Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Examiner: B. Weiss

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**

- Attempt all questions

Class Teacher:
(Please tick or highlight)

- ✓ Mr Barrett
- O Mr Fletcher
- O Mr Weiss

Ben L

Student Number 194 88152

(To be used by the exam markers only.)

Question No	1	2	3	4	5	6	7	8	Total	Total
Mark	11/15	14/15	13/15	14/15	12/15	15/15	8/15 +3	15/15	102/120	/100

H.

Question 1

(a) Find the following integrals:

(i) $\displaystyle\int \tan^3 x\, dx$ 3

(ii) $\displaystyle\int \frac{dx}{x^2 - 6x + 13}$ 2

(b) Evaluate

(i) $\displaystyle\int_0^1 \frac{x}{\sqrt{4 - x^2}}\, dx$ 3

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} x \cos^2 x\, dx$ 3

(c) (i) Show that if $I_n = \displaystyle\int_0^1 x^n e^{-x}\, dx$, then $I_n = n \cdot I_{n-1} - \dfrac{1}{e}$ 2

(ii) Hence find $\displaystyle\int_0^1 x^3 e^{-x}\, dx$. 2

Question 2 (Start a new page)

(a) Find $\sqrt{6i - 8}$, and hence solve the equation $2z^2 - (3 + i)\, z + 2 = 0$. 4

(b) Solve $3x^3 - 10x^2 + 7x + 10 = 0$ given that $x = 2 - i$ is a root of the equation. 3

(c) The polynomial equation $P(x) = x^3 + px^2 + q = 0$ has roots α, β and γ. Form the polynomial equation with roots given by

(i) $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$ 2

(ii) α^2, β^2 and γ^2 2

(d) Use the method of cylindrical shells to find the voume of the solid generated by rotating the region bounded by $y = \ln x$, the x-axis and the lines $x = 1$ and $x = e$, about the y-axis. 4

Question 3 (Start a new page)

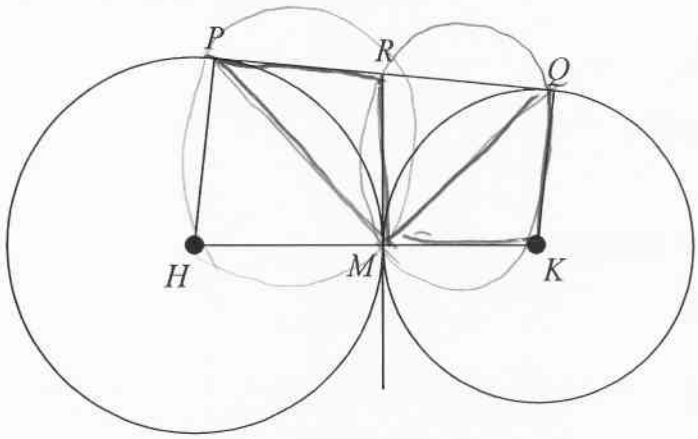
(a) (i) Prove that the equation of the tangent at the point $\left(t, \dfrac{1}{t}\right)$ to the hyperbola $xy = 1$ is $x + t^2y = 2t$. **2**

(ii) The tangent at a point P on the hyperbola $xy = 1$ meets the y-axis at A, and the normal at P meets the x-axis at B. Find the equation of the locus of the midpoint of AB as P moves on the hyperbola. (Draw a diagram) **3**

(b) $P(a\cos\alpha,\ b\sin\alpha)$ and $Q(a\cos\beta,\ b\sin\beta)$ are the endpoints of a focal chord of the curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **3**

Show that $e = \dfrac{\sin(\alpha - \beta)}{\sin\alpha - \sin\beta}$.

(c) Shown below are two circles with centres H and K which touch at M. PQ and RM are common tangents.



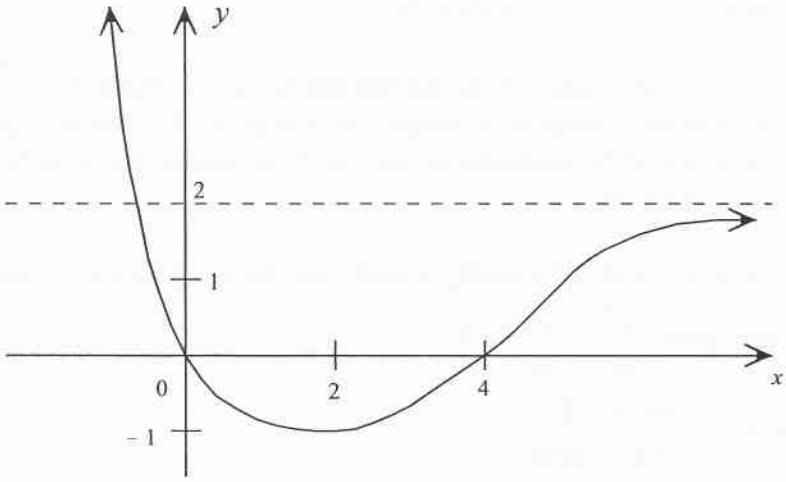
(i) Show that quadrilaterals $HPRM$ and $MRQK$ are cyclic. **2**

