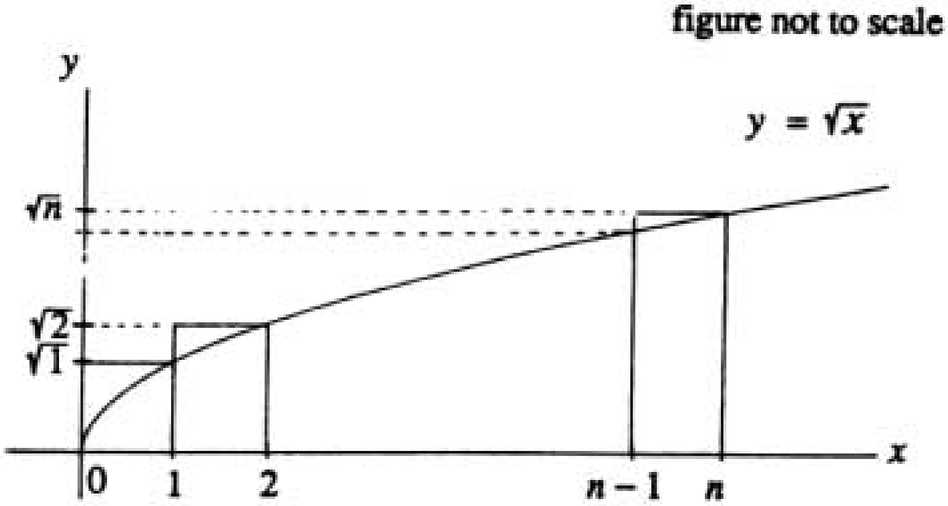
(i) Show that the equation of motion of the mass is $\ddot{x} = g - kv^2$, where g is the acceleration due to gravity.

(ii) Find v^2 as a function of x. Hint: $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2) = v\frac{dv}{dx}$.

(iii) Find the velocity V as the mass hits the ground in terms of g, k and h.

(iv) Find the velocity of the mass as it hits the ground if air resistance is neglected.

(b)



Consider the graph of the function $y = \sqrt{x}$.

(i) Show that this curve is increasing for all $x \geqq 0$.

(ii) Hence show that $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \geqq \int_0^n \sqrt{x}\, dx = \frac{2}{3}n\sqrt{n}$.

(iii) Use mathematical induction to show that $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \leqq \frac{4n+3}{6}\sqrt{n}$, for all integers $n \geqq 1$.

(iv) Use parts **(ii)** and **(iii)** to estimate $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{10\,000}$ to the nearest hundred.

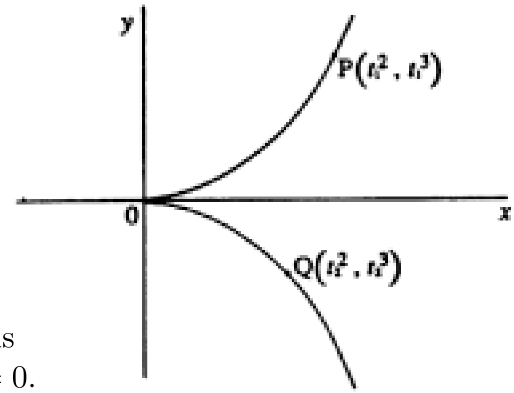
8. **(a)** You are given that $2\cos A \sin B = \sin(A + B) - \sin(A - B)$.

Let $S = 1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta$.

(i) Prove that $S \sin\frac{\theta}{2} = \sin\frac{7\theta}{2}$.

(ii) Hence show that if $\theta = \frac{2\pi}{7}$, then $1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta = 0$.

(iii) By writing S in terms of $\cos\theta$, prove that $\cos\frac{2\pi}{7}$ is a solution of the polynomial equation $8x^3 + 4x^2 - 4x - 1 = 0$.



(b) Consider the curve defined parametrically by $x = t^2, y = t^3$.

Let $P(t_1^2, t_1^3)$ and $Q(t_2^2, t_2^3)$ be two distinct points on the curve.

(i) Write down the equation of the curve in terms of x and y only.

(ii) Show that the equation of the chord PQ is given by $(t_1 + t_2)y - (t_1^2 + t_1 t_2 + t_2^2)x + t_1^2 t_2^2 = 0$.

(iii) Hence, or otherwise, show that the equation of the tangent to the curve at a point corresponding to t, where $t \neq 0$, is given by $2y - 3tx + t^3 = 0$.

(iv) Let $R(x_0, y_0)$ be a point in the plane such that $x_0^3 > y_0^2 > 0$. Prove that there are precisely 3 tangents from R to the curve and sketch this on a diagram.

NSW HSC 4 Unit Mathematics Examination 1991

1. **(a)** Find: **(i)** $\int \frac{t-1}{t^3}\, dt$ **(ii)** $\int \frac{e^x}{e^{2x}+9}\, dx$, using the substitution $u = e^x$.

(b) **(i)** Evaluate $\int_0^1 \frac{5}{(2t+1)(2-t)}\, dt$.

(ii) By using the substitution $t = \tan\frac{\theta}{2}$ and **(i)**, evaluate $\int_0^{\pi/2} \frac{d\theta}{3\sin\theta+4\cos\theta}$.

(c) Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$ where n is a non-negative integer.

(i) Show that $I_n = (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x\, dx$ when $n \geqq 2$.

(ii) Deduce that $I_n = \frac{n-1}{n} I_{n-2}$ when $n \geqq 2$.

(iii) Evaluate I_4.

2. **(a)** Plot on an Argand diagram the points P, Q and R which correspond to the complex numbers $2i\sqrt{3} - i$, and $-\sqrt{3} - i$, respectively.

Prove that P, Q and R are the vertices of an equilateral triangle.

(b) Let $z_1 = \cos\theta_1 + i\sin\theta_1$ and $z_2 = \cos\theta_2 + i\sin\theta_2$, where θ_1 and θ_2 are real.

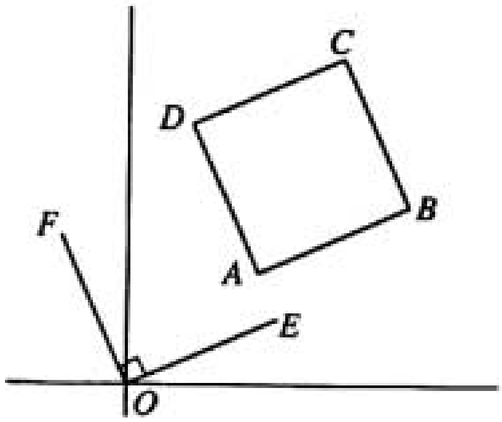
Show that: **(i)** $\frac{1}{z_1} = \cos\theta_1 - i\sin\theta_1$

(ii) $z_1 z_2 = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$.

(c) **(i)** Find all pairs of integers x and y such that $(x+iy)^2 = -3-4i$.

(ii) Using **(i)**, or otherwise, solve the quadratic equation $z^2 - 2z + (3+i) = 0$.

(d) In the Argand diagram, $ABCD$ is a square and OE and OF are parallel and equal in length to AB and AD respectively. The vertices A and B correspond to the complex numbers w_1 and w_2 respectively.



(i) Explain why the point E corresponds to $w_2 - w_1$.

(ii) What complex number corresponds to the point F?

(iii) What complex number corresponds to the vertex D?

3. **(a)** The hyperbola $\mathcal{H}$ has equation $xy = 4$.

