

Question 1 (12 marks) Start a NEW booklet.

Marks

- a) Find the remainder when the polynomial $P(x) = 2x^3 - 4x + 3$ is divided by $(x+1)$. 1
- b) Using the points $A(-4,1)$ and $B(2,-2)$ find the coordinates of the point $P(x,y)$ that divides the interval AB **externally** in the ratio 5:2 . 2
- c) Solve $\frac{2}{x-1} \leq 3$. 3
- d) i) Show that $f(x) = \frac{x^3}{1+x^2}$ is an odd function. 1
- ii) Hence or otherwise evaluate $\int_{-1}^1 \frac{x^3}{1+x^2} dx$. 1
- e) Neatly sketch $y = 3\cos^{-1} 2x$ clearly showing the domain and range. 2
- f) Show that $\frac{d}{dx}(\sec x) = \sec x \tan x$. 2

Question 2 (12 marks) Start a NEW booklet.

Marks

- a) Find $\int \frac{1}{\sqrt{3-x^2}} dx$. 1
- b) The parametric equations of a curve are given by $x = 6r$, $y = \frac{3}{r}$.
Find the cartesian equation of the curve. 1
- c) Given the function $f : y = \frac{x-1}{x+2}$, find its inverse function $f^{-1}(x)$ in terms of x and **state** the range of the inverse function. 3
- d) Solve $\sin 2\theta = \sin \theta$ for $0 \leq \theta \leq 2\pi$. 3
- e) If α, β and γ are the roots of $2x^3 + 12x^2 - 6x + 1 = 0$, find
- i) $\alpha + \beta + \gamma$ 1
- ii) $\alpha\beta\gamma$ 1
- iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ 2

Question 3 (12 marks) Start a NEW booklet.**Marks**

- a) Find $\lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\tan 3\theta}$ 2
- b) Use the substitution $u = 5 - x$ to evaluate $\int_1^5 x\sqrt{5-x} \, dx$ 3
- c) i) Express $\cos\theta - \sqrt{3}\sin\theta$ in the form $R\cos(\theta + \alpha)$, 2
 where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.
- ii) Hence solve $\cos\theta - \sqrt{3}\sin\theta = 2$ for $0 \leq \theta \leq 2\pi$. 2
- d) A particle is moving in a straight line x centimetres from the origin O.
 After t minutes its displacement is given by $x = 3 - 5\cos 2t$.
- i) Show that its acceleration is given by $\ddot{x} = -4(x - 3)$. 2
- ii) Assuming it is moving in simple harmonic motion find its centre of motion. 1

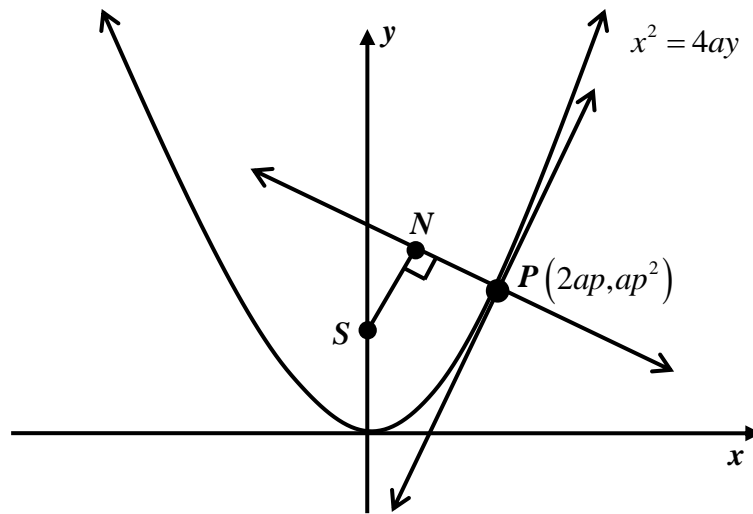
Question 4 (12 marks) Start a NEW booklet.

Marks

- a) The acceleration of a body P is given by $\frac{d^2x}{dt^2} = 18x^3 + 18x$, where x is the displacement of P from O at time t . The velocity is v .