(ii) Prove tha triangles PRM and MKQ are similar. **2**

(d) Show that the polynomial equation $4x^3 + 20x^2 - 23x + 6 = 0$ has a double root, and find the value of each of its roots. **3**

Question 4

(a) The diagram shows the graph of $y = f(x)$.



Sketch on separate diagrams, the following curves, indicating clearly any turning points and asymptotes.

(i) $y = \dfrac{1}{f(x)}$ **1**

(ii) $y = [f(x)]^2$ **2**

Draw neat sketches of the following:

(b) $y = x \sin x$ **2**

(c) $y = \sin^{-1}(\sin x)$ **2**

(d) $y = x^2 - \dfrac{1}{x}$ **3**

(e) (i) $f(x) = \dfrac{x^2 - 4}{x - 3}$ **3**

(ii) $[f(x)]^2 = \dfrac{x^2 - 4}{x - 3}$ **2**

Question 5 (Start a new page)

(a) (i) Express the complex number $z = -\sqrt{3} + i$ in mod-arg form. **1**

(ii) Hence, or otherwise, show that $z^7 + 64z = 0$. **2**

(b) Find the equation, in Cartesian form, of the locus of the point z if

$$\text{Re}\left[\frac{z-4}{z}\right] = 0$$ **3**

(c) Sketch the region S in the complex plane, where

$$S = \left\{ |z| \le 1 \quad \text{and} \quad 0 \le \arg z < \frac{\pi}{3} \right\}$$ **2**

(d) (i) Use de Moivre's theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$. **3**

(ii) Hence express $\tan 5\theta$ as a rational function of t, where $t = \tan\theta$. **2**

(iii) Find $\tan\frac{\pi}{5} \cdot \tan\frac{2\pi}{5} \cdot \tan\frac{3\pi}{5} \cdot \tan\frac{4\pi}{5}$ **2**

Question 6 (Start a new page)

(a) A particle of mass 1 kg is projected upwards with initial speed 10 ms^{-1}. **5**
The air resistance is given by $R = \frac{1}{10}V^2$.
Take the acceleration due to gravity to be 10 ms^{-2}.
Find the maximum height reached, and the time taken to reach this height.

(b) Find the largest coefficient in the expansion of $(2x+3)^{21}$. **3**

(c) If $x^m y^n = k$, where k is a constant, show that $\frac{dy}{dx} = -\frac{my}{nx}$. **3**

(d) Use the expansion of $(1+x)^{2n}$ to show that

(i) $\binom{2n}{1} + \binom{2n}{2} + \binom{2n}{3} + \ldots + \binom{2n}{2n} = 4^n - 1$ **2**

(ii) Use the identity $(1+x)^{2n} \equiv (1+x)^n(1+x)^n$ to show that **2**

$$\binom{2n}{2} = 2\cdot\binom{n}{2} + \binom{n}{1}^2$$

Question 7 (Start a new page)

(a) Use the process of mathematical induction to show that

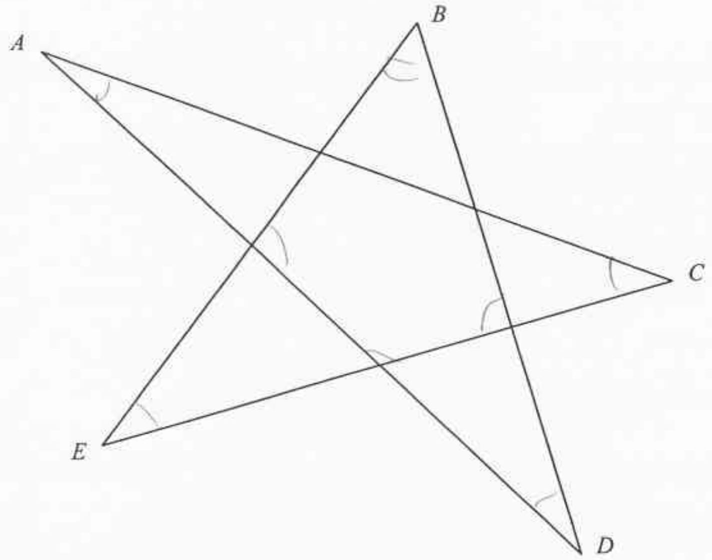
$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$$ 5

(b) With the aid of a diagram, show that the area enclosed by the ellipse 4

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by $A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx.$

Hence show that the area of the ellipse is πab.