(i) Sketch $\mathcal{H}$ and indicate on your diagram the positions and coordinates of all points at which $\mathcal{H}$ intersects the axes of symmetry.

(ii) Show that the equation of the tangent to $\mathcal{H}$ at $P(2t, 2/t)$, where $t \neq 0$, is $x + t^2 y = 4t$.

(iii) If $s \neq 0$ and $s^2 \neq t^2$, show that the tangents to $\mathcal{H}$ at P and $Q(2s, 2/s)$ intersect at $M\left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$.

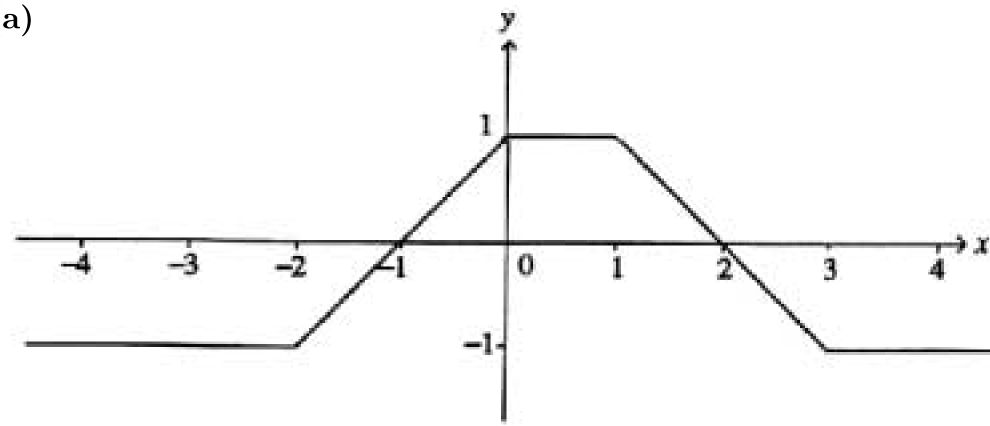
(iv) Suppose that in **(iii)** the parameter $s = -1/t$. Show that the locus of M is a straight line through, but excluding, the origin.

(b) Let $C_1 \equiv x^2 + 3y^2 - 1, C_2 \equiv 4x^2 + y^2 - 1$, and let λ be a real number.

(i) Show that $C_1 + \lambda C_2 = 0$ is the equation of a curve through the points of intersection of the ellipses $C_1 = 0$ and $C_2 = 0$.

(ii) Determine the values of λ for which $C_1 + \lambda C_2 = 0$ is the equation of an ellipse.

4. (a)



The diagram is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = -f(x)$ **(ii)** $y = |f(x)|$ **(iii)** $y = f(|x|)$ **(iv)** $y = \sin^{-1}(f(x))$.

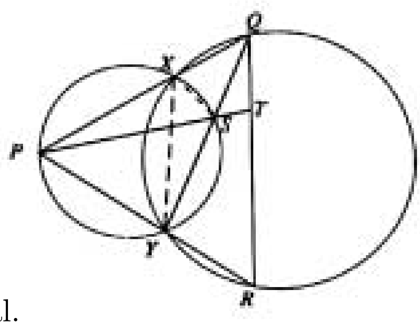
(b) The even function g is defined by $g(x) = \begin{cases} 4e^{-x} - 6e^{-2x} & \text{for } x \geqq 0; \\ g(-x) & \text{for } x < 0. \end{cases}$

(i) Show that the curve of $y = g(x)$ has a maximum turning point at $(\ln 3, 2/3)$.

(ii) Sketch the curve $y = g(x)$ and label the turning points, any points of inflexion, asymptotes, and points of intersection with the axes.

(iii) Discuss the behaviour of the curve $y = g(x)$ at $x = 0$.

5. (a) In the diagram, the circles $XPYS$ and $XYRQ$ intersect at the points X and Y, and PXQ, PYR, QSY, PST and QTR are straight lines.



(i) Explain why $\angle STQ = \angle YRQ + \angle YPS$.

(ii) Show that $\angle TRQ + \angle YPS + \angle SXQ = \pi$.

(iii) Deduce that $STQX$ is a cyclic quadrilateral.

(iv) Let $\angle QPY = \alpha$ and $\angle PQY = \beta$. Show that $\angle STQ = \alpha + \beta$.

(b) A drinking glass having the form of a right circular cylinder of radius a and height h, is filled with water. The glass is slowly tilted over, spilling water out of it, until it reaches the position where the water's surface bisects the base of the glass. Figure 1 shows this position.

Figure 1

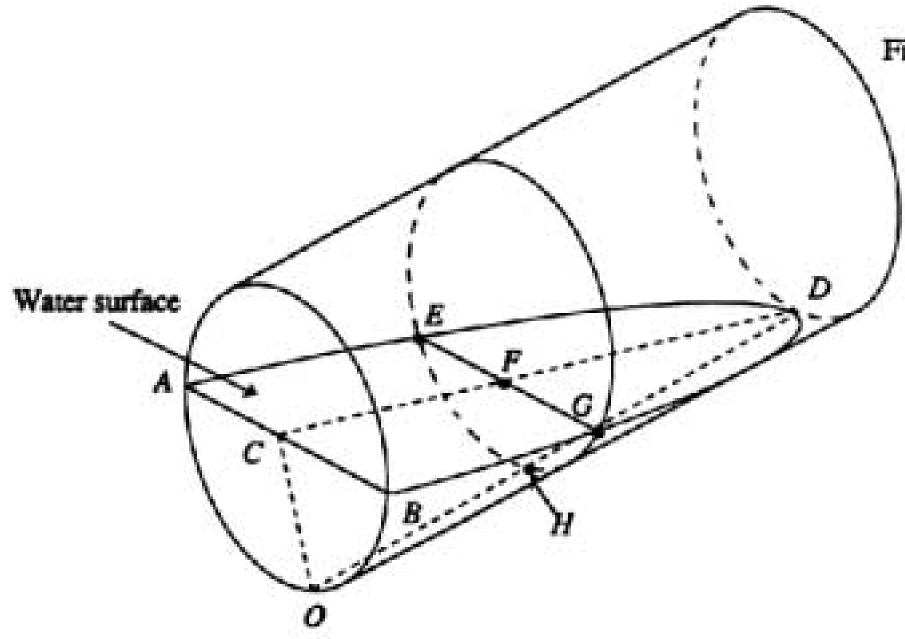
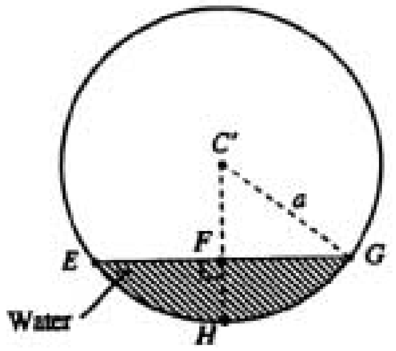
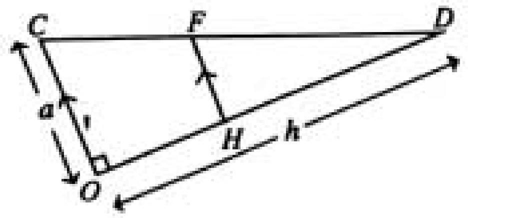


Figure 2



Note: $EG \perp C'H$ at F

Figure 3



Note: $FH \parallel CO$, $CO = a$, and $OD = h$

In Figure 1, AB is a diameter of the circular base with centre C, O is the lowest point on the base, and D is the point where the water's surface touches the rim of the glass.