Given that $t = 0, x = 0, v = 3$ and that $v > 0$ throughout the motion:

- i) Find v in terms of x . 2
- ii) Show that $x = \tan 3t$. 2
- b) $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. SN is perpendicular to the normal at P , where S is the focus of the parabola and N the foot of the perpendicular from S to the normal.



- i) Show that the equation of the normal at P is $x + py = 2ap + ap^3$. 2
- ii) Find the equation of SN . 2
- iii) Show that the coordinates of the point N are $(ap, ap^2 + a)$. 2
- iv) Find the locus of N as P moves on the parabola. 2

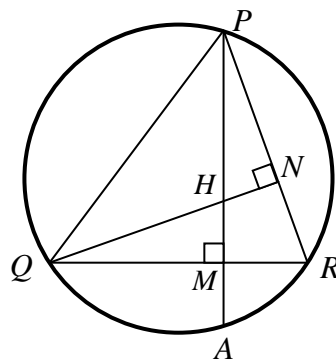
Question 5 (12 marks) Start a NEW booklet.

Marks

- a) i) Find the values of a and b such that $x^2 - 4x + 7 = (x - a)^2 + b$. 1
- ii) Hence, state the largest positive domain for which $y = x^2 - 4x + 7$ has an inverse function. 1

- b) Evaluate $\sin\left(\tan^{-1}\frac{1}{2}\right)$ in exact form. 2

- c) The altitudes PM and QN of an acute angled triangle PQR meet at H . PM produced cuts the circle PQR at A . [A larger diagram is included at the end of the exam paper, **remove** it, use it and **submit** it with your **solutions**]



- i) Explain why $PQMN$ is a cyclic quadrilateral. 1
- ii) With the aid of congruent triangles prove that $HM = MA$. 3
- d) The rate of growth of the number of bacteria in a colony is proportional to the excess of the colony's population over 5000 and is given by $\frac{dN}{dt} = k(N - 5000)$.
- i) Show that $N = 5000 + Ae^{kt}$ is a solution to the differential equation. 1
- ii) If the initial population is 15 000 and reaches 20 000 after 2 days find the values of A and k . 2
- iii) Hence, calculate the expected population after 7 days. 1

Question 6 (12 marks) Start a NEW booklet.**Marks**

a) Using $t = \tan \frac{\theta}{2}$, prove that $\frac{1 + \cos \theta}{1 - \cos \theta} = \cot^2 \frac{\theta}{2}$. 2

b) Air is being pumped into a spherical balloon at the rate of $20 \text{ cm}^3 / \text{s}$. 4

Find the rate of increase of the balloon's surface area when the radius is 5cm.

$$\left[V = \frac{4}{3} \pi r^3, SA = 4\pi r^2 \right]$$

c) For the function $y = \frac{x^2 - 4}{x^2 - 1}$

i) Write down the equations of any horizontal and vertical asymptotes. 2

ii) Find any stationary points and determine their nature. 2

iii) Neatly sketch the graph showing the above features. 2

Question 7 (12 marks) Start a NEW booklet.

Marks

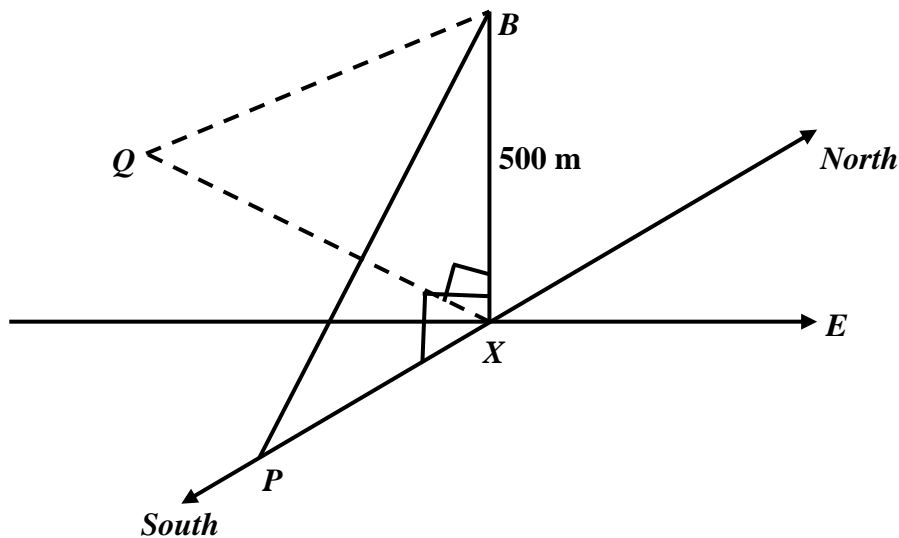
- a) Prove by mathematical induction that for any positive integer $n \geq 1$.