(c) 2



Prove that $\angle A + \angle B + \angle C + \angle D + \angle E = 180°$

(d) The base of a solid is the region bounded by the parabolas $x = y^2$ and $x = 4 - 3y^2$, and the cross-sections perpendicular to the x-axis are squares.

(i) Draw a neat sketch of this solid. 1

(ii) Find the volume of the solid. 3

$3y^2 = 4 - x$

$y^2 = \frac{4-x}{3}$

$y = \sqrt{\frac{4-x}{3}}$

Question 8 (Start a new page)

(a) If a and b are positive numbers such that $a + b = 1$, prove that

(i) $a + b \geq 2\sqrt{ab}$ 1

(ii) $\frac{1}{a} + \frac{1}{b} \geq 4$ 2

(iii) $a^2 + b^2 \geq \frac{1}{2}$ 2

(iv) $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) \geq 9$ 2

(b) A particle is projected from ground level so that it just clears two poles of height h at distances of b and c metres from the point of projection. If v m/s is the velocity of projection, and θ is the angle of projection to the horizontal:

(i) Show that $y = x\tan\theta - \frac{gx^2}{2v^2} \cdot \sec^2\theta$ 2

(ii) Show that $v^2 = \frac{(b+c)g\sec^2\theta}{2\tan\theta}$ 3

(iii) Hence or otherwise show that $\tan\theta = \frac{h(b+c)}{bc}$ 3

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Q1) a) i/ $\int \tan^3 x\,dx$

$= \int \tan x(\sec^2 x - 1)\,dx$ 1

$= \dfrac{\tan^2 x}{2} - \log(\cos x) + C$ 1 1

ii/ $\int \dfrac{dx}{x^2-6x+13}$

$= \int \dfrac{dx}{(x-3)^2+4}$ 1

$= \dfrac{1}{2}\tan^{-1}\left(\dfrac{x-3}{2}\right) + C$ 1

b) i/ $\int_0^1 \dfrac{x}{\sqrt{4-x^2}}\,dx$

Let $u = 4 - x^2$ 1

$du = -2x\,dx$

$= -\dfrac{1}{2}\int_4^3 u^{-1/2}\,du$ 1

$= \dfrac{1}{2}\int_3^4 u^{-1/2}\,du$

$= \dfrac{1}{2}\left[2u^{1/2}\right]_3^4$

$= 2 - \sqrt{3}$ 1

ii/ $\int_0^{\pi/2} x\cos^2 x\,dx$ (with $dv = \cos^2 x\,dx$, $u = x$)

$= \int_0^{\pi/2} x\left(\dfrac{1}{2} + \dfrac{\cos 2x}{2}\right)dx$ 1

$= \left.\dfrac{x^2}{4}\right]_0^{\pi/2} + \int_0^{\pi/2} \dfrac{x\cos 2x}{2}\,dx$

$= \dfrac{\pi^2}{16} + \left.\dfrac{x\sin 2x}{4}\right]_0^{\pi/2} - \int_0^{\pi/2}\dfrac{\sin 2x}{4}\,dx$ 1

$= \dfrac{\pi^2}{16} + \left.\dfrac{\cos 2x}{8}\right]_0^{\pi/2}$

$= \dfrac{\pi^2}{16} + \dfrac{-1}{8} - \dfrac{1}{8}$

$= \dfrac{\pi^2 - 4}{16}$

c) i/ $I_n = \int_0^1 x^n e^{-x}\,dx$ (with $u = x^n$, $dv = e^{-x}\,dx$)