Figure 2 shows a cross-section of the titled glass parallel to its base. The centre of this circular section is C' and EFG shows the water level. The section cuts the lines CD and OD of Figure 1 in F and H respectively.

Figure 3 shows the section COD of the tilted glass.

(i) Use Figure 3 to show that $FH = \frac{a}{h}(h - x)$, where $OH = x$.

(ii) Use Figure 2 to show that $C'F = \frac{ax}{h}$ and $\angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$.

(iii) Use **(ii)** to show that the area of the shaded segment EGH is $a^2\left[\cos^{-1}\left(\frac{x}{h}\right) - \left(\frac{x}{h}\right)\sqrt{1 - \left(\frac{x}{h}\right)^2}\right]$.

(iv) Given that $\int \cos^{-1}\theta \, d\theta = \theta\cos^{-1}\theta - \sqrt{1 - \theta^2}$, find the volume of water in the tilted glass of Figure 1.

6. **(a)** A nine-member Fund Raising Committee consists of four students, three teachers and two parents. The Committee meets around a circular table.

(i) How many different arrangements of the nine members around the table are possible if the students sit together as a group and so do the teachers, but no teacher sits next to a student?

(ii) One student and one parent are related. Given that all arrangements in **(i)** are equally likely, what is the probability that these two members sit next to each other?

(b) When a jet aircraft touches down, two different retarding forces combine to bring it to rest. If the aircraft has mass M kg and speed v m/s there is a constant frictional force of $M/4$ newtons and a force of $Mv^2/108$ newtons due to the reverse thrust of the engines. The reverse thrust of the engines does not take effect until 20 seconds after touchdown.

Let x be the distance in metres of the jet from its point of touchdown and let t be the time in seconds after touchdown.

(i) Show that $\frac{d^2x}{dt^2} = -\frac{1}{4}$ for $0 < t < 20$ and that for $t > 20$, and until the jet stops, $\frac{d^2x}{dt^2} = -\frac{1}{108}(27 + v^2)$.

(ii) Prove the identity $\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx}$.

(iii) If the jet's speed at touchdown is 60 m/s show that v=55 and $x = 1150$ at the instant the reverse thrust of the engines takes effect.

(iv) Show that when $t > 20, x = 1150 + 54\{\ln(27 + 55^2) - \ln(27 + v^2)\}$.

(v) Calculate how far from the touchdown point the jet comes to rest. Give your answer to the nearest metre.

7. **(a)** **(i)** By assuming that $\cos(A + B) = \cos A\cos B - \sin A\sin B$, prove the identity $\cos^3 A - \frac{3}{4}\cos A = \frac{1}{4}\cos 3A$.

(ii) Show that $x = 2\sqrt{2}\cos A$ satisfies the cubic equation $x^3 - 6x = -2$ provided $\cos 3A = -1/(2\sqrt{2})$.

(iii) Using **(ii)**, find the three roots of the equation $x^3 - 6x + 2 = 0$. Give your answers to four decimal places.

(b) Let $x = \alpha$ be a root of the quartic polynomial $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$ where $(2 + B)^2 \neq 4A^2$.

(i) Show that α cannot be $0, 1$ or -1.

(ii) Show that $x = 1/\alpha$ is a root.

(iii) Deduce that if α is a multiple root, then its multiplicity is 2 and $4B = 8 + A^2$.

8. (a) Let $f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } 0 < x \leqq \pi/2; \\ 1 & \text{for } x = 0. \end{cases}$

(i) Find the derivative of f for $0 < x < \pi/2$ and prove that f' is negative in this interval.

(ii) Sketch the graph of $y = f(x)$ for $0 \leqq x \leqq \pi/2$ and deduce that $\sin x \geqq 2x/\pi$ in this interval.

(b) (i) Suppose $x > 0, y > 0$. Prove that $\frac{1}{x} + \frac{1}{y} \geqq \frac{4}{s}$ where $s = x + y$.

Hint: Consider $f(x) = \frac{1}{x} + \frac{1}{s-x}$.

(ii) Suppose $x_i > 0$ for $i = 1, 2, \ldots, n$ where $n \geqq 2$.

Show by mathematical induction, or otherwise, that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geqq \frac{n^2}{s}$ where s denotes the sum $x_1 + x_2 + \cdots + x_n$.

NSW HSC 4 Unit Mathematics Examination 1992

1. **(a)** Find: **(i)** $\int \tan\theta \sec^2\theta \, d\theta$ **(ii)** $\int \frac{2x+6}{x^2+6x+1}\, dx$.

(b) Evaluate $\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{dx}{\sqrt{(x-1)(3-x)}}$ by using the substitution $u = x - 2$.

(c) Evaluate $\int_0^1 \frac{5(1-t)}{(t+1)(3-2t)}\, dt$.

(d) **(i)** Find $\int xe^x \, dx$ **(ii)** Evaluate $\int_0^1 2x^3 e^{x^2}\, dx$.

2. **(a)** The points A and B represent the complex numbers $3-2i$ and $1+i$ respectively.

(i) Plot the points A and B on an Argand diagram and mark the point P such that $OAPB$ is a parallelogram.

(ii) What complex number does P represent?

(b) Let $z = a + ib$ where $a^2 + b^2 \neq 0$.

(i) Show that if $\Im(z) > 0$ then $\Im\left(\frac{1}{z}\right) < 0$.

(ii) Prove that $\left|\frac{1}{z}\right| = \frac{1}{|z|}$.

(c) Describe and sketch the locus of those points z such that:

(i) $|z - i| = |z + i|$ **(ii)** $|z - i| = \sqrt{2}\, |z + i|$.

(d) It is given that $1 + i$ is a root of $P(z) = 2z^3 - 3z^2 + rz + s$ where r and s are real numbers.

(i) Explain why $1 - i$ is also a root of $P(z)$.

(ii) Factorize $P(z)$ over the real numbers.

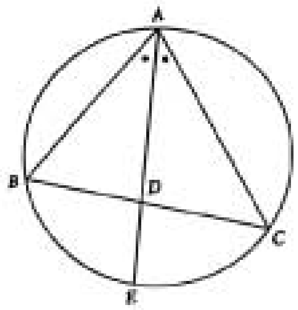
3. **(a)** The ellipse $\mathcal{E}$ has equation $\frac{x^2}{100} + \frac{y^2}{75} = 1$.

(i) Sketch he curve $\mathcal{E}$, showing on your diagram the coordinates of the foci and the equation of each directrix.

(ii) Find the equation of the normal to the ellipse at the point $P(5, 7.5)$.

(iii) Find the equation of the circle that is tangential to the ellipse at P and $Q(5, -7.5)$.

(b) In the diagram, the bisector AD of $\angle BAC$ has been extended to intersect the circle ABC at E.



Copy the diagram.

(i) Prove that the triangles ABE and ADE are similar.

(ii) Show that $AB.AC = AD.AE$.

(iii) Prove that $AD^2 = AB.AC - BD.DC$.

4. **(a)** Each of the following statements is either true or false. Write TRUE or FALSE for each statement and give brief reasons for your answers. (You are not asked to find the primitive functions.)

(i) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \; d\theta = 0$ **(ii)** $\int_0^{\pi} \sin^7\theta \; d\theta = 0.$

(iii) $\int_{-1}^{1} e^{-x^2} \; dx = 0$ **(iv)** $\int_0^{\frac{\pi}{2}} (\sin^8\theta - \cos^8\theta) \; d\theta = 0$

(v) For $n = 1, 2, 3, \ldots, \int_0^1 \frac{dt}{1+t^n} \leq \int_0^1 \frac{dt}{1+t^{n+1}}$.

