Name:________________________ Maths Class Teacher:____________________

SYDNEY TECHNICAL HIGH SCHOOL

Extension 2 Mathematics

Trial HSC

August 2013

General Instructions

- **Reading time – 5 minutes**
- Working time – **3 hours**
- Write using **black or blue pen**
- Board-approved calculators may be used
- **All necessary working** should be shown in question 11 -16 . Marks may be deducted for careless or badly arranged work.
- Start each question on a **new page.**
- Place your papers **in order** with the question paper on top and staple or pin them.

Total Marks - 100

Section 1 – Multiple Choice

10 marks

Attempt Questions 1 – 10

Allow 15 minutes for this section

Section 11

90 Marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

Section 1

10 marks

Attempt Questions 1 – 10

Allow about 15 minutes for this section

Select the alternate A, B, C or D that best answers the question and indicate your choice on your multiple choice answer sheet.

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1. $z_1 = 1 + 2i$ and $z_2 = 3 - i$.
   The value of $z_1^{\,2} \div \overline{z_2}$

   A. $\frac{19+7i}{10}$
   B. $\frac{-5+15i}{8}$
   C. $\frac{1+3i}{2}$
   D. $\frac{3i-1}{2}$

---

2. By considering the graphs of $y = 3x^2 - 2x - 2$ and $y = |3x|$ the solution to $3x^2 - 2x - 2 \leq |3x|$ is,

   A. $-\frac{1}{3} \leq x \leq 2$
   B. $-1 \leq x \leq \frac{3}{2}$
   C. $-\frac{1}{3} \leq x \leq \frac{3}{2}$
   D. $-1 \leq x \leq 2$

---

3. Consider the hyperbola with the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

   What are the coordinates of the foci of the hyperbola?

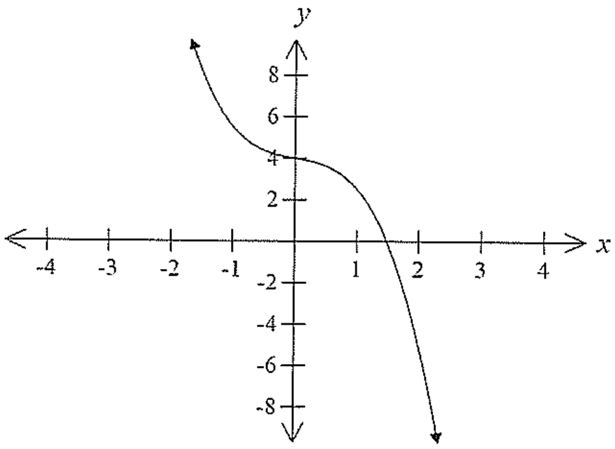
   (A) $(\pm 4, 0)$ (B) $(0, \pm 4)$
   (C) $(0, \pm 5)$ (D) $(\pm 5, 0)$

---

4. The roots of $x^3 + 5x^2 + 11 = 0$ are α, β and γ, the value of $\alpha^2 + \beta^2 + \gamma^2$ is

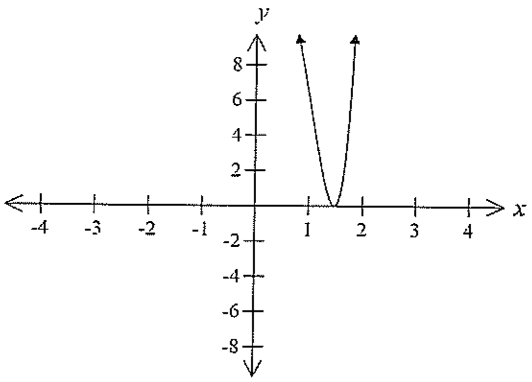
   A. -5
   B. 25
   C. 0
   D. 3

5. The diagram below shows the graph of the function $y = f(x)$.



Which diagram represents the graph of $y^2 = f(x)$?

A



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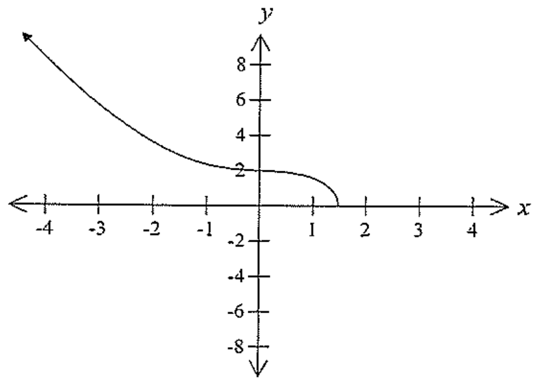
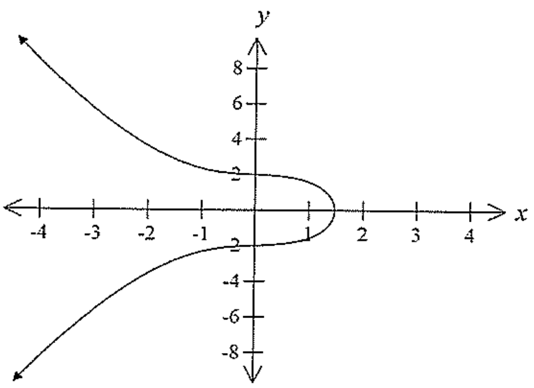
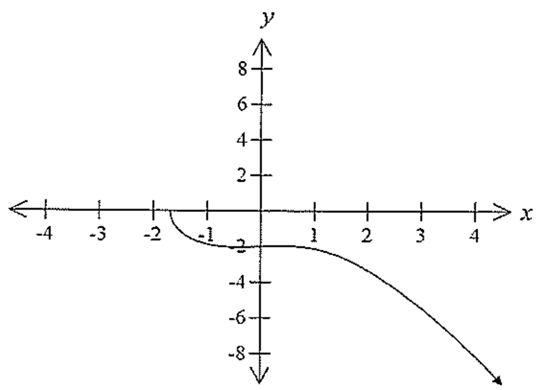


