POLYNOMIALS

EXERCISE 1

1. For each of the following polynomials, state the constant term, the leading term, the leading coefficient and weather the polynomial is monic, the degree.

(a) $3x^2$ (b) 7 (c) 0 (d) $1 - x$
(e) $x^2 - 2x - 3$ (f) $(x + 2)^2 - (x - 2)^2$ (g) $x^3(x^2 - 7)$
(h) $(x + 1)^3 - (x - 1)^3$ (i) $x(x^2 + x - 1) - x(x^2 - x + 1)$
(j) $(x + 1)^3 - (x - 1)^3$

2. Explain why each of the following is not a polynomial.

(a) $7x^2 - 5x^{-3} + 3$ (b) $4\sqrt{x} - 2x - 5$ (c) $(x + 3)^3 - (x - 1)^{\frac{2}{3}}$

3. If $A(x) = 2x + 3$, $B(x) = x^2 - x + 2$, $C(x) = (x - 2)^2$, write down the polynomials:

(a) $A(x) + B(x)$ (b) $B(x) - C(x)$ (c) $A(x) . B(x)$
(d) $A(x)[B(x) + C(x)]$

4. State the smallest field (Q, R, C) over which the polynomials are defined.

(a) $3x^2 - 7x + 2$ (b) $(3 + 2i)x^3 - 7x + 5$ (c) $2\sqrt{3}x^4 + (1 + \sqrt{2})x - 8$
(d) $3 + 4x - 5\sqrt[3]{2}x^3$ (e) $ix + 7$ (f) $\frac{1}{4}x + 8\sqrt{3}$
(g) 3 (h) $\frac{2}{3}x^7 - 5ix + 9$ (i) $(2 + 3\sqrt{2})x^2 + (3 + 7\sqrt{2})$

5. (a) If $x^2 = a(x - 1)^2 + b(x - 1) + c$ find a, b, c
(b) If $2(x - 1)^2 = c(x^2 + 1) + dx$ find c, d
(c) If $a_0 + a_1x + a_2x^2 = 3x^2 - 5$ evaluate a_0, a_1, a_2
(d) If the polynomials $(a + 2b) + (b + 3c)x - (2b + 5c)s^2$, $5 + 11x - 19x^2$ are equal, find a, b, c

6. (a) If $P(x) = x^2 - 4$, $Q(x) = x + 2$ and $P(x) = Q(x) . R(x)$, find $R(x)$
(b) If $a(x) = x^3 - 8$, $b(x) = x - 2$ and $q(x) = x^2 - 2x + 2$, find $r(x)$ where $a(x) = b(x) . q(x) + r(x)$
(c) If $A(x) = 4x^2 + 3x - 1$, $B(x) = 4x - 1$, $C(x) = kx + l$, where k, l are constants, find k, l if $A(x) = B(x) . C$

7. For the polynomial $(a + 3)x^8 - (7 - 2b)x^5 + (12 - 6c)$, find the constants a, b, c if the polynomial is:

(a) monic (b) of zero degree (c) a zero polynomial

EXERCISE 2

1. Reduce where possible each of the following polynomials into irreducible factors over the field of:
(i) rationals Q (ii) reals R (iii) complex numbers C

(a)	$x^4 - 1$	(b)	$x^4 - 9$	(c)	$x^2 - x - 6$
(d)	$x^2 + 4x + 2$	(e)	$x^2 + 2x + 5$	(f)	$3x^2 - 2x - 4$
(g)	$3x^2 - 2x + 4$	(h)	$x^3 - 27$	(i)	$x^3 + 27$

2. Verify that $(x + 2)$ is a factor of $x^3 - x^2 + 4$ and hence find the prime factors over each of the fields Q, R, C.

3. Show that:
(a) $x^4 - 7x^2 + 1 = (x^2 + 1)^2 - 9x^2$ (b) $x^6 - 64 = (x^3 - 8)(x^3 + 8)$
(c) $x^4 - x^2 - 12 = (x^2 - 4)(x^2 + 3)$ (d) $x^4 - x^2 + 4x - 4 = x^4 - (x - 2)^2$
and hence reduce each of these polynomials into prime polynomials over the field of real numbers.