(b) Let $f(x) = \ln(1+x) - \ln(1-x)$ where $-1 < x < 1$.

(i) Show that $f'(x) > 0$ for $-1 < x < 1$.

(ii) On the same diagram sketch $y = \ln(1+x)$ for $x > -1$.

$y = \ln(1-x)$ for $x < 1$.

and $y = f(x)$ for $-1 < x < 1$.

Clearly label the three graphs.

(iii) Find an expression for the inverse function $y = f^{-1}(x)$.

5. **(a)** The solid $\mathcal{S}$ is a rectangular prism of dimensions $a \times a \times 2a$ from which right square pyramids of base $a \times a$ and height a have been removed from each end. The solid $\mathcal{T}$ is a wedge that has been obtained by slicing a right circular cylinder of radius a at $45°$ through a diameter AB of its base.

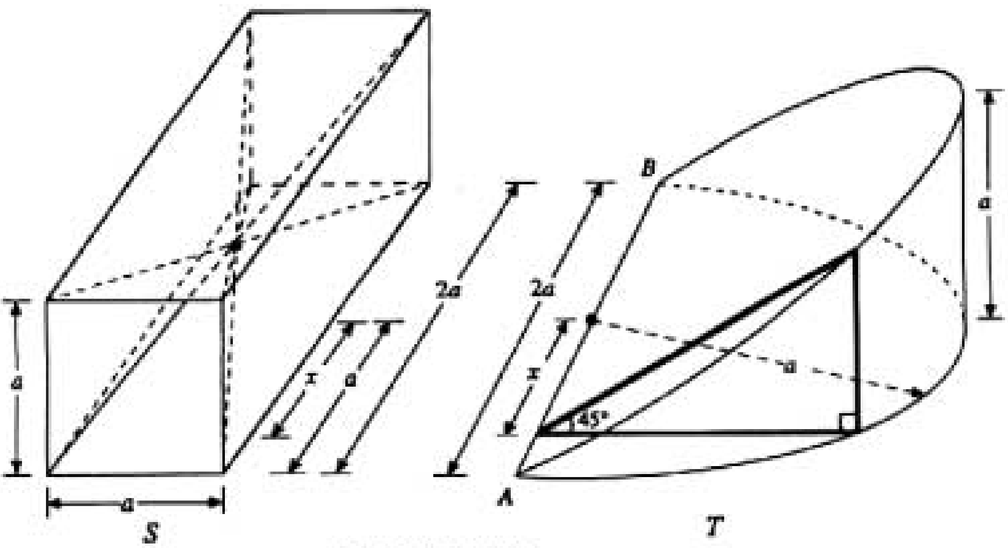
Consider a cross-section of $\mathcal{S}$ which is parallel to its square base at distance x from its centre, and a corresponding cross-section of $\mathcal{T}$ which is perpendicular to AB and at distance x from its centre.

(i) The triangular cross-section of $\mathcal{T}$ is shown on the diagram on the next page. Show that it has area $\frac{1}{2}(a^2 - x^2)$.

(ii) Draw the cross-section of $\mathcal{S}$ and calculate its area.

(iii) Express the volumes of $\mathcal{S}$ and $\mathcal{T}$ as definite integrals.

(iv) What is the relationship between the volumes of $\mathcal{S}$ and $\mathcal{T}$? (There is no need to evaluate either integral.)



NOT TO SCALE

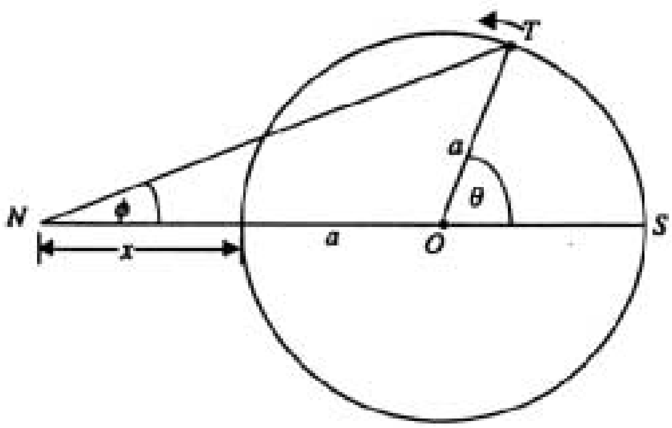
(b) An object is fired vertically upwards with initial speed 400m/s from the surface of the Earth.

Assume that the acceleration due to gravity at height x above the Earth's surface is $10/\left[1+\frac{x}{R}\right]^2$ m/s^2 where the radius of the Earth, $R = 6.4 \times 10^6$ m.

(i) Show that $\frac{d}{dx}(\frac{1}{2}v^2) = -10/\left[1+\frac{x}{R}\right]^2$ where v is the speed of the object at height x. (Neglect air resistance.)

(ii) Calculate the maximum height the object reaches. Give your answer to the nearest metre.

6. (a)



The diagram shows a model train $\mathcal{T}$ that is moving around a circular track, centre O and radius a metres. The train is moving at a constant speed of u m/s. The point N is in the same plane as the track and is x metres from the nearest point on the track. The line NO produced meets the track at S.

Let $\angle TNS = \phi$ and $\angle TOS = \theta$ as in the diagram.

(i) Express $\frac{d\theta}{dt}$ in terms of a and u.

(ii) Show that $a\sin(\theta - \phi) - (x + a)\sin\phi = 0.$ and deduce that $\frac{d\phi}{dt} = \frac{u\cos(\theta-\phi)}{(x+a)\cos\phi + a\cos(\theta-\phi)}$

(iii) Show that $\frac{d\phi}{dt} = 0$ when NT is tangential to the track.

(iv) Suppose that $x = a$.

Show that the train's angular velocity about N when $\theta = \frac{\pi}{2}$ is $\frac{3}{5}$ times the angular velocity about N when $\theta = 0$.

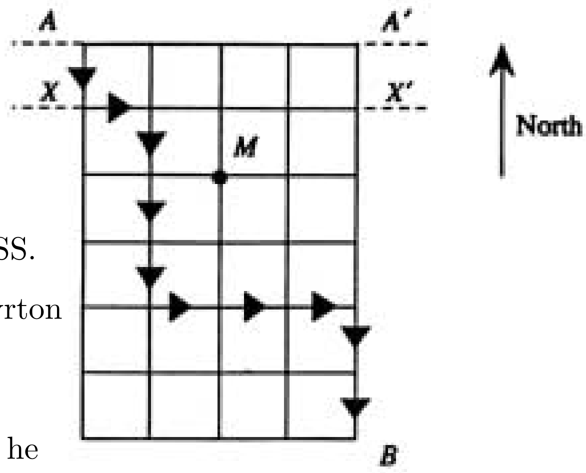
(b) Let n be an integer with $n \geq 2$.

(i) For $i = 1, 2, \ldots, n$ suppose x_i is a real number satisfying $0 < x_i < \pi$.

Use mathematical induction to show that there exist real numbers $a_1, a_2, \ldots, a_n$ such that $|a_i| \leq 1$ for $i = 1, 2, \ldots, n$, and such that $\sin(x_1 + x_2 + \cdots + x_n) = a_1 \sin x_1 + a_2 \sin x_2 + \cdots + a_n \sin x_n$.