$= -x^n e^{-x}\Big]_0^1 + \int_0^1 n x^{n-1} e^{-x}\,dx$ 1

$= -e^{-1} + n\int_0^1 x^{n-1}e^{-x}\,dx$

$\therefore I_n = nI_{n-1} - \dfrac{1}{e}$ 1

ii/ $I_0 = \int_0^1 e^{-x}\,dx$

$= -\dfrac{1}{e^x}\Big]_0^1$

$= 1 - \dfrac{1}{e}$

$I_1 = I_0 - \dfrac{1}{e}$

$= 1 - \dfrac{1}{e} - \dfrac{1}{e}$

$= 1 - \dfrac{2}{e}$ 1

$I_2 = 2I_1 - \dfrac{1}{e}$

$= 2\left(1 - \dfrac{2}{e}\right) - \dfrac{1}{e}$

$= 2 - \dfrac{5}{e}$

$I_3 = 3I_2 - \dfrac{1}{e}$

$= 3\left(2 - \dfrac{5}{e}\right) - \dfrac{1}{e}$

$= 6 - \dfrac{16}{e}$

$\therefore \int_0^1 x^3 e^{-x}\,dx$

$= 6 - \dfrac{16}{e}$ 1

Q2) a) $\sqrt{6i-8} = a + ib$

$a^2 - b^2 = -8$

$2ab = 6$

$a = \dfrac{b}{3}$ 1

$\dfrac{b^2}{9} - b^2 = -8$

$-\dfrac{8b^2}{9} = 8$

$b = \pm 3$

$\therefore a = \pm 1$ 1

$2z^2 - (3+i)z + 2 = 0$

$x = \dfrac{(3+i) \pm \sqrt{9+6i-1-4.2.2}}{4}$ 1

$= \dfrac{3+i \pm \sqrt{6i-8}}{4}$

$= \dfrac{3+i \pm (1+3i)}{4}$

$= 1+i$ or $\dfrac{1-i}{2}$ 1

b) If $2-i$ is a root so is $2+i$

$(x-(2+i))(x-(2-i))$

$= x^2 - (2-i)x - (2+i)x + 5$

$= x^2 - 4x + 5$ 1

$$\begin{array}{r} 3x + 2 \\ x^2-4x+5 \overline{\smash{)}\,3x^3 - 10x^2 + 7x + 10} \\ 3x^3 - 12x^2 + 15x \\ \hline 2x^2 - 8x + 10 \end{array}$$ 1

$\therefore$ roots are $x = (2-i), (2+i), -\dfrac{2}{3}$ 1

c) $P(x) = x^3 + px^2 + q = 0$

$P\left(\dfrac{1}{x}\right) = \dfrac{1}{x^3} + \dfrac{p}{x} + \dfrac{q}{1} = 0$ 1

$= 1 + px^2 + qx^3 = 0$

$\therefore qx^3 + px^2 + 1 = 0$ 1

ii) $P(x) = x^3 + px^2 + q = 0$

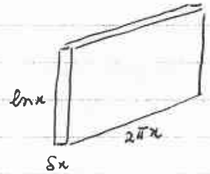
$P(\sqrt{x}) = x^{3/2} + px + q = 0$ 1

$\therefore x^{3/2} = -(px+q)$

$x^3 = p^2x^2 + 2pqx + q^2$

$\therefore x^3 - p^2x^2 - 2pqx - q^2 = 0.$ 1

d)



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Volume of slice $2\pi x\,\delta x\,\ln x$