4

$$\frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \dots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$

- b) From a balloon 500 metres above a road junction X , the angle of depression to a point P , which lies due south of the junction X is 42° .

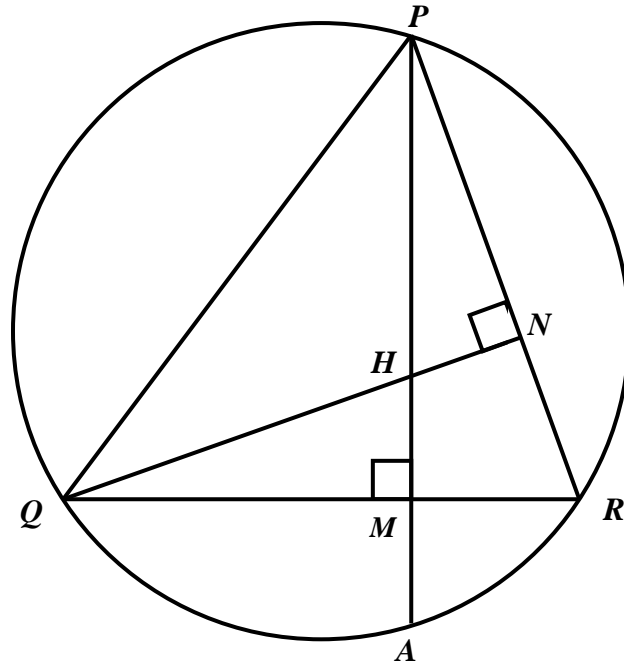
To a point Q , which lies at a bearing of 280° from the road junction, the angle of depression to the balloon is 32° .



- i) Clearly explain why $\angle QXP = 100^\circ$. 1
- ii) Calculate the distance PQ (correct to the nearest metre). 3
- c) A particle is moving in a straight line with simple harmonic motion. The velocity of the particle is respectively $\sqrt{20}ms^{-1}$ and $4ms^{-1}$ at distances of 1 metre and 2 metres from the centre of motion. Find the period and amplitude of the motion. 4

END OF EXAM

Question 5c (additional diagram) Candidate Number: _____



1a) $P_{-1} = -2 + 4 + 3 = 5$

b) $x = \frac{5 \times 2 + -2 \times -4}{5 - 2}$ $y = \frac{5 \times -2 + -2 \times 1}{5 - 2}$

$= 6$ $= -4$

$\therefore P_{x,y} = 6, -4$

c) CV1: $2 = 3x - 1 \rightarrow x = \frac{5}{3}$

CV2: $x = 1$

On testing $x < 1$ and $x \geq \frac{5}{3}$, since $x \neq 1$

d) i) An odd function exists if $f(-x) = -f(x)$

$$f(-x) = \frac{-x^3}{1 + (-x)^2}$$

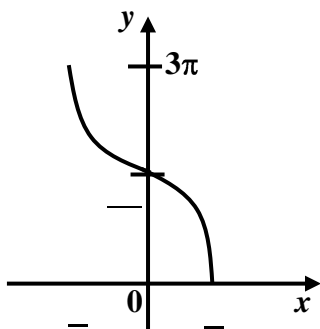
$$= \frac{-x^3}{1 + x^2} = -f(x)$$

\therefore an odd function.

ii) $\therefore \int_{-1}^1 \frac{x^3}{1+x^2} dx = 0$

an odd function with symmetrical limits.