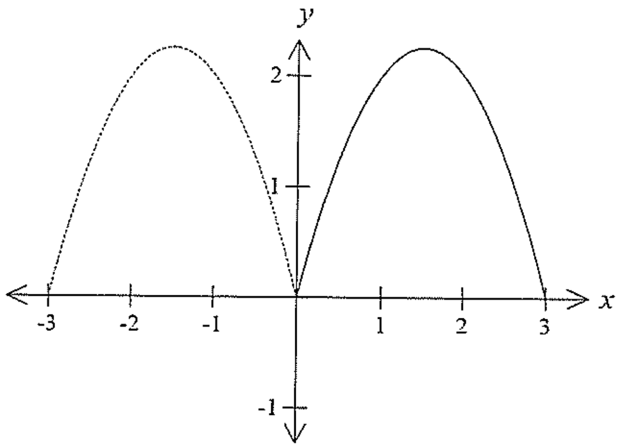
C



D



6. The area between the curve $y=3x-x^2$, the x-axis, $x=0$ and $x=3$, is rotated about the y-axis to form a solid.



The volume of this solid can be found by using the integral with the method of cylindrical shells?

A $6\pi\int_0^3 \sqrt{9-4y}\ dy$

B $\pi\int_0^3 (3x^2-x^3)^2\, dx$

C $2\pi\int_{-3}^3 3x^2-x^3\, dx$

D $2\pi\int_0^3 3x^2-x^3\, dx$

---

7. The polynomial equation $x^3-3x^2-x+2=0$ has roots α, β and γ. Which of the following polynomial equations have roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$?

A $x^3-x^2-3x+1=0$

B $x^3-2x^2-3x+1=0$

C $2x^3-x^2-3x+1=0$

D $2x^3-2x^2-3x+1=0$

---

8. What is the derivative of $\sin^{-1}x - \sqrt{1-x^2}$?

A $\frac{\sqrt{1+x}}{\sqrt{1-x}}$

B $\frac{\sqrt{1+x}}{1-x}$

C $\frac{1+x}{\sqrt{1-x}}$

D $\frac{1+x}{1-x}$

---

9. If $p + q = 1$ and $p^2 + q^2 = 2$, the value of $p^3 + q^3$ is

A. 1 ½

B. 2 ½

C. 3 ½

D. 2

---

10. Which of the following statements is incorrect?

A $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} sin^7\theta \, d\theta = 0$

B $\int_{-1}^{1} e^{-x^2} \, dx = 0$

C $\int_{0}^{\frac{\pi}{2}} sin^8\theta - cos^8\theta \, d\theta = 0$

D $\int_{-2}^{2} \frac{x^3}{1+x^2} \, dx = 0$

End of section 1

Section II

90 Marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

START EACH QUESTION ON A **NEW** PAGE

Question 11 **Start a new page**

a. Find,

i. $\int \frac{4x-16}{x^2-8x+20} \quad dx$ 2

ii. $\int \frac{1}{x^2-8x+20} \quad dx$ 2

iii. $\int \frac{x}{\sqrt{x-1}} \quad dx$ 3

b. Evaluate,

i. $\int_0^{0.5} sin^{-1}x \quad dx$ 2

ii. $\int_0^{\frac{\pi}{4}} xsec^2x \quad dx$ 2

c. i. Find the value of ***a*** and ***b*** such that

$$\frac{x}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$$ 2

ii. Hence, evaluate $\int_0^1 \frac{x}{(x+1)(x+2)} \quad dx$ 2

Question 12 **Start a new page**

a.

i. Find the two square roots of $2i$ 3

ii. Solve $x^2 + 2x + \left(1 - \frac{i}{2}\right) = 0$ 2

b. Find α and β given that $z^3 + 3z + 2i = (z - \alpha)^2(z - \beta)$ 2

c.

i. On a Argand diagram, sketch the locus of the point P representing the complex number z which moves so that $|z - 2| = 1$ 2

ii. Find the range of possible values of $|z|$ and $\arg(z)$ 2

iii. The points P_1 and P_2 such that OP_1 and OP_2 are tangents to the locus, (O is the origin) represent the complex numbers z_1 and z_2 respectively. Express z_1 and z_2 in modulus – argument form 2

iv. Evaluate $z_1^{20} + z_2^{20}$, give your answer in its simplest form. 2

Question 13 **Start a new page**

a. Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ by using the substitution $t = \tan\frac{\theta}{2}$ 3