Volume $= 2\pi\int_1^e x\ln x\,dx$ 1

$= 2\pi\left[\dfrac{x^2}{2}\ln x\right]_1^e - \dfrac{1.2\pi}{2}\int_1^e x\,dx$ 1

$= 2\pi\left[\dfrac{e^2}{2}\right] - \pi\left[\dfrac{x^2}{2}\right]_1^e$

$= \pi e^2 - \dfrac{\pi e^2}{2} + \dfrac{\pi}{2}$

$= \dfrac{\pi}{2}(e^2+1)\ u^3$ 1

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Q3) a) i/ $y = \frac{1}{x}$

$\frac{dy}{dx} = \frac{-1}{x^2}$

at $x = t$ $\frac{dy}{dx} = -\frac{1}{t^2}$ 1

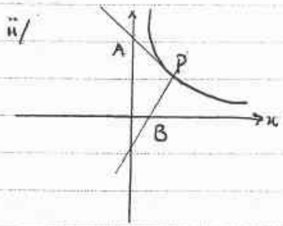
∴ eqn of tangent

$y - \frac{1}{t} = \frac{-1}{t^2}(x - t)$

$t^2 y - t = -x + t$

$x + t^2 y = 2t$ 1

ii/



tangent meets y axis at

$\left(0, \frac{2}{t}\right)$

normal meets x axis at

$t^2 x = y + t^3 - \frac{1}{t}$

$\left(t - \frac{1}{t^3}, 0\right)$ 1

Mid point $\left(\frac{t^4 - 1}{2t^3}, \frac{1}{t}\right)$ 1

$y = \frac{1}{t}$

$x = \frac{\cancel{t}^4 - 1}{2\cancel{t}^3}$

$= \frac{\frac{1}{y^4} - 1}{\frac{2}{y^3}}$

∴ $x = \frac{1 - y^4}{2y}$

$2xy + y^4 - 1 = 0$ 1

d) $P(x) = 4x^3 + 20x^2 - 23x + 6$

$P'(x) = 12x^2 + 40x - 23$

$= (6x + 23)(2x - 1)$

∴ $(6x + 23)(2x - 1) = 0$

when $x = \frac{1}{2}$ or $-\frac{23}{6}$ 1

$P(\frac{1}{2}) = 0 = P'(\frac{1}{2})$

∴ $x = \frac{1}{2}$ is the double root 1

Using product of roots

$\frac{1}{2} \cdot \frac{1}{2} \cdot \alpha = -\frac{6}{4}$

$\alpha = -6$

∴ roots are $\frac{1}{2}, \frac{1}{2}, -6$. 1

Q3)

b) grad of m $= \frac{b(\sin\alpha - \sin\beta)}{a(\cos\alpha - \cos\beta)}$

eqn of PQ

$y - b\sin\alpha = \frac{b}{a}\left(\frac{s\alpha - s\beta}{c\alpha - \cos\beta}\right)(x - a\cos\alpha)$

Now focal chord ⇒ $(ae, 0)$ satisfies eqn 1

$-\cancel{b}\sin\alpha = \frac{\cancel{b}(s\alpha - s\beta)}{\cancel{a}(c\alpha - \cos\beta)}(\cancel{a}e - \cancel{a}\cos\alpha)$

∴ $-\sin\alpha\frac{(c\alpha - c\beta)}{\sin\alpha - \sin\beta} = e - \cos\alpha$

$-\frac{(\sin\alpha\cos\alpha - \sin\alpha\cos\beta)}{\sin\alpha - \sin\beta} + \cos\alpha = e$

∴ $e = \frac{-\cancel{\sin\alpha\cos\alpha} + \sin\alpha\cos\beta + \cancel{\sin\alpha\cos\alpha} - \cos\alpha\sin\beta}{\sin\alpha - \sin\beta}$ 2

$= \frac{\sin(\alpha - \beta)}{\sin\alpha - \sin\beta}$

c) i/ In HPRM

∠HPR = ∠RMH (angle between radius + tangent) 1

∴ HPRM is a cyclic quad (opp angles add to 180°) 1

Similarly MRQK

ii/ Join PM and QM

In △PRM and MQK

∠PRM = ∠MKQ (ext angle of a cyclic quad)

PR = RM (tangents from an ext pt) 1

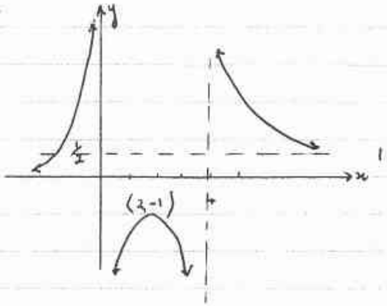
KM = KQ (radii)

∴ △ isos

∠RPM = ∠RMP = ∠KMQ 1

∴ △PRM ||| △MQK (equiangular)

Q4) a) i/



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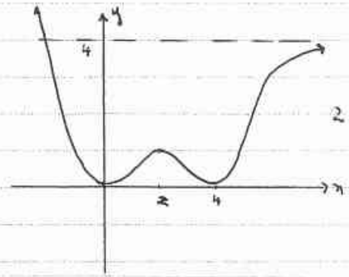
ii/



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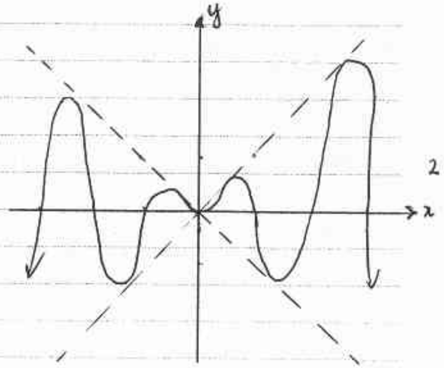
b)



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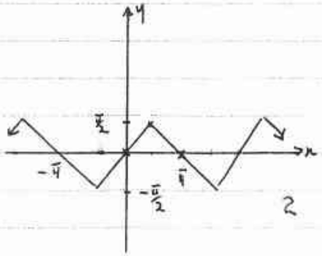
c)



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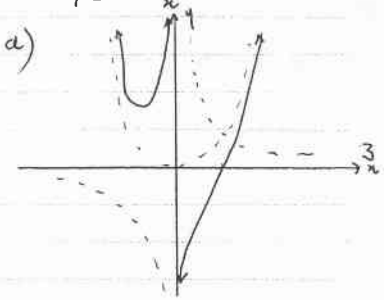
d) $y = x^2 - \frac{1}{x}$