e)



f) $\frac{d}{dx} \sec x = \frac{d}{dx} \cos x^{-1}$

$= -\cos x^{-2} \times -\sin x$

$= \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \tan x \sec x$

2a) $\int \frac{1}{\sqrt{3-x^2}} dx = \sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

b) $x = 6r \rightarrow r = \frac{x}{6}$

$\therefore y = \frac{3}{r} \rightarrow y = \frac{3}{x/6}$

Hence $xy = 18$

c) $f^{-1}(x) : x = \frac{y-1}{y+2}$

$xy + 2x = y - 1$

$2x + 1 = y - xy$

$2x + 1 = y(1 - x)$

$\therefore y = \frac{2x + 1}{1 - x}$

Range is all real y except $y = -2$

d) $\sin 2\theta = \sin \theta$ for $0 \leq \theta \leq 2\pi$

$2\sin \theta \cos \theta - \sin \theta = 0$

$2\sin \theta (2\cos \theta - 1) = 0$

$\therefore \sin \theta = 0$ or $\cos \theta = \frac{1}{2}$

$\therefore \theta = 0, \pi, 2\pi$ or $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$$\text{e) i) } \alpha + \beta + \gamma = \frac{-b}{a} = -6 \quad \checkmark$$

$$\text{ii) } \alpha\beta\gamma = \frac{d}{a} = \frac{-1}{2} \quad \checkmark$$

$$\text{iii) } \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} \quad \checkmark$$

$$= \frac{-6/2}{-1/2} = 6 \quad \checkmark$$

$$\text{3a) } \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{\tan 3\theta} = \lim_{\theta \rightarrow 0} \frac{2}{3} \left[\frac{\sin 2\theta}{2\theta} \times \frac{3\theta}{\tan 3\theta} \right] \quad \checkmark$$

$$= \frac{2}{3} \quad \checkmark$$

b) When $u = 5 - x \rightarrow x = 5 - u$
 When $x = 1, u = 4; x = 5, u = 0.$
 $du = -dx$

$$I = - \int_4^0 5 - u \sqrt{u} \, du \quad \checkmark$$

$$= \int_0^4 5u^{\frac{1}{2}} - u^{\frac{3}{2}} \, du$$

$$= \left[\frac{10u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5} \right]_0^4 \quad \checkmark$$

$$= \left[\frac{10 \cdot 4^{\frac{3}{2}}}{3} - \frac{2 \cdot 4^{\frac{5}{2}}}{5} \right] - 0$$

$$= \frac{208}{15} \quad \checkmark$$

c) i) $\cos\theta - \sqrt{3}\sin\theta = R\cos\theta + \alpha$
 $= R\cos\alpha\cos\theta - R\sin\alpha\sin\theta$

$$\therefore R\cos\alpha = 1 \text{ and } R\sin\alpha = \sqrt{3} \quad \checkmark$$

$$\therefore R^2 \cos^2\alpha + R^2 \sin^2\alpha = 1 + 3$$

$$\therefore R^2 = 4 \rightarrow R = 2$$

$$\text{and hence } \alpha = \frac{\pi}{3} \quad \checkmark$$

$$\therefore \cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right)$$

ii) $\therefore 2\cos\left(\theta + \frac{\pi}{3}\right) = 2$

$$\therefore \cos\left(\theta + \frac{\pi}{3}\right) = 1 \quad \checkmark$$

$$\therefore \theta + \frac{\pi}{3} = 0, 2\pi \rightarrow \theta = \frac{5\pi}{3} \text{ only.} \quad \checkmark$$