(ii) Deduce that $\sin nx \leq n \sin x$ whenever $0 < x < \pi$.

7. **(a)** The diagram shows the road grid of a city.



Ayrton drives exactly 10 blocks from his home, A, to his workplace, B, which is 6 blocks south (S) and 4 blocks east (E). The route on the diagram is SESSSEEESS.

(i) By how many different routes can Ayrton drive to work?

(ii) By how many different routes can Ayrton drive to work on those days that he wishes to stop at the shop marked M?

(iii) The street marked AA' is made one-way westward. How many different routes can Ayrton follow if he cannot drive along AA'?

(iv) Suppose that instead of AA' the street marked XX' is made one-way westward. How many different routes can Ayrton follow if he cannot drive along XX'?

(b) Suppose that $z^7 = 1$ where $z \neq 1$.

(i) Deduce that $z^3 + z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} = 0$.

(ii) By letting $x = z + \frac{1}{z}$ reduce the equation in **(i)** to a cubic equation in x.

(iii) Hence deduce that $\cos\frac{\pi}{7} + \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{8}$.

8. **(a)** Consider the function $f(x) = e^x\left(1 - \frac{x}{10}\right)^{10}$.

(i) Find the turning points of the graph of $y = f(x)$.

(ii) Sketch the curve $y = f(x)$ and label the turning points and any asymptotes.

(iii) From your graph deduce that $e^x \leq \left(1 - \frac{x}{10}\right)^{-10}$ for $x < 10$.

(iv) Using **(iii)**, show that $\left(\frac{11}{10}\right)^{10} \leq e \leq \left(\frac{10}{9}\right)^{10}$.

(b) Let n be a positive integer and let x be any positive approximation to $\sqrt{n}$. Choose y so that $xy = n$.

(i) Prove that $\frac{x+y}{2} \geq \sqrt{n}$.

(ii) Suppose that $x > \sqrt{n}$.

Show that $\frac{x+y}{2}$ is a closer approximation to $\sqrt{n}$ than x is.

(iii) Suppose $x < \sqrt{n}$.

How large must x be in terms of n for $\frac{x+y}{2}$ to be a closer approximation to $\sqrt{n}$ than x is?

NSW HSC 4 Unit Mathematics Examination 1993

1. (a) Evaluate $\int_3^8 \frac{x}{(x+1)-\sqrt{x+1}}\,dx$ by using the substitution $x+1=u^2$.

(b) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ by using the substitution $t=\tan\frac{\theta}{2}$.

(c) Evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$.

(d) (i) Find real numbers a, b and c such that $\frac{4x+3}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$.

(ii) Hence find $\int \frac{4x+3}{(x^2+1)(x+2)}\,dx$.

2. (a) (i) On an Argand diagram shade in the region determined by the inequalities

$$2 \le \Im(z) \le 4 \text{ and } \frac{\pi}{6} \le \arg z \le \frac{\pi}{4}.$$

(ii) Let z_0 be the complex number of maximum modulus satisfying the inequalities of **(i)**. Express z_0 in the form $a+ib$.

(b) Let θ be a real number and consider $(\cos\theta + i\sin\theta)^3$.

(i) Prove $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^3\theta$.

(ii) Find a similar expression for $\sin 3\theta$.

(c) Find the equation, in Cartesian form, of the locus of the point z if $\Re\left(\frac{z-4}{z}\right) = 0$.

(d) By substituting appropriate values of z_1 and z_2 into the equation $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$ show that $\frac{\pi}{4} = \tan^{-1} 2 - \tan^{-1}\frac{1}{3}$.

(e) Let P, Q and R represent the complex numbers w_1, w_2, and w_3 respectively. What geometric properties characterize triangle PQR if $w_2 - w_1 = i(w_3 - w_1)$? Give reasons for your answer.

3. (a) Consider the point $P\left(ct, \frac{c}{t}\right)$, where $t \ne \pm 1$, which lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the equation of the tangent to the hyperbola at P is $x + t^2 y = 2ct$.

(ii) Let the tangent to the hyperbola at P intersect the coordinate axes at A and B. Show that $PA = PB$.

(iii) Let the normal to the hyperbola at P meet the axes of symmetry of the hyperbola at C and D. Show that $PC = PD = PA$.

[You may assume that the equation of the normal is $t^3x - ty = c(t^4 - 1)$.]

(iv) Sketch a graph of the hyperbola showing the results proved so far.

(v) Explain why A, B, C and D must be the vertices of a square.

(b) Let $R(x_0, y_0), P(x_1, y_1)$, and $Q(x_2, y_2)$ be points on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

(i) If d is the distance between the points R and P, show that

$$-\frac{d^2}{2} = x_0x_1 + y_0y_1 + g(x_0 + x_1) + f(y_0 + y_1) + c.$$

(ii) Suppose Q is also distance d from R. Explain why the equation of the chord PQ is

$$-\frac{d^2}{2} = x_0x + y_0y + g(x_0 + x) + f(y_0 + y) + c.$$

4. (a) Let $f(x) = \frac{1-x}{x}$. On separate diagrams sketch the graphs of the following functions.

For each graph label any asymptote.

(i) $y = f(x)$ **(ii)** $y = f(|x|)$ **(iii)** $y = e^{f(x)}$ **(iv)** $y^2 = f(x)$.

Discuss the behaviour of the curve of **(iv)** at $x = 1$.

(b) Suppose k is a constant greater than 1. Let $f(x) = \frac{1}{1+(\tan x)^k}$ where $0 \leq x \leq \frac{\pi}{2}$.

[You may assume $f\left(\frac{\pi}{2}\right) = 0$.]

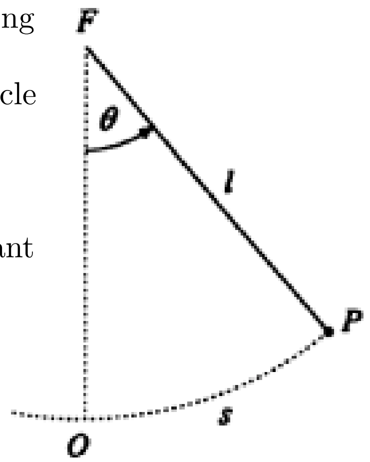
(i) Show that $f(x) + f\left(\frac{\pi}{2} - x\right) = 1$ for $0 \leq x \leq \frac{\pi}{2}$.

(ii) Sketch $y = f(x)$ for $0 \leq x \leq \frac{\pi}{2}$.

[There is no need to find $f'(x)$ but assume $y = f(x)$ has a horizontal tangent at $x = 0$. Your graph should exhibit the property of **(b) (i)**.]

(iii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^k}$.

5. (a) The diagram shows a simple pendulum consisting of a particle P, of mass m kg, which is attached to a fixed point F by a string of length l metres. The particle P moves along a circular arc in a fixed vertical plane through F. The point O is the lowest point of the arc, $\angle OFP = \theta$, and the arc length $OP = s$ metres. The time t is measured in seconds and g m/s^2 is the constant gravitational acceleration.



(i) Show that the tangential acceleration of P is given by $\frac{d^2s}{dt^2} = l\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right)$, where $\dot{\theta} = \frac{d\theta}{dt}$.

(ii) Show that the equation of motion of the pendulum is $l\frac{d}{d\theta}\left(\frac{1}{2}\dot{\theta}^2\right) = -g\sin\theta$.

(iii) Suppose the pendulum is given an initial angular velocity of $\sqrt{\frac{g}{l}}$ radians/second at $\theta = 0$.