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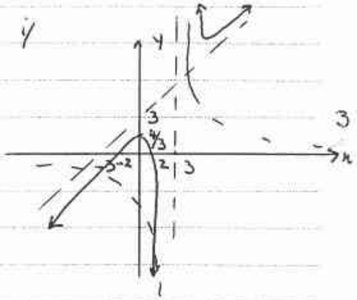
e) $f(x) = \frac{x^2-4}{x-3}$

$= x + 3 + \frac{5}{x-3}$

i/



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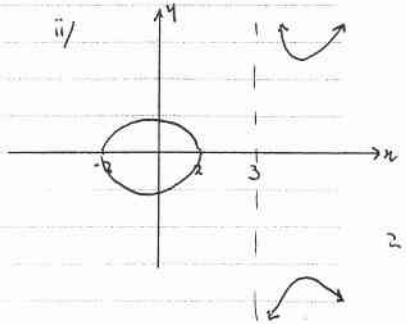
ii/



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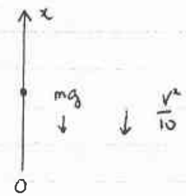
Q6)

a/



$m\ddot{x} = -mg - \frac{v^2}{10}$ 1

$\ddot{x} = -g - \frac{v^2}{10}$ $(\because m = 1)$

$\frac{dv}{dt} = -g - \frac{v^2}{10}$

$\therefore t = \int \frac{dv}{-g - \frac{v^2}{10}}$ 1

$= -10 \int \frac{dv}{100 + v^2}$

$= -\tan^{-1}\left(\frac{v}{10}\right) + c_1$

when $v = 10$, $t = 0 \Rightarrow c_1 = \frac{\pi}{4}$

$\therefore \tan^{-1}\frac{v}{10} = \frac{\pi}{4} - t$ 1

$\frac{v}{10} = \tan\left(\frac{\pi}{4} - t\right)$

when $v = 0$ $t = \frac{\pi}{4}$

$\therefore x = \int 10 \tan\left(\frac{\pi}{4} - t\right) dt$ 1

$= 10 \ln \cos\left(\frac{\pi}{4} - t\right) + c_2$

when $t = 0$, $x = 0$

$\therefore c_2 = -10 \ln \frac{1}{\sqrt{2}} = 5 \ln 2$

$\therefore$ when $x = \frac{\pi}{4}$ $x = 5 \ln 2$ metres.

1

b) $(2x+3)^{21}$

$\frac{T_{r+1}}{T_r} = \frac{{}^{21}C_r (2x)^{21-r} 3^r}{{}^{21}C_{r-1} (2x)^{22-r} 3^{r-1}}$ 1

$= \dfrac{\frac{21!}{r!(21-r)!} \times 3}{\frac{21!}{r!(22-r)!} \times 2}$

$= \frac{22-r}{r} \times \frac{3}{2}$

Now $\frac{66-3r}{2r} \geq 1$ 1

$66 - 3r \geq 2r$

$r \leq 13\frac{1}{5}$

$\therefore r = 13$

$\therefore T_{14} = {}^{21}C_{13}\, 2^8\, 3^{13}$ 1

c) $x^m y^n = k$

$\therefore x^m n y^{n-1} \frac{dy}{dx} + m x^{m-1} y^n = 0$ 1

$x^m n y^{n-1} \frac{dy}{dx} = -m x^{m-1} y^n$ 1

$\therefore \frac{dy}{dx} = \frac{-m x^{m-1} y^n}{n x^m y^{n-1}}$

$= \frac{-my}{nx}$ 1

d) i/ $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + \ldots + {}^{2n}C_{2n} x^{2n}$