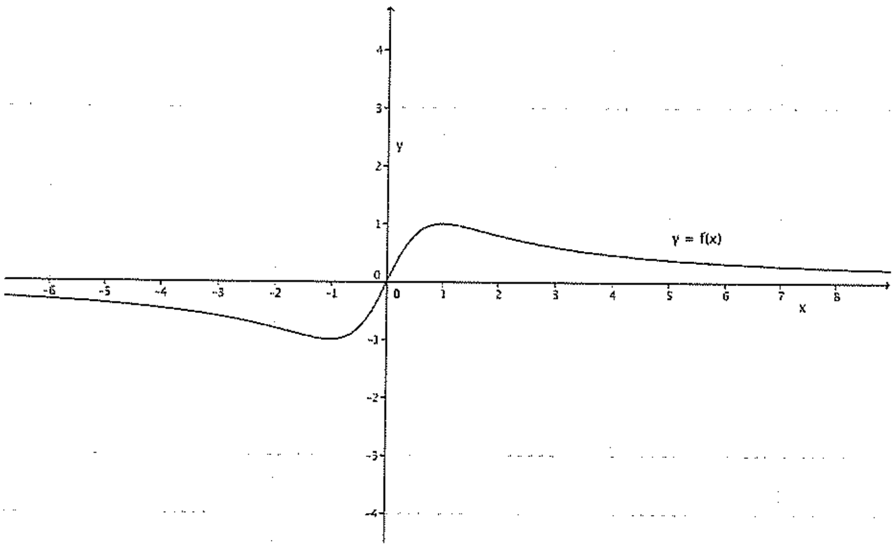
b.

i. If $I_n = \int_0^1 x^n e^{-x}\,dx$, where n is a positive integer,

show that $I_n = nI_{n-1} - \frac{1}{e}$ 2

ii. Hence, evaluate $\int_0^1 x^3\, e^{-x}\,dx$ 2

c. The diagram below is of the function $f(x) = \frac{2x}{x^2+1}$



Sketch the following on separate number planes, without the use of calculus 8

i. $y = f(|x|)$

ii. $|y| = f(x)$

iii. $y \times f(x) = 1$

iv. $y = e^{f(x)}$

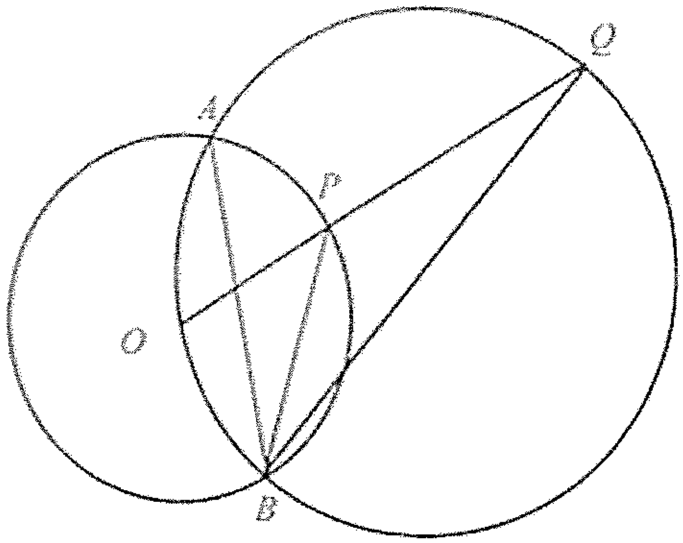
Question 14 **Start a new page**

a. $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$. Given that $1 + i$ is a zero of $P(x)$, find all the zeros of $P(x)$ 3

b. Two sides of a triangle are in the ratio 3 : 1 and the angles opposite these sides differ by $\frac{\pi}{6}$. Show that the smaller of the two angles is, 4

$$\tan^{-1}\frac{1}{6-\sqrt{3}}$$

c.



In the diagram above, the centre O of the small circle APB lies on the circumference of the larger circle AQB. The points O, P and Q are collinear.

i. Let angle OAB = x, show that angle OQB = x 1

ii. Let angle ABP = y, find an expression for angle OPB 2

iii. Prove that BP bisects $\angle ABQ$ 2

d. Show that the polynomial $P(x) = x^n - x^{n-1} - 1$, where $n > 1$ cannot have a repeated root. 3

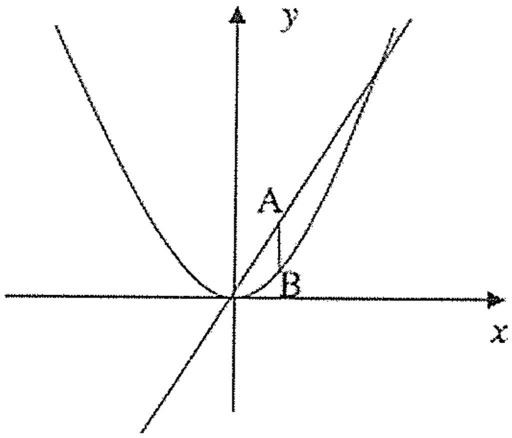
Question 15 **Start a new page**

a.

i. Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ Intersect at right angles, in the first quadrant. 3

ii. Find the equation of the circle through the points of intersection of the two conics. 1

b. The base of a solid is the region enclosed by $y = 2x$ and $y = x^2$. 5
Cross sections taken perpendicular to the x – axis are semi – circles with the diameter in the base of the solid (as indicated the diameter AB of the semicircle is perpendicular to the x axis; the semicircle is perpendicular to the xy plane)

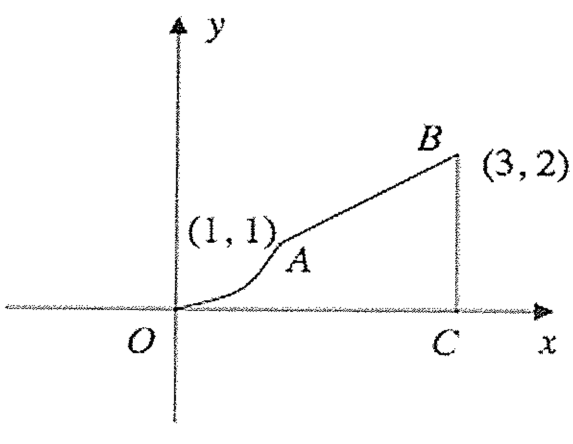


Find the volume of the solid

c. OA is an arc of the parabola $y = x^2$. The region OABC is rotated about the y – axis forming a bowl.

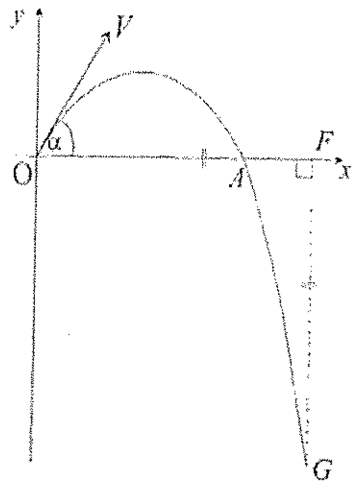
i. By using cylindrical shells determine the volume of the solid formed 5

ii. Hence, find the holding capacity of the bowl. 1



Question 16 **Start a new page**

a.



$OF = FG$

In the diagram above, a projectile is fired from a point O at the top of a vertical cliff.

Its initial speed is $V\,m/s$ and its angle of elevation is α. Let the acceleration due to gravity be g m/s^2.

i. By using the equation of motion $\ddot{x} = 0$ and $\ddot{y} = -g$, derive the expressions for the horizontal and vertical displacements after t seconds. 2

ii. Let G be the point on the projectile's path where the distance below the origin equals the distance to the right of the origin. That is, OF = FG on the diagram above.

α. Prove that the time taken for the projectile to reach G is

$$\frac{2V(\sin\alpha + \cos\alpha)}{g} \text{ seconds.}$$ 2

β. Show that OF = $\frac{V^2}{g}(\sin 2\alpha + \cos 2\alpha + 1)$ metres. 2

γ. Let A be the point on the projectile's path where it is level with the point of projection. 4

If OF = $\frac{4}{3}OA$, find α, to the nearest degree.

You may assume that the distance OA is given by $OA = \frac{V^2 \sin 2\alpha}{g}$ metres.

b. Define $f^{(n)}(x)$ to be $f(f \ldots (f(x)) \ldots)$ where f is repeated n times 5

That is,

$f^{(1)}(x) = f(x)$

$f^{(2)}(x) = f(f(x))$.

$f^{(3)}(x) = f(f(f(x)))$ etc

Let $f(x) = \frac{x}{\sqrt{1+x^2}}$

Prove by mathematical induction that $f^{(n)}(x) = \frac{x}{\sqrt{1+nx^2}}$

End of examination

Section I - Multiple Choice (1 mark each)

1. D 2. D 3. D 4. B 5. C

6. D 7. C 8. A 9. B 10. B

Section II

Question 11

a. i. $\int \frac{4x-16}{x^2-8x+20}\,dx = 2\ln(x^2-8x+20) + C$

ii. $\int \frac{1}{x^2-8x+20}\,dx = \int \frac{1}{(x-4)^2+4}\,dx$

$= \frac{1}{2}\tan^{-1}\left(\frac{x-4}{2}\right) + C$

iii. $\int \frac{x}{\sqrt{x-1}}\,dx$ let $x-1=u^2$, $x = u^2+1$, $dx = 2u\,du$, $u=\sqrt{x-1}$

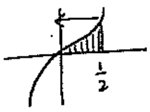
$\int \frac{u^2+1}{\sqrt{u^2}}\cdot 2u\,du$

$= 2\int u^2+1\,du$

$= 2\left[\frac{u^3}{3}+u\right]$

$= \frac{2}{3}(x-1)\sqrt{x-1} + 2\sqrt{x-1} + C$

b. i. $\int_0^{0.5} \sin^{-1}x\,dx$



$= \frac{\pi}{6}\times\frac{1}{2} - \int_0^{\pi/6}\sin y\,dy$

$= \frac{\pi}{12} + [\cos y]_0^{\pi/6}$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