4. Reduce each polynomial into irreducible factors over:
(i) Q (ii) R (iii) C

(a)	$x^4 - 6x^2 + 8$	(b)	$x^4 + x^2 - 20$	(c)	$x^4 - 10x^2 + 21$
(d)	$x^4 - 3x^2 - 28$	(e)	$x^4 + 7x^2 + 6$	(f)	$x^6 - 9x^3 + 8$
(g)	$x^8 - 6x^4 + 5$	(h)	$(x^2 + 3x)^2 - (x^2 + 3x) + 20$		
(i)	$(2x^2 + 3x)^2 + 3(2x^2 + 3x) + 2$			(j)	$(x^2 + 5x)^2 - 4$

EXERCISE 3

1. Show that $x^2 + 6x + 8$ properly divides $P(x) = x^3 + 5x^2 + 2x - 8$, and hence reduce $P(x)$ into Irreducible factors over the field of rationals.

2. Find polynomials $Q(x)$, $R(x)$ over the field F stated, such that $A(x) = Q(x) . B(x) + R(x)$
Where $A(x)$, $B(x)$ are polynomials over F and $R(x) = 0$ or $0 \leq \deg R(x) \leq \deg B(x)$ when:

(a)	$A(x) = x^3 - 5x^2 + 7x - 2$,	$B(x) = x + 3$;	polynomials over Q
(b)	$A(x) = 3x^4 + 7x^3 - 8$,	$B(x) = x^2 + 2$;	polynomials over Q
(c)	$A(x) = 2\sqrt{3}x^5 - 4x^2 - 31$,	$B(x) = x - \sqrt{3}$;	polynomials over R
(d)	$A(x) = x^3 + 5ix^2 - 7ix - 3$,	$B(x) = x - 2i$;	polynomials over C

EXERCISE 4

1. Without dividing, find the remainder when each of the following is divided by the linear polynomial.
   (a) $(x^3 - 5x^2 - 4x + 7)$ by $(x - 1)$ (b) $(x^4 - 9x^2 + 3x + 2)$ by $(x + 3)$
   (c) $(x^2 + x - 1)$ by $(x + i)$ (d) $(x^3 + x^2 + x + 1)$ by $(x + 1 + i)$

2. Show that $(2x - 1)$ is a factor of $6x^3 + 13x^2 + 2x - 5$. Hence factorise the polynomial to irreducible factors over the rational field Q. What are the zeros of this polynomial in Q?

3. (a) If $x^3 - ax^2 + 1$ leaves a remainder of -2 when divided by $(x - 1)$, find a.
   (b) Find a if $(x + 1)$ divides $x^3 - ax^2 - 2ax + 8$.
   (c) $f(y) = y^3 - by^2 - by + 2b$. Find b if $f(y)$ leaves a remainder of 12 when divided by $(y - 2)$.
   (d) The polynomial $p(s) = ax^2 + bx - 6$ is divisible by $x + 1$. If $a + b = 20$, find a and b.
   (e) $x^3 + ax^2 - 2x + b$ has $x + 1$ as a factor and leaves a remainder of 4 when divided by $x - 3$. Find a and b.

4. A polynomial P(x) has roots 2, $1 + i$, $1 - i$. What are three factors of P(x)? If it is known that P(x) is a monic polynomial of degree 3, find P(x). If P(x) is a polynomial of degree 3 with a leading coefficient of 5, find P(x).

5. Determine the rational zeros of the polynomial $f(x) = 4x^4 + 8x^3 + 7x^2 + 8x + 3$. Hence find the decomposition of $f(x)$ over the field of (a) Q (b) C.

EXERCISE 5

1. Use the factor theorem to show that $x - i$ is a factor of $P(x) = x^4 + 3x^3 + 6x^2 + 3x + 5$.
   Name one other factor of P(x). Hence reduce P(x) to linear factors over the complex field.

2. Show that $1 - 2i$ is a zero of the polynomial $p(x) = x^3 - 5x^2 + 11x - 15$. Hence resolve p($x$) into irreducible factors over the field of (a) C (b) R.