d) i) $x = 3 - 5\cos 2t$

$$\therefore \dot{x} = 10\sin 2t$$

$$\ddot{x} = 20\cos 2t \quad \checkmark$$

$$\ddot{x} = 20\left(\frac{3-x}{5}\right) = 4(3-x) \quad \checkmark$$

$$\therefore \ddot{x} = -4(x-3)$$

ii) Centre of motion is at $x = 3.$ \checkmark

4a) i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18x^3 + 18x$

$$\therefore \frac{1}{2}v^2 = \frac{9x^4}{2} + 9x^2 + c$$

$$v^2 = 9x^4 + 18x^2 + d$$

when $x = 0, v = 3$

$$\therefore 9 = 0 + 0 + d \rightarrow d = 9 \quad \checkmark$$

$$v^2 = 9x^4 + 18x^2 + 9$$

$$= 9x^4 + 2x^2 + 1$$

$$= 9x^2 + 1^2$$

$$\therefore v = 3x^2 + 1, \text{ since } v > 0. \quad \checkmark$$

ii) $\frac{dx}{dt} = 3x^2 + 1$

$$\therefore \frac{dt}{dx} = \frac{1}{3x^2 + 1}$$

$$\therefore t = \frac{1}{3} \tan^{-1} x + c \quad \checkmark$$

$$\therefore 0 = \frac{1}{3} \tan^{-1} 0 + c \rightarrow c = 0$$

$$\therefore t = \frac{1}{3} \tan^{-1} x$$

$$\therefore 3t = \tan^{-1} x \rightarrow x = \tan 3t \quad \checkmark$$

b) i) $x^2 = 4ay \rightarrow y = \frac{x^2}{4a}$

$$\therefore \frac{dy}{dx} = \frac{x}{2a} \rightarrow m_1 = \frac{2ap}{2a} = p$$

$$\therefore \text{gradient of the normal at } P \text{ is } m_2 = \frac{-1}{p} \quad \checkmark$$

\therefore Equation of the normal:

$$y - ap^2 = \frac{-1}{p} (x - 2ap) \quad \checkmark$$

$$py - ap^3 = -x + 2ap$$

$$\therefore x + py = 2ap + ap^3$$

ii) gradient of SN is p and S has coordinates $(0, a)$ \checkmark

$$\therefore \text{eq}^n \text{ of } SN \text{ is } y - a = p(x - 0) \rightarrow y = px + a \quad \checkmark$$

iii) $x + py = 2ap + ap^3$ --- [1]

$$y = px + a$$
 --- [2]

$$\therefore [2] \text{ in } [1] \rightarrow x + p(px + a) = 2ap + ap^3$$

$$\therefore x + p^2x + ap = 2ap + ap^3$$

$$\therefore x(1 + p^2) = ap(1 + p^2) \quad \checkmark$$

$$\therefore x = ap$$

$$\text{and } y = ap^2 + a \quad \checkmark$$

$$\therefore N \text{ is } (ap, ap^2 + a)$$

iv) $x = ap \rightarrow p = \frac{x}{a}$ \checkmark

$$\therefore y = a \left(\frac{x}{a} \right)^2 + a \rightarrow y = \frac{x^2}{a} + a \quad \checkmark$$

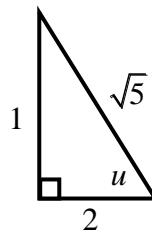
$$x^2 = a(y - a)$$

5a) i) $x^2 - 4x + 7 = x - a^2 + b$
 $x^2 - 4x + 4 + 3 = x - a^2 + b$
 $x - 2^2 + 3 = x - a^2 + b$
 $\therefore a = 2, b = 3 \quad \checkmark$

ii) all $x \geq 2$ \checkmark

b) Let $u = \tan^{-1} \frac{1}{2}$ and evaluate $\sin u$

$$u = \tan^{-1} \frac{1}{2} \rightarrow \tan u = \frac{1}{2} \quad \checkmark$$



$$\sin u = \frac{1}{\sqrt{5}} \quad \checkmark$$

c) i) $\angle QMP = \angle PNQ$ { both 90° }
 $\therefore PQMN$ is a cyclic quad - \angle 's in same segment standing on same arc or chord PQ . \checkmark

ii) Join QA : $\angle AQR = \angle APR$ { $= \theta$ say }
 { \angle 's in same segment on arc AR }
 Also $\angle MQN = \angle MPN$ { $= \theta$ }
 { \angle 's in same segment on arc MN } \checkmark

\therefore in Δ 's QHM and QAM

$\angle QMH = \angle QMA$ { both 90° }
 $\angle HQM = \angle AQM$ { proven above ($= \theta$) }
 $QM = QM$ { common } \checkmark