$= \frac{\pi + 6\sqrt{3} - 12}{12}$

ii. $\int_0^{\pi/4} x\sec^2 x\,dx$

$= x\tan x - \int 1\tan x\,dx$

$= \left[x\tan x + \ln(\cos x)\right]_0^{\pi/4}$

$= \left[\pi/4 \times 1 + \ln(1/\sqrt{2}) - (0 + \ln 1)\right]$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

c. i. $\frac{x}{(x+1)(x+2)} = \frac{a}{x+1} + \frac{b}{x+2}$

$x = a(x+2) + b(x+1)$

let $x=-1$ $-1 = a(1)$ $\therefore a = -1$

let $x=-2$ $-2 = b(-1)$ $\therefore b = 2$

ii. $\int_0^1 \frac{-1}{x+1} + \frac{2}{x+2}\,dx$

$= -\ln(x+1) + 2\ln(x+2)\Big]_0^1$

$= 2\ln(x+2) - \ln(x+1)\Big]_0^1$

$= 2\ln 3 - \ln 2 - [2\ln 2 - 0]$

$= 2\ln 3 - 3\ln 2$

$= \ln 9 - \ln 8$

$= \ln(9/8)$.

a. i. $\sqrt{2i} = x + iy$

$2i = (x+iy)^2$

$= x^2 + 2xyi - y^2$

equating reals : $x^2 - y^2 = 0$

Im : $2xy = 2$

$xy = 1$

By inspection $x = 1, y = 1$ or $x = -1, y = -1$

$\therefore \sqrt{2i} = 1 + i$ or $-1 - i$

ii. $x^2 + 2x + (1 - i/2) = 0$

$$x = \frac{-2 \pm \sqrt{4 - 4 \times 1 \times (1 - i/2)}}{2}$$

$$= \frac{-2 \pm \sqrt{2i}}{2}$$

$$= \frac{-2 \pm (x+i)}{2} \text{ or } \frac{-2 \pm (-x-i)}{2}$$

$$x = \frac{-1+i}{2} \text{ or } \frac{-3-i}{2}$$

b. $z^3 + 3z + 2i = (z-\alpha)^2(z-\beta)$

sum : $2\alpha + \beta = \frac{-b}{a} = 0$ — ①

double : $\alpha^2 + 2\alpha\beta = \frac{c}{a} = 3$ — ②

triple : $\alpha^2\beta = -2i$ — ③

Solving ① and ②

$2\alpha\beta + \alpha^2 = 3$

$2\alpha(-2\alpha) + \alpha^2 = 3$

$-3\alpha^2 = 3$

$\alpha^2 = -1$

$\alpha = \pm i \quad \therefore \beta = \mp 2i$

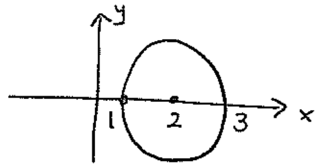
$\alpha = -i \quad \beta = 2i$ works when tested

($\alpha = i$ and $\beta = -2i$ doesn't)

$\therefore \alpha = -i, \beta = 2i$

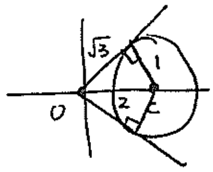
c. $|z - 2| = 1$

i.



ii. $1 \leq |z| \leq 3$ (using values on x-axis)

ii.



$-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$

(iii) $P_1 \rightarrow z_1 = r\,\text{cis}\,\theta_1$

$= \sqrt{3}\,\text{cis}\frac{\pi}{6}$ ← answer here

$= \left(\frac{3}{2} + \frac{\sqrt{3}i}{2}\right)$ ← can accept if u want

$P_2 \rightarrow z_2 = r\,\text{cis}\,\theta_2$

$= \sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$ ← answer

$= \left(\frac{3}{2} - \frac{\sqrt{3}i}{2}\right)$

iv) $z_1^{20} + z_2^{20} = \left(\sqrt{3}\,\text{cis}\,\frac{\pi}{6}\right)^{20} + \left(\sqrt{3}\,\text{cis}\,-\frac{\pi}{6}\right)^{20}$

$= \sqrt{3}^{20}\text{cis}\,\frac{20\pi}{6} + \sqrt{3}^{20}\text{cis}\left(\frac{-20\pi}{6}\right.$

$= -3^{10}$

Student Name: ______________________ Teacher Name: ______________________

Question 13

a. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta}$ let $t = \tan\theta/2$

$d\theta = \frac{2\,dt}{t^2+1}$

$\theta = \pi/2 \quad t = 1$
$\theta = 0 \quad t = 0$

$= \int_0^1 \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{t^2+1}$

$= \int_0^1 \frac{2\,dt}{2(1+t^2)+1-t^2}$

$= \int_0^1 \frac{2\,dt}{3+t^2}$

$= 2/\sqrt{3}\left[\tan^{-1}(t/\sqrt{3})\right]_0^1$

$= 2/\sqrt{3}\left[\pi/6 - 0\right]$

$= \frac{\pi}{3\sqrt{3}}$

b. $I_n = \int_0^1 x^n e^{-x}\,dx$

$I_n = \left[\frac{x^n \cdot e^{-x}}{-1}\right]_0^1 - \int_0^1 \frac{n x^{n-1} \cdot e^{-x}}{-1}\,dx$