Show that $\frac{1}{2}l\dot{\theta}^2 = g\left(\cos\theta - \frac{1}{2}\right)$.

Hence deduce that the maximum value of θ attained by the pendulum is $\frac{\pi}{3}$.

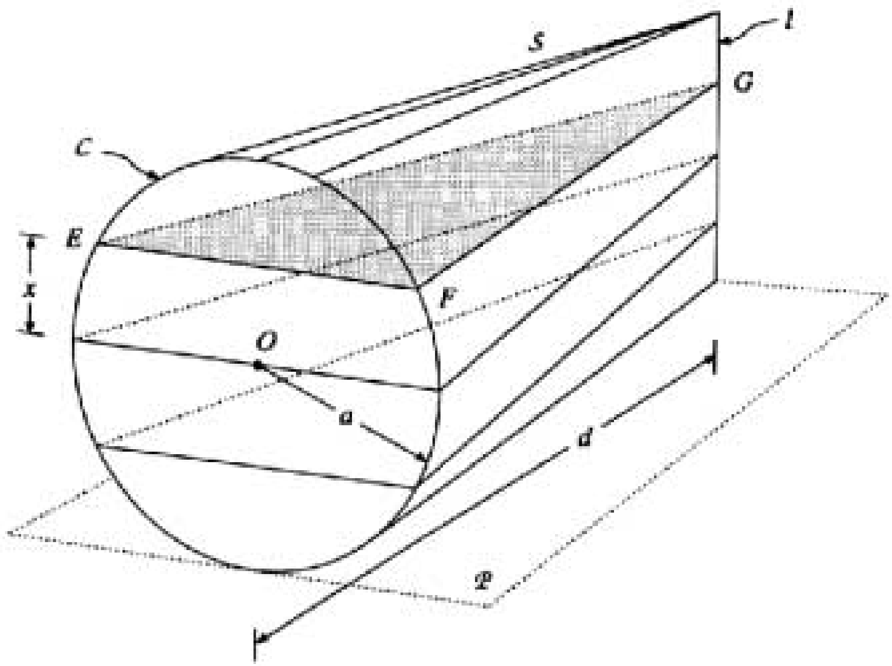
(iv) Suppose that on the initial upward swing the angular velocity is better approximated from the equation $\frac{1}{2}l\dot{\theta}^2 = g\left(\cos\theta - \frac{1}{2}\right) - \frac{g}{10}(2\sin\theta - \theta)$.

Use one application of Newton's method to find the maximum value of θ attained by the pendulum. Take $\theta = \frac{\pi}{3}$ as the first approximation.

(b) **(i)** If a is a multiple root of the polynomial equation $P(x) = 0$, prove that $P'(a) = 0$.

(ii) Find all roots of the equation $18x^3 + 3x^2 - 28x + 12 = 0$ if two of the roots are equal.

6. **(a)**

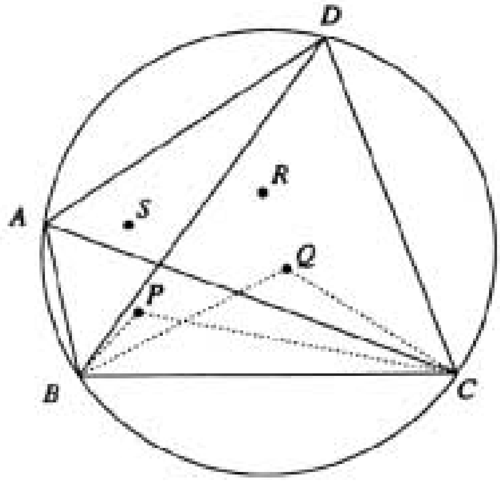


The solid $\mathcal{S}$ is generated by moving a straight edge so that it is always parallel to a fixed plane $\mathcal{P}$. It is constrained to pass through a circle $\mathcal{C}$ and line segment l. The circle $\mathcal{C}$, which forms a base for $\mathcal{S}$, has radius a and the line segment l is distance d from $\mathcal{C}$. Both $\mathcal{C}$ and l are perpendicular to $\mathcal{P}$ and sit on $\mathcal{P}$ in such a way that the perpendicular to $\mathcal{C}$ at its centre O bisects l.

(i) Calculate the area of the triangular cross-section EFG which is parallel to $\mathcal{P}$ and distance x from the centre O of $\mathcal{C}$.

(ii) Calculate the volume of $\mathcal{S}$.

(b) In the diagram $ABCD$ is a cyclic quadrilateral and P, Q, R and S are the incentres of triangles ABC, BCD, CDA and DAB respectively. The incentre of a triangle is the point of intersection of the bisectors of its three angles. Thus, for example, BP bisects $\angle ABC$ and CP bisects $\angle ACB$; similarly BQ bisects $\angle DBC$ and CQ bisects $\angle DCB$.



(i) Copy the diagram

(ii) Prove that $\angle PBQ (= \angle PBC - \angle QBC) = \frac{1}{2}\angle ABD$.

(iii) Prove that $\angle PCQ = \angle PBQ$ and hence explain why $BCQP$ must be a cyclic quadrilateral.

(iv) Prove that $\angle SPQ = \angle BAS + \angle BCQ$.

(v) Deduce that $SPQR$ is a rectangle.

7. (a) In the next 7 days, called day 1, day 2, ... , day 7, Esther and George must each take 3 days in a row off work. They choose their consecutive 3 days randomly and independently of each other.

(i) Show that the probability that they both have day 1 off together is $\frac{1}{25}$.

(ii) What is the probability that day 2 is the first day that they both have off together?

(iii) Find the probability that Esther and George have at least one day off together.

(b) For $n = 1, 2, 3, \ldots$, let $s_n = 1 + \sum_{r=1}^{n} \frac{1}{r!}$.

(i) Prove by mathematical induction that $e - s_n = e \int_0^1 \frac{x^n}{n!} e^{-x}\, dx$.

(ii) From **(i)**, deduce that $0 < e - s_n < \frac{3}{(n+1)!}$ for $n = 1, 2, 3, \ldots$.

[Remember that $e < 3$ and $e^{-x} \leq 1$ for $x \geq 0$.]

(iii) By using the inequality of **(ii)**, deduce that $(e - s_n)n!$ is not an integer for $n = 2, 3, 4, \ldots$.

(iv) Show that there cannot exist positive integers p and q such that $e = \frac{p}{q}$.

8. **(a)** Let the points $A_1, A_2, \ldots, A_n$ represent the nth roots of unity, $w_1, w_2, \ldots, w_n$, and suppose P represents any complex number z such that $|z| = 1$.

(i) Prove that $w_1 + w_2 + \cdots + w_n = 0$.

(ii) Show that $PA_i^2 = (z - w_i)(\overline{z} - \overline{w}_i)$ for $i = 1, 2, \ldots, n$.

(iii) Prove that $\sum\limits_{i=1}^{n} PA_i^2 = 2n$.

(b) Let $f(x) = 1 + x^2$ and let x_1 be a real number. For $n = 1, 2, 3, \ldots,$ define $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

[You may assume that $f'(x_n) \neq 0$.]

(i) Show that $|x_{n+1} - x_n| \geq 1$ for $n = 1, 2, 3, \ldots$.

(ii) Graph the function $y = \cot\theta$ for $0 < \theta < \pi$.

(iii) Using your graph from **(ii)**, show that there exists a real number θ_n such that $x_n = \cot\theta_n$ where $0 < \theta_n < \pi$.