$\therefore \Delta QHM \equiv \Delta QAM$ (AAS)

Hence $HM = MA$ { corresponding sides in $\equiv \Delta$'s } \checkmark

d) i) $N = 5000 + Ae^{kt} \rightarrow Ae^{kt} = N - 5000$

$$LHS = \frac{d}{dt} (5000 + Ae^{kt})$$

$$= k Ae^{kt} \quad \checkmark$$

$$= k(N - 5000) = RHS$$

ii) $t = 0 \rightarrow N = 15000$

$$15000 = 5000 + Ae^0 \rightarrow A = 10000 \quad \checkmark$$

$$t = 2 \rightarrow N = 20000$$

$$20000 = 5000 + 10000e^{2k}$$

$$\therefore e^{2k} = \frac{3}{2} \rightarrow 2k = \ln \frac{3}{2}$$

$$k = \frac{1}{2} \ln \frac{3}{2} \approx 0.202733$$

iii) When $t = 7$, $N = 5000 + 10000e^{7k}$

$$\approx 46335$$

6a) Using $t = \tan \frac{\theta}{2}$:

$$LHS = \frac{1 + \frac{1-t^2}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}}$$

$$= \frac{1+t^2 + 1-t^2}{1+t^2 - 1-t^2}$$

$$= \frac{2}{2t^2} = \frac{1}{t^2}$$

$$= \frac{1}{\tan^2 \frac{\theta}{2}} = \cot^2 \frac{\theta}{2}$$

b) $\frac{dV}{dt} = 20$, $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = \pi r^2$, $A = 4\pi r^2$, $\frac{dA}{dr} = 8\pi r$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \rightarrow 20 = 4\pi r^2 \times \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = \frac{5}{\pi r^2}$$

Also $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$$= 8\pi r \times \frac{5}{\pi r^2} = \frac{40}{r}$$

and when $r = 5$, $\frac{dA}{dt} = 8 \text{ cm}^2 / \text{s}$.

c) i) Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$

ii) $\frac{dy}{dx} = \frac{x^2 - 1 \cdot 2x - x^2 - 4 \cdot 2x}{x^2 - 1^2}$

$$= \frac{6x}{x^2 - 1^2}$$

Stationary points when $\frac{dy}{dx} = 0$

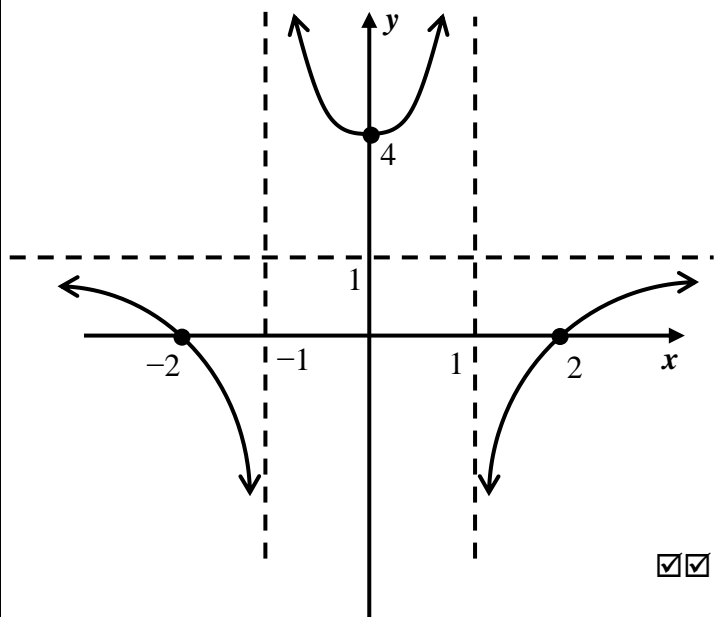
\therefore when $6x = 0 \rightarrow x = 0$

at $x = 0$, $y = 4$

x	$-1/2$	0	$1/2$
y'	-ve	0	+ve

\therefore local minimum turning point at $(0, 4)$

iii)



$$7a) \frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \dots + \frac{1}{4n-3} \frac{1}{4n+1} = \frac{n}{4n+1}$$

$$\text{When } n = 1, LHS = \frac{1}{1 \times 5} = \frac{1}{5}; RHS = \frac{1}{4 \times 1 + 1} = \frac{1}{5}$$

∴ true for $n = 1$.