3. Show that $1 + i$ is a zero of the polynomial $x^3 - x^2 + 2$ and find the other two zeros in C.

4. If $x - 2 - 2i$ is a factor of $x^3 - ax^2 + 20x - 24$, find a and the other zeros over C.

5. (a) $3 + \sqrt{5}$ is one root of a polynomial P(x), which has rational coefficients.
   State one other root of P(x)
   (b) $\sqrt{3}$ is one root of a polynomial Q(x) with rational coefficients. State one other root of Q(x).
   (c) $3 - 2\sqrt{5}$ is one root of a polynomial R(x) with rational coefficients. State one other root of R(x).

6. Find the polynomial of the least degree, with integral coeffiencts which have no common factor, having the roots (a) $\sqrt{5}$ and $2 - i$ (b) $3 - \sqrt{2}$ and $3 - 2i$

EXERCISE 6

1. Find the roots of the polynomials:
   (a) $16x^3 - 12x^2 + 1$ given that it has a 2-fold root
   (b) $x^4 - 6x^3 + 12x^2 - 10x + 3$ given that it has a root of multiplicity 3
   (c) $x^5 + 2x^4 - 2x^3 - 8s^2 - 7x - 2$, if it has a 4-fold root

2. Given that the polynomial $P(x) = x^4 + x^2 + 6x + 4$ has a rational root of multiplicity 2, find all the roots of $P(x)$ over the complex field.

3. Find the value of c if the polynomial $5x^5 - 3x^3 + c$ has a repeated positive root.

4. Prove that $ax^2 + bx + c$ has a double root if $b^2 - 4ac = 0$.

5. Find the condition that $f(x) = x^3 - 3ax + b$ has repeated factors.

EXERCISE 7

1. If α, β, γ are the roots of the polynomial $x^3 - x^2 + 5x - 3$ in the field of complex numbers, find the value of:

(a)	$\Sigma\alpha$	(b)	$\Sigma\alpha\beta$	(c)	$\Sigma\alpha\beta\gamma$	(d)	$\Sigma\alpha^2$
(e)	$\Sigma\alpha^3$	(f)	$\Sigma\alpha^4$	(g)	$\Sigma\alpha^5$	(h)	$\Sigma\frac{1}{\alpha}$
(i)	$\Sigma\alpha^{-2}$	(k)	$\Sigma\alpha^2\beta^2$	(k)	$(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$		

2. If p, q, r are the zeros in the field of complex numbers of the polynomial $2x^2 - 3x$ - , find the values of:
   (a) $(p + 1)(q + 1)(r + 1)$ (b) $(p + q - r)(q + r - p)(p + r - q)$
   (c) $\Sigma (pq)^{-1}$

3. If α, β, γ are the roots in a field F of the polynomial $x^3 + ax + b$, which is reducible to linear factors of F, find in terms of a and b:

(a)	$\Sigma\alpha$	(b)	$\Sigma\alpha^2$	(c)	$\Sigma\alpha^3$	(d)	$\Sigma\alpha^4$
(e)	$\Sigma\alpha^2\beta$	(f)	$\Sigma\alpha^2\beta^2$	(g)	$\Sigma\alpha^3(\beta + \gamma)$	(h)	$\Sigma(\alpha + \beta)^{-1}$

4. If a, b, c, d are the roots of the polynomial $y^4 - 4y^2 - y + 2$, find the values of:

(a)	Σa	(b)	Σab	(c)	Σabc	(d)	$abcd$
(e)	Σa^{-1}	(f)	Σa^2				

   (g) Σa^2b [first prove $\Sigma a^2b = (\Sigma a)(\Sigma ab) - 3(\Sigma abc)$]
   (h) Σa^3 [first prove $\Sigma a^3 = (\Sigma a^2)(\Sigma a) - \Sigma a^2b$]
   (l) Use the factor theorem to factorise $y^4 - 4y^2 - y + 2$ and state in which fields the results in (a) to (k) are valid.