(iv) Deduce that $\cot\theta_{n+1} = \cot 2\theta_n$ for $n = 1, 2, 3, \ldots$.

[You may assume that $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$.]

(v) Find all points x_1 such that, for some $n, x_1 = x_{n+1}$.

NSW HSC 4 Unit Mathematics Examination 1994

1. **(a)** Find **(i)** $\int \frac{4x-12}{x^2-6x+13}\,dx$ **(ii)** $\int \frac{1}{x^2-6x+13}\,dx$.

(b) Evaluate $\int_{\frac{3}{2}}^{3} \sqrt{9-u^2}\,du$. **(c)** Evaluate $\int_1^2 \frac{11-2t}{(2t-1)(3-t)}\,dt$.

(d) Evaluate $\int_0^{\pi} e^{2x}\sin x\,dx$.

2. **(a)** Let $z = a + ib$, where a and b are real. Find:

(i) $\Im(4i - z)$

(ii) $\overline{(3iz)}$ in the form $x + iy$, where x and y are real

(iii) $\tan\theta$, where $\theta = \arg(z^2)$.

(b) Express in modulus-argument form:

(i) $-1 + i$ **(ii)** $(-1 + i)^n$, where n is a positive integer.

(c) **(i)** On the same diagram, draw a neat sketch of the locus specified by each of the following:

(α) $|z - (3 + 2i)| = 2$ **(β)** $|z + 3| = |z - 5|$.

(ii) Hence write down all the values of z which satisfy simultaneously

$$|z - (3 + 2i)| = 2 \quad \text{and} \quad |z + 3| = |z - 5|$$

(iii) Use your diagram in **(i)** to determine the values of k for which the simultaneous equations $|z - (3 + 2i)| = 2$ and $|z - 2i| = k$ have exactly one solution for z.

3. **(a)** The hyperbola $\mathcal{H} : 16x^2 - 9y^2 = 144$ has foci $S(5, 0)$ and $S'(-5, 0)$.

The directrices are $x = \frac{9}{5}$ and $x = -\frac{9}{5}$.

(i) Find the equation of each asymptote of $\mathcal{H}$.

(ii) Sketch $\mathcal{H}$ and indicate on your diagram the foci, directrices, and asymptotes.

(iii) By differentiation, find the gradient of the tangent to $\mathcal{H}$ at $P(3\sec\theta, 4\tan\theta)$.

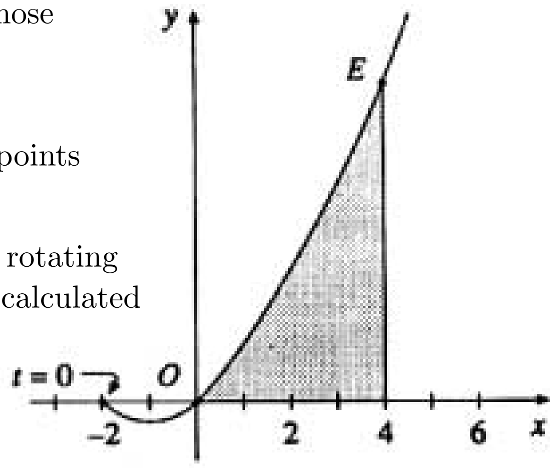
(iv) Show that the tangent to $\mathcal{H}$ at P has equation $4x = (3\sin\theta)y + 12\cos\theta$.

(v) Given that $0 < \theta < \frac{\pi}{2}$, show that Q, the point of intersection of the tangent to $\mathcal{H}$ at P and the nearer directrix, has y coordinate $\frac{12-20\cos\theta}{5\sin\theta}$.

(vi) Calculate the gradient of SP and SQ.

(vii) Determine whether $\angle PSQ$ is a right angle.

(b) The graph shows part of the curve whose parametric equations are
$x = t^2 + t - 2, y = t^3 - t, t \geq 0$.



(i) Find the values of t corresponding to points O and E on the curve.

(ii) The volume V of the solid formed by rotating the shaded area about the y axis is to be calculated using cylindrical shells.

Express V in the form
$V = 2\pi \int_a^b f(t)\ dt$.

Specify the limits of integration a and b and function $f(t)$. You may leave $f(t)$ in unexpanded form. Do NOT evaluate this integral.

4. **(a)** Find α and β, given that $z^3 + 3z + 2i = (z - \alpha)^2(z - \beta)$.

(b) Alex decides to go bungy-jumping. This involves being tied to a bridge at a point O by an elastic cable of length l metres, and then falling vertically from rest from this point.

After Alex free-falls l metres, she is slowed down by the cable, which exerts a force, in newtons, of Mgk times the distance greater than l that she has fallen (where M is her mass in kilograms, g m/s^2 is the constant acceleration due to gravity, and k is a constant).

Let x m be the distance Alex has fallen, and let v m/s be her speed at x. You may assume that her acceleration is given by $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

(i) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g$ when $x \leq l$,

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - gk(x - l) \text{ when } x > l.$$

(ii) Show that $v^2 = 2gl$ when Alex first passes $x = l$.

(iii) Show that $v^2 = 2gx - kg(x - l)^2$ for $x > l$.

(iv) Show that Alex's fall is halted first at $x = l + \frac{1}{k} + \sqrt{\frac{2l}{k} + \left(\frac{1}{k}\right)^2}$.

(v) Suppose $\frac{1}{k} = \frac{l}{4}$.

Show that O must be at least $2l$ metres above any obstruction on Alex's path.

5. **(a)** Let $f(x) = \frac{(x-2)(x+1)}{5-x}$ for $x \neq 5$.

(i) Show that $f(x) = -x - 4 + \frac{18}{5-x}$.

(ii) Explain why the graph of $y = f(x)$ approaches that of $y = -x - 4$ as $x \to \infty$ and as $x \to -\infty$.

(iii) Find the values of x for which $f(x)$ is positive, and the values of x for which $f(x)$ is negative.

(iv) Using part **(i)**, show that the graph of $y = f(x)$ has two stationary points. (There is no need to find the y coordinates of the stationary points.)

(v) Sketch the curve $y = f(x)$. Label all asymptotes, and show the x intercepts.

(b) A jar contains w white and r red jellybeans. Three jellybeans are taken at random from the jar and eaten.

(i) Write down an expression, in terms of w and r, for the probability that these 3 jellybeans were white.

Gary observed that if the jar had initially contained $(w + 1)$ white and r red jellybeans, then the probability that the 3 eaten jellybeans were white would have been double that in part **(i)**.

(ii) Show that $r = \frac{w^2-w-2}{5-w}$.

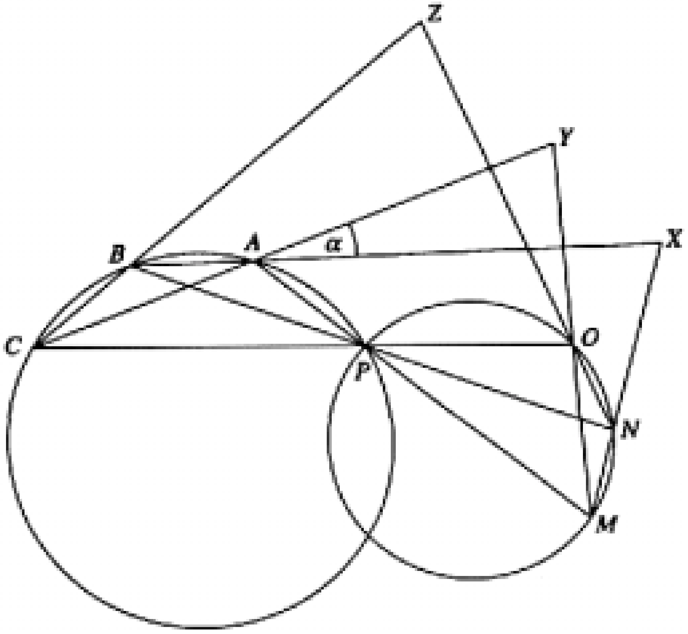
(iii) Using part **(a)** **(v)**, or otherwise, determine all possible numbers of white and red jellybeans.