Assume true for $n = k$

$$\text{i.e. } \frac{1}{1 \times 5} + \frac{1}{5 \times 9} + \frac{1}{9 \times 13} + \dots + \frac{1}{4k-3} \frac{1}{4k+1} = \frac{k}{4k+1}$$

Prove true for $n = k + 1$

$$\text{i.e. } S_k + T_{k+1} = S_{k+1}$$

$$S_k = \frac{k}{4k+1}, T_{k+1} = \frac{1}{4k+1} \frac{1}{4k+5}, S_{k+1} = \frac{k+1}{4k+5} \quad \checkmark$$

$$LHS = S_k + T_{k+1}$$

$$LHS = \frac{k}{4k+1} + \frac{1}{4k+1} \frac{1}{4k+5}$$

$$= \frac{k}{4k+1} \frac{4k+5}{4k+5} + \frac{1}{4k+1} \frac{1}{4k+5} \quad \checkmark$$

$$= \frac{4k^2 + 5k + 1}{4k+1} \frac{1}{4k+5}$$

$$= \frac{4k+1}{4k+1} \frac{k+1}{4k+5}$$

$$= \frac{k+1}{4k+5} = S_{k+1} = RHS \quad \checkmark$$

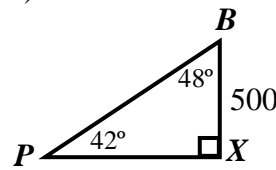
∴ If true for $n = k$, then true for $n = k + 1$.

Hence by the principle of mathematical induction,
the result is true for all $n \geq 1$.

b) i) P is due south of X , i.e. at a bearing of 180° .
 Q is at a bearing of 280° .

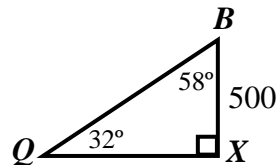
∴ the angle between P and Q on the ground is
 $280^\circ - 180^\circ = 100^\circ$.

ii)



$$\tan 48^\circ = \frac{PX}{500}$$

$$\therefore PX = 500 \tan 48^\circ \quad \checkmark$$



$$\tan 58^\circ = \frac{QX}{500}$$

$$\therefore QX = 500 \tan 58^\circ$$

∴ using ΔQXP :

$$PQ^2 = PX^2 + QX^2 - 2 \cdot PX \cdot QX \cdot \cos 100^\circ \quad \checkmark$$

$$= 500^2 \tan^2 48^\circ + 500^2 \tan^2 58^\circ - 2 \times 500 \tan 48^\circ \times 500 \tan 58^\circ \times \cos 100^\circ$$

$$\therefore PQ^2 = 1102949.62 \rightarrow PQ \approx 1050 \text{ m} \quad \checkmark$$

c) $v^2 = n^2 a^2 - x^2$; $v = \sqrt{20}$, $x = 1$; $v = 4$, $x = 2$.

$$20 = n^2 a^2 - 1 \text{ ---- } \boxed{1}$$

$$16 = n^2 a^2 - 4 \text{ ---- } \boxed{2} \quad \checkmark$$

$$\boxed{1} \div \boxed{2} \rightarrow \frac{5}{4} = \frac{a^2 - 1}{a^2 - 4} \quad \checkmark$$

$$\therefore 5a^2 - 20 = 4a^2 - 4$$

$$\therefore a^2 = 16 \rightarrow a = 4$$

$$\therefore \frac{16}{12} = \frac{4}{3} = n^2$$

$$\therefore n = \frac{2}{\sqrt{3}}$$

$$\text{hence the period} = \frac{2\pi}{n} = \frac{2\pi}{\frac{2}{\sqrt{3}}} = \pi\sqrt{3} \text{ seconds} \quad \checkmark$$

$$\text{amplitude} = 4 \text{ metres.} \quad \checkmark$$