EXERCISE 8

1. If α, β, γ are the roots of $x^3 + 2x^2 - 2x + 3 = 0$, form the equation whose roots are:
   (a) $\frac{2}{\alpha}, \frac{2}{\beta}, \frac{2}{\gamma}$ (b) $2\alpha, 2\beta, 2\gamma$ (c) $\alpha + 1, \beta + 1, \gamma + 1$
   (d) $\alpha - 2, \beta - 2, \gamma - 2$ (e) $\alpha^2, \beta^2, \gamma^2$ (f) $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$

2. If α, β, γ are the roots of $2x^3 + 3x^2 - x - 1 = 0$, form the equation whose roots are:
   (a) $\alpha + 2, \beta + 2, \gamma + 2$ (b) $\frac{1}{\alpha+2}, \frac{1}{\beta+2}, \frac{1}{\gamma+2}$ [Hint: use (a)]
   (c) $\alpha^2, \beta^2, \gamma^2$

3. If α, β, γ are the roots of $ax^3 + bx^2 + cx + \mathrm{d} = 0$, form the equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ and hence evaluate:
   (a) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (b) $\alpha^{-1}\beta^{-1} + \beta^{-1}\gamma^{-1} + \gamma^{-1}\alpha^{-1}$
   (c) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$

4. If α, β, γ, δ are the roots of $x^4 - x^2 + 2x + 3 = 0$, form the equation whose roots are:
   (a) $2\alpha, 2\beta, 2\gamma, 2\delta$ (b) $\alpha^{-1}, \beta^{-1}, \gamma^{-1}, \delta^{-1}$ (c) $\alpha^2, \beta^2, \gamma^2, \delta^2$

5. α, β, γ are the roots of $x^3 - 2x + 3 = 0$
   (a) From the equation whose roots are $\alpha^2, \beta^2, \gamma^2$
   (b) Using the result of (a), now form the equation with roots $\alpha^2 + 1, \beta^2 + 1, \gamma^2 + 1$
   (c) Hence evaluate $(\alpha^2 + 1)(\beta^2 + 1)(\gamma^2 + 1)$

6. Use the forth degree equation $\mathrm{a}x^4 + bx^3 + cx^2 + dx + e = 0$ whose roots are α, β, γ, δ to verify that:
   (a) The equation $\mathrm{P}\left(\frac{x}{m}\right) = 0$ has roots m times those of $\mathrm{P}(x) = 0$
   (b) The equation $\mathrm{P}\left(\frac{1}{x}\right) = 0$ has roots which are reciprocals of those $\mathrm{P}(x) = 0$
   (c) The equation $\mathrm{P}(x + k) = 0$ has roots which are k less than those of $\mathrm{P}(x) = 0$

EXERCISE 9

1. Express cos 5θ as a polynomial in cos θ, and obtain a similar expression for sin 5θ as a polynomial in sin θ.
   (a) Solve the equation cos 5θ = 1 for 0 < θ 2π and hence find the roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$. Hence prove that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$ and $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$.
   (b) Solve the equation cos 5θ = 0 for 0 < θ 2π and hence find the roots of the equation $16x^4 - 20x^5 + 5 = 0$. Determine the exact value for $\cos\frac{\pi}{10}\cos\frac{3\pi}{10}$ and $\cos^2\frac{\pi}{10} - \cos^2\frac{3\pi}{10}$.
   (c) Solve the equation sin 5θ = 1 for 0 < θ 2π and hence find the roots of the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$. Determine the exact value of $\sin\frac{\pi}{10}\sin\frac{3\pi}{10}$.
   (d) Show that the roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $x = \pm\sin\frac{\pi}{5}\sin\frac{2\pi}{5}$.and prove that $\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \frac{5}{4}$.

2. Prove that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta - 7\cos\theta$.
   (a) Hence find the roots of the equation $64x^6 - 112x^4 + 56x^2 - 7 = 0$. Deduce that $\cos\frac{\pi}{14} . \cos\frac{3\pi}{14} . \cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$ and $\cos^2\frac{\pi}{14} + \cos^2\frac{3\pi}{14} + \cos^2\frac{5\pi}{14} = \frac{7}{8}$.
   (b) Solve the equation cos 7θ = 1 for 0 < θ 2π, and hence find the roots of the equation $64x^7 - 112x^5 + 56x^3 - 7 = 1$. Deduce that $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$ and $\cos\frac{\pi}{7} . \cos\frac{\pi}{7} . \cos\frac{5\pi}{7} = -\frac{5}{4}$.