6. **(a)** **(i)** Given that $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, explain why
$\int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx < \int_0^{\frac{\pi}{2}} e^{-2x/\pi}\, dx.$

(ii) Show that $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}\, dx = \int_0^{\frac{\pi}{2}} e^{-\sin x}\, dx.$

(iii) Hence show that $\int_0^{\pi} e^{-\sin x}\, dx < \frac{\pi}{e}(e - 1).$

(b)



Circles $PABC$ and $PMNO$ intersect at P, and APM, BPN, and CPO are straight lines. BA and MN produced meet at X, CA and MO produced meet at Y, and CB and NO produced meet at Z, as in the diagram. Let $\angle YAX = \alpha$.

(i) Prove that $\angle BPC = \alpha$.

(ii) Prove that $\angle OMN = \alpha$.

(iii) Prove that $XYAM$ is a cyclic quadrilateral.

(iv) Prove that $\angle XYM = \angle BCP$.

(v) Prove that X, Y, and Z are collinear.

7. (a) (i) It is known that if $f'(x) \geq 0$ and $f(0) = 0$, then $f(x) \geq 0$ for $x > 0$.

Show that $\sin x - x + \frac{x^3}{6} \geq 0$ for $x > 0$, and hence show that $\frac{\sin x}{x} \geq 1 - \frac{x^2}{6}$ for $x > 0$.

Let the points $A_0, A_1, A_2, \ldots, A_{n-1}$ represent the nth roots of unity on an Argand diagram, where A_k represents $\cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$.

Let P be the regular polygon $A_0 A_1 \cdots A_{n-1}$.

(ii) Show that the area of P is $\frac{n}{2} \sin \frac{2\pi}{n}$.

(iii) Using part **(i)**, or otherwise, show that for all $n \geq 26, P$ covers more than 99% of the unit circle.

(b) For all integers $n \geq 1$, let $t_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}$.

That is: $t_1 = \frac{1}{2}$

$t_2 = \frac{1}{3} + \frac{1}{4}$

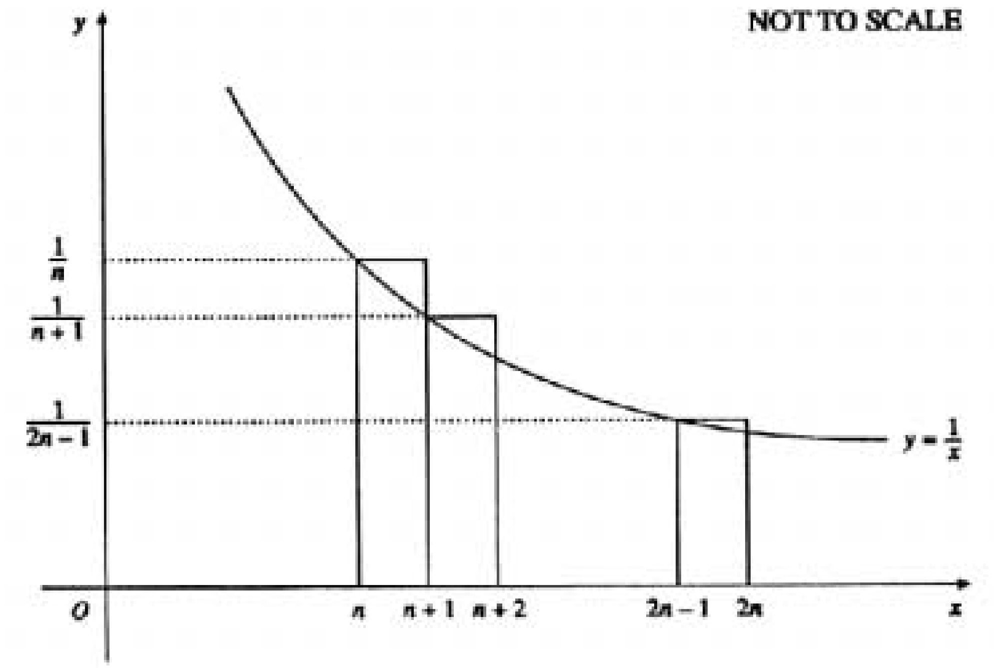
$t_3 = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$

. . . .

. . . .

(i) Show that $t_n + \frac{1}{2n} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$

(ii)



The diagram above shows the graph of the function $y = \frac{1}{x}$ for $n \leq x \leq 2n$.

Use the diagram to show that $t_n + \frac{1}{2n} > \ln 2$.

[Note that it can similarly be shown that $t_n < \ln 2$.]

(iii) For all integers $n \geq 1$, let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$.

That is: $s_1 = 1 - \frac{1}{2}$

$s_2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$s_3 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}$

. . . .

. . . .

Prove by mathematical induction that $s_n = t_n$.

(iv) Hence find, to three decimal places, the value of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{9999} - \frac{1}{10000}$.

8. **(a)** Suppose $a > 0, b > 0, c > 0$.

(i) Prove that $a^2 + b^2 \geq 2ab$.

(ii) Hence prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$.

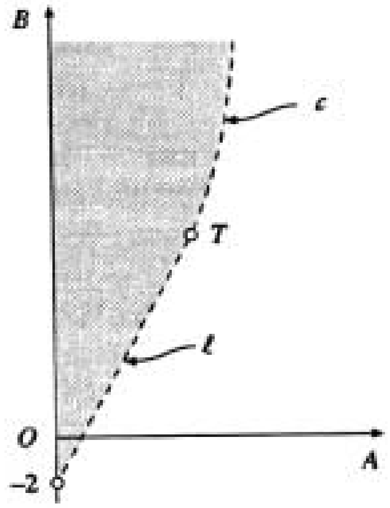
(iii) Given $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ prove that $a^3 + b^3 + c^3 \geq 3abc$.

In parts **(iv)** and **(v)**, assume $x > 0, y > 0, z > 0$.

(iv) By making suitable substitutions into **(iii)**, show that $x + y + z \geq 3(xyz)^{1/3}$.

(v) Suppose $(1 + x)(1 + y)(1 + z) = 8$. Prove that $xyz \leq 1$.

(b) Let $x = \alpha$ be a root of the quartic polynomial $P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$, where A and B are real. Note that α may be complex.



(i) Show that $\alpha \neq 0$.

(ii) Show that $x = \alpha$ is a root of $Q(x) = x^2 + \frac{1}{x^2} + A\left(x + \frac{1}{x}\right) + B$.

(iii) With $u = x + \frac{1}{x}$, show that $Q(x)$ becomes $R(u) = u^2 + Au + (B - 2)$.

(iv) For certain values of A and B, $P(x)$ has no real roots. Let $\mathcal{D}$ be the region of the AB plane where $P(x)$ has no real roots and $A \geq 0$.

The region $\mathcal{D}$ is shaded in the figure. Specify the bounding straight-line segment ℓ and the curved segment c. Determine the coordinates of T.