$= \left[-1^n e^{-1} - 0\right] + n\int_0^1 x^{n-1} e^{-x}\,dx$

$= \frac{-1}{e} + n I_{n-1}$

$= n I_{n-1} - 1/e$ as req.

ii. $I_3 = 3I_2 - 1/e$

$= 3\left[2I_1 - 1/e\right] - 1/e$

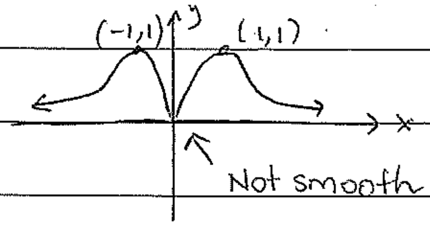
$= 3\left[2(I_0 - 1/e) - 1/e\right] - 1/e$

$I_0 = \int_0^1 e^{-x}\,dx = 1 - 1/e$

$\therefore I_3 = 3\left[2 - 5/e\right] - 1/e$

$= 6 - \frac{16}{e}$

c. i. $y = f|x|$



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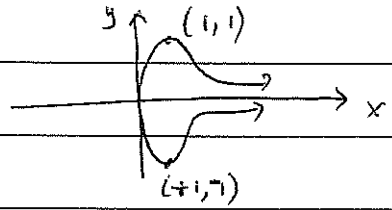
ii. $|y| = f(x)$ $\therefore f(x) \geq 0$

$y^2 = [f(x)]^2$



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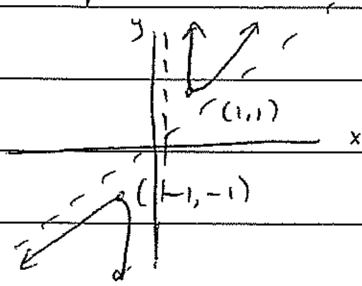
iii. $y \times f(x) = 1$



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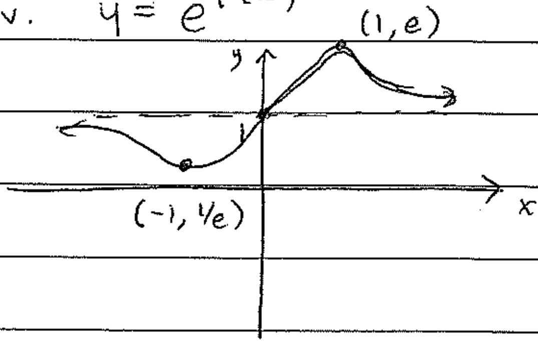
iv. $y = e^{f(x)}$



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Student Name: ______________________ Teacher Name: ______________________

Question 14.

a. $P(x) = x^4 - x^3 - 2x^2 + 6x - 4$

if $1+i$ is a factor so is

$1-i$ (conjugate pairs)

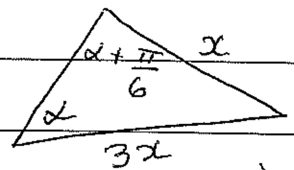
$\therefore x^2 - 2x + 2$ is a factor

$$\begin{array}{r} x^2 + x - 2 \\ x^2 - 2x + 2 \overline{\smash{)} x^4 - x^3 - 2x^2 + 6x - 4} \end{array}$$

$P(x) = (x+2)(x-1)(x^2 - 2x + 2)$

zeros are: $-2, 1, 1 \pm i$

b. let smaller angle be α



sine rule

$$\frac{\sin(\alpha + \pi/6)}{3x} = \frac{\sin\alpha}{x}$$

$$\sin(\alpha + \pi/6) = 3\sin\alpha$$

$$\sin\alpha \frac{\sqrt{3}}{2} + \cos\alpha . \frac{1}{2} = 3\sin\alpha$$

$$\cos\alpha = 2\sin\alpha\left(3 - \sqrt{3}/2\right)$$

$$\cot\alpha = 6 - \sqrt{3}$$

$$\therefore \tan\alpha = \frac{1}{6 - \sqrt{3}}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{1}{6 - \sqrt{3}}\right)$$

c. i. let $\angle OAB = x$

$\angle OQB = x$ (~~angle at~~ circumf on arc QB circle AQB)

$OA = OB$ radii $\therefore \triangle OAB$ is isosceles with $\angle OAB = \angle OBA$

$\therefore \angle OBA = x$

ii. $\angle ABP = y$

$\therefore \angle OBP = x + y$

$OP = OB$ radii circle centre O

$\angle OPB = \angle OBP$

$= x + y$

iii. $\angle OPB = \angle PQB + \angle PBQ$

$\therefore \angle PBQ = y$

and $\angle PBQ = \angle ABP$ $(= y)$

$\therefore$ PB bisects $\angle ABQ$.