when $x = 1$

$2^{2n} = 1 + {}^{2n}C_1 + \ldots + {}^{2n}C_{2n}$

$\therefore 4^n - 1 = {}^{2n}C_1 + \ldots + {}^{2n}C_{2n}$

ii/ $(1+x)^{2n} = (1+x)^n (1+x)^n$

term in x^2 on RHS

$= {}^nC_2 x^2 + {}^nC_1 x \, {}^nC_1 x + {}^nC_2 x$

equating coeff of x^2

Q5) a) i) $z = -\sqrt{3} + i$

$= 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$ 1

ii) $z^7 = 2^7 \operatorname{cis}\left(\frac{35\pi}{6}\right)$

$= 128\left(\cos -\frac{\pi}{6} + i \sin -\frac{\pi}{6}\right)$

$= 128\left(\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}\right)$ 1

$64z = 2 \cdot 64 \operatorname{cis} \frac{5\pi}{6}$

$= 128\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

$= 128\left(-\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

$= -128 \cos \frac{\pi}{6} + 128 \sin \frac{\pi}{6}$ 1

$\therefore z^7 + 64z = 0.$ 1

b) $\operatorname{Re}\left(\frac{z-4}{z}\right) = 0$

$\frac{z-4}{z} = \frac{x-4+iy}{x+iy} \times \frac{x-iy}{x-iy}$ 1

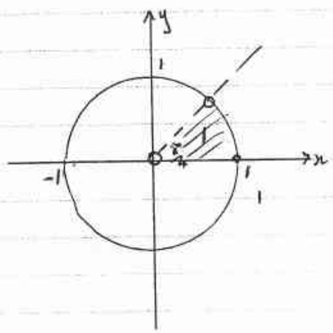
$= \frac{x^2 - 4x + y^2 + i(xy - xy + 4)}{x^2 + y^2}$ 1

$\therefore x^2 - 4x + y^2 = 0$

$(x-2)^2 + y^2 = 4$

circle centre $(2, 0)$ $r = 2$ 1

c)



Q5)

d) i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i \sin 5\theta$

$= \cos^5\theta + 5i\cos^4\theta \sin\theta + 10 \cos^3\theta (i\sin\theta)^2 + 10 \cos^2\theta (i\sin\theta)^3$

$+ 5 \cos\theta (i \sin\theta)^4 + (i \sin\theta)^5$

$= \cos^5\theta + 5i \cos^4\theta \sin\theta - 10 \cos^3\theta \sin^2\theta - 10 i\cos^2\theta \sin^3\theta$

$+ 5 \cos\theta \sin^4\theta + i \sin^5\theta$

$\therefore \cos 5\theta = \cos^5\theta - 10 \cos^3\theta \sin^2\theta + 5 \cos\theta \sin^4\theta$

$\sin 5\theta = 5 \sin\theta \cos^4\theta - 10 \sin^3\theta \cos^2\theta + \sin^5\theta$ 3

ii) Now $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5c^4 s - 10c^2 s^3 + s^5}{c^5 - 10c^3 s^2 + 5s^4 c}$ $\div c^5$

$= \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ let $\tan\theta = t$ 2

iii) Let $\tan 5\theta = 0$

then $5\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi \ldots$

$\therefore \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \ldots$

$\tan 5\theta = 0$ has an infinite no. of soln^s

but $\frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4} = 0$

has only 5 roots

$t(5 - 10t^2 + 10^4) = 0$

$\tan\theta = 0$ $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$, $\tan\frac{4\pi}{5}$ 2

product of roots is 5

$\therefore \tan\frac{\pi}{5} \cdot \tan\frac{2\pi}{5} \tan\frac{3\pi}{5} \tan\frac{4\pi}{5} = 5$

Q7) a) $\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$

Step 1 Let $n = 1$

L.H.S. $= \frac{1}{2!} = \frac{1}{2}$ RHS $= 1 - \frac{1}{2!} = \frac{1}{2}$ 1

$\therefore$ true for $n = 1$

Step 2. Assume true for $n = k$

ie

Prove true for $n = k+1$

$\sum_{k=1}^{n+1} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$

$= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!}$

$= 1 - \frac{n+2-n-1}{(n+2)!}$ 3

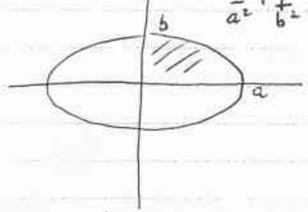
$= 1 - \frac{1}{(n+2)!}$

$=$ RHS.

So if true for n it is true for $n+1$

Step 3 It is true for $n=1$ and as it works for n and $n+1$ then it is true by math induction 1

b)



1

Area of ellipse $= 4 \times$ shaded area

$A = 4\int_0^a y\,dx$ 1

$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

as $y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$y = \frac{b}{a}\sqrt{a^2-x^2}$ 1

$A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a} \times \frac{\pi a^2}{4}$ ($\frac{1}{4}$ of circle)

$= \pi ab \ u^2$ 1

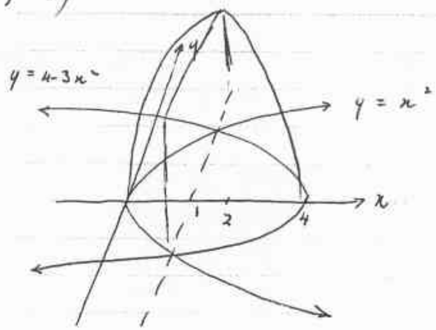
c) $\hat{F} = D\hat{F}C = \hat{A} + \hat{C}$ (ext angle of $\triangle$)