d. $P(x) = x^n - x^{n-1} - 1$, $n > 1$

repeated root when $P(\alpha) = P'(\alpha) = 0$

$P'(x) = nx^{n-1} - (n-1)x^{n-2}$ IF $P'(\alpha) = 0$

$$n\alpha^{n-1} - (n-1)\alpha^{n-2} = 0$$

$$\alpha^{n-2}\left[\alpha n - n + 1\right] = 0$$

$\alpha = 0$ but $P(\alpha) \neq 0$

$\alpha n = n - 1$

$\alpha = \frac{n-1}{n}$

Now $P\left(\frac{n-1}{n}\right) = \left(\frac{n-1}{n}\right)^n - \left(\frac{n-1}{n}\right)^{n-1} - 1$

$$0 = \frac{(n-1)^{n-1}\left[n - 1 - n\right]}{n^n} - 1$$

$$1 = \frac{-1(n-1)^{n-1}}{n^n} \quad \text{but } n > 1$$

$\therefore \frac{-1(n-1)^{n-1}}{n^n} < 0 \quad \therefore \alpha \neq \frac{n-1}{n}$

$\therefore$ can't have a repeated root.

a. $4x^2 + 9y^2 = 36$ ① and $4x^2 - y^2 = 4$ ②

$10y^2 = 32$

$y^2 = 3.2 \quad y = \pm\sqrt{3.2} = \pm 4\sqrt{0.2}$

$4x^2 - y^2 = 4 \Rightarrow 4x^2 - 3.2 = 4$

$4x^2 = 7.2$

$x^2 = 1.8$

$x = \pm 3\sqrt{0.2}$

For $4x^2 + 9y^2 = 36$

$8x + 18y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-4x}{9y}$ at $(\pm 3\sqrt{0.2}, \pm 4\sqrt{0.2})$

take $(3\sqrt{0.2}, 4\sqrt{0.2})$ 1st quad

$M_T = -\frac{1}{3}$

For $4x^2 - y^2 = 4$

$8x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{4x}{y}$ at $(\pm 3\sqrt{0.2}, \pm 4\sqrt{0.2})$

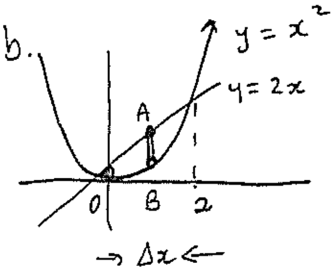
$M_T = 3$

∴ as gradients of tangents are negative reciprocals tangents are ⊥.

ii distance from $(0,0)$ to $(3\sqrt{0.2}, 4\sqrt{0.2})$ $= 5$ (this is the radius of circle as dist ⊥)

∴ Locus $x^2 + y^2 = 5$

b.



diameter $2x - x^2$

radius $= x - \frac{1}{2}x^2$

Area cross section

$A = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi\left(x - \frac{1}{2}x^2\right)^2$

$\Delta V = \frac{1}{2}\pi\left(x - \frac{1}{2}x\right)^2 \Delta x$

Total Volume $= \lim_{\Delta x \to 0} \sum_{x=0}^{2} \frac{\pi}{2}\left(x - \frac{x^2}{2}\right)\Delta x$

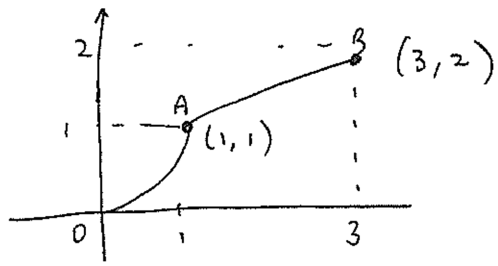
$V = \frac{\pi}{2}\int_0^2 \left(x^2 - x^3 + \frac{x^4}{4}\right)dx$

$= \frac{\pi}{2}\left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{20}\right]_0^2$

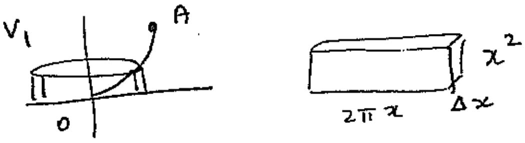
$= \frac{\pi}{2}\left[\frac{8}{3} - 4 + \frac{32}{20} - (0)\right]$

$= \frac{2\pi}{15}\ u^3$

c.



i. consider V_1



$\Delta V = 2\pi x^3 \Delta x$

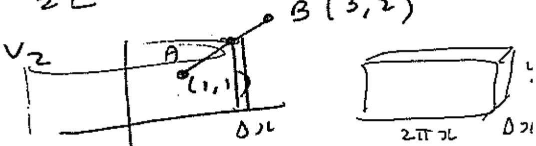
$\therefore V_1 = \lim_{\Delta x \to 0} \sum_0^1 2\pi x^3 \Delta x$

$= 2\pi \int_0^1 x^3\, dx$

$= \frac{2\pi}{4}\left[x^4\right]_0^1$

$= \frac{\pi}{2}\left[1 - 0\right] = \frac{\pi}{2}\ u^3$

consider V_2



equation Line AB $\Rightarrow$ $y = \frac{x+1}{2}$

$\Delta V = 2\pi x\left[\frac{x+1}{2}\right]\Delta x$

$V_2 = \lim_{\Delta x \to 0} \sum_1^3 2\pi x\left(\frac{x+1}{2}\right) dx$

$= \pi\int_1^3 x^2 + x\ dx$

$= \pi\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^3$

$= \pi\left[9 + \frac{9}{4} - \left(\frac{1}{3} + \frac{1}{2}\right)\right] = \frac{38\pi}{3}$