$\hat{G} = D\hat{G}F = \hat{B} + \hat{E}$ (" " ") 1

$\hat{D} + \hat{F} + \hat{G} = 180°$ (angle sum of $\triangle$)

$\therefore \hat{A} + \hat{B} + \hat{C} + \hat{D} + \hat{E} = 180$ 1

Q7) d)



Curves meet at $x = 1$ 1

From $x = 0$ to 1 the cross-section with co-ord x has $A = (2y)^2 = 4y^2$

$= 4x$

$\therefore V = \int_0^1 4x\,dx$

$= 2$ 1

from $x = 1$ to 4 the cross section ~~to~~ co-ord x has $A = (2y)^2 = 4y^2$

$= \frac{4}{3}(4-x)$

$\therefore V = \frac{4}{3}\int_1^4 4 - x\,dx$

$= \frac{4}{3}\left[4x - \frac{x^2}{2}\right]_1^4$

$= \frac{4}{3}\left(16 - 8 - 4 + \frac{1}{2}\right)$

$= 6$ 1

$\therefore$ total volume is $8 \ u^3$

1

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Q8) a) ii/ $\frac{1}{a}+\frac{1}{b} = \frac{b+a}{ab}$

as $a+b=1$

$1 = a+b \geq 2\sqrt{ab}$

$\frac{1}{2} \geq \sqrt{ab}$

$ab \leq \frac{1}{4}$

$\therefore \frac{1}{ab} \geq 4$

$a+b \geq 4$ 2

iii/ $a^2+b^2 = a^2+(1-a)^2$

$= 2a^2-2a+1$

$= 2(a-\frac{1}{2})^2+\frac{1}{2} \geq \frac{1}{2}$

equality holds when $a=\frac{1}{2}$ 2

iv/ $(1+\frac{1}{a})(1+\frac{1}{b}) = (1+\frac{1}{a})(1+\frac{1}{1-b})$

$= 1+\frac{1}{a}+\frac{1}{1-a}+\frac{1}{a(1-a)}$

$= 1+\frac{2}{a(1-a)}$

Now $a(1-a) = -a^2+a = -(a-\frac{1}{2})^2+\frac{1}{4} \leq \frac{1}{4}$

and $1+\frac{2}{a(1-a)} \geq 1+\frac{2}{\frac{1}{4}}$

$\geq 9.$ 2

i/ for $a,b>0$

$a+b \geq 2\sqrt{ab}$

$(a+b)^2 \geq 4ab$ 1

$a^2-2ab+b^2 \geq 0$

$(a-b)^2 \geq 0$

true

Q8)

b)

i/ from $x = vt\cos\theta$ & $y = vt\sin\theta - \frac{1}{2}gt^2$ 1

$\therefore t = \frac{x}{v\cos\theta}$

$y = v\sin\theta \cdot \frac{x}{v\cos\theta} - \frac{1}{2}g\frac{x^2}{v^2\cos^2\theta}$

$= x\tan\theta - \frac{gx^2\sec^2\theta}{2v^2}$ 1

ii/ passes thru (b,h) and (c,h)

$\therefore h = b\tan\theta - \frac{gb^2}{2v^2}\sec^2\theta$ ①

and $h = c\tan\theta - \frac{gc^2}{2v^2}\sec^2\theta$ ② 1

$\therefore b\tan\theta - \frac{gb^2}{2v^2}\sec^2\theta = c\tan\theta - \frac{gc^2}{2v^2}\sec^2\theta$

$(b-c)\tan\theta = (b^2-c^2)\frac{g}{2v^2}\sec^2\theta$ 1

$\therefore \tan\theta = (b+c)\frac{g}{2v^2}\sec^2\theta$

$\therefore v^2 = \frac{(b+c)g\sec^2\theta}{2\tan\theta}$ 1

iii/ Subst into ① 1

$h = b\tan\theta - \frac{gb^2\sec^2\theta}{\cancel{2}} \cdot \frac{\cancel{2}\tan\theta}{(b+c)g\sec^2\theta}$

$= b\tan\theta - \frac{b^2\tan\theta}{b+c}$ 1

$h(b+c) = (b^2+bc)\tan\theta - b^2\tan\theta$

$= bc\tan\theta$

$\therefore \tan\theta = \frac{h(b+c)}{bc}$ 1

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