∴ Volume bowl $= \frac{\pi}{2} + \frac{38\pi}{3} = \frac{79\pi}{6}$

ii. Capacity $= \pi R^2 H - \frac{79\pi}{3}$

$= 18\pi - \frac{79\pi}{3} = \frac{29\pi}{6}\ u^3$

$\alpha = 71°34 \doteqdot 12$ (nearest degree)

a - H

1. $\boxed{\ddot{x} = 0}$

$\dot{x} = c_1$

$t=0 \quad \dot{x} = v\cos\alpha$

$\boxed{\dot{x} = v\cos\alpha}$

$x = v\cos\alpha t + c_2$

$t=0 \quad x=0 \therefore c_2 = 0$

$\boxed{x = v\cos\alpha t}$

V

$\boxed{\ddot{y} = -g}$

$\dot{y} = -gt + c_3 \quad t=0, \ \dot{y} = v\sin\alpha$

$\boxed{\dot{y} = -gt + v\sin\alpha}$

$y = -g\frac{t^2}{2} + v\sin\alpha t + c_4$

$t=0 \ y=0 \quad \therefore c_4 = 0$

$\boxed{y = v\sin\alpha t - \frac{gt^2}{2}}$

ii. $OA = FG$ hence

i) $v\sin\alpha t - g\frac{t^2}{2} = -v\cos\alpha t$

$g\frac{t^2}{2} = v\sin\alpha t + v\cos\alpha t$

$\div t$ as $t \neq 0$

$g\frac{t}{2} = v\sin\alpha + v\cos\alpha$

$t = \frac{2v}{g}(\sin\alpha + \cos\alpha)$ sec.

β) $OF = v\cos\alpha t$

$= v\cos\alpha\left[\frac{2v}{g}(\sin\alpha + \cos\alpha)\right]$

$= \frac{v^2}{g}\left(2\sin\alpha\cos\alpha + 2\cos^2\alpha\right)$

$= \frac{v^2}{g}\left(\sin 2\alpha + \cos 2\alpha + 1\right)$ m

γ) Several sol^n possible :

$OF = \frac{4}{3} OA$

$\frac{v^2}{g}(\sin 2\alpha + \cos 2\alpha + 1) = \frac{4}{3}\frac{v^2}{g}\sin 2\alpha$

$\sin 2\alpha + \cos 2\alpha + 1 = \frac{4}{3}\sin 2\alpha$

$\cos 2\alpha + 1 = \frac{1}{3}\sin 2\alpha$

$\sin 2\alpha = 3\cos 2\alpha + 3$

ie $\sin 2\alpha - 3\cos 2\alpha = 3$

let $\sin 2\alpha - 3\cos 2\alpha = R\sin(2\alpha - \theta)$

$= \sqrt{10}\sin(2\alpha - \tan^{-1}(3))$

$= \sqrt{10}\sin(2\alpha - 71°34')$

$\sqrt{10}\sin(2\alpha - 71°34') = 3$

$2\alpha - 71°34' = \sin^{-1}(3/\sqrt{10})$

b. $f^n(x) = \frac{x}{\sqrt{1+nx^2}} \qquad f(x) = \frac{x}{\sqrt{1+x^2}}$

test $n=1$

$f^1(x) = \frac{x}{\sqrt{1+1\times x^2}}$

$= \frac{x}{\sqrt{1+x^2}}$

$= f(x) \quad \therefore$ true for $n=1$

Assume true for $n=K$

ie $f^K(x) = f(f(f\ldots f(x))) = \frac{x}{\sqrt{1+Kx^2}}$

Prove true for $n = K+1$

ie $f^{K+1}(x) = f(f(f(f\ldots f(x))$ ($K+1$ times)

$= f\left[f^K(x)\right]$

$= f\left[\frac{x}{\sqrt{1+Kx^2}}\right]$

$= \frac{\frac{x}{\sqrt{1+Kx^2}}}{\sqrt{1+\left(\frac{x}{\sqrt{1+Kx^2}}\right)^2}}$

$=$

$= \frac{\frac{x}{\sqrt{1+Kx^2}}}{\sqrt{1+\frac{x^2}{1+Kx^2}}}$

$= \frac{x}{\sqrt{1+Kx^2}\sqrt{1+\frac{x^2}{1+Kx^2}}}$

$= \frac{x}{\sqrt{1+Kx^2 + \frac{x^2(1+Kx^2)}{1+Kx^2}}}$

$= \frac{x}{\sqrt{1+Kx^2+x^2}}$

$= \frac{x}{\sqrt{1+(K+1)x^2}}$

$\therefore$ true $n=K+1$ if true $n=K$.

ie. As true for $n=1$, also true for $n=2,3,4$ etc hence by M.I true all positive integer n.