

2009

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

# MATHEMATICS EXTENSION 2

## General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using blue or black pen
- Board approved calculators may be used
- Write your name on each page
- Each question is to be started on a new page.
- This examination paper must NOT be removed from the examination room
- There is a total of eight questions.
- Each question is worth 15 marks.
- Marks may be deducted for careless or badly arranged work.

**Question 1** (15 marks) Start a NEW page.**Marks**

a) Find  $\int \cos x \sin^4 x \, dx$ . 1

b) Find  $\int \frac{dx}{x^2 - 4x + 8}$ . 2

c) Use the substitution  $u = x - 2$  to find the exact value of  $\int_1^3 x(x-2)^5 \, dx$ . 3

d) i) Find the values of  $A$ ,  $B$  and  $C$  so that 2

$$\frac{5}{(x^2 + 4)(x + 1)} \equiv \frac{Ax + B}{x^2 + 4} + \frac{C}{x + 1}.$$

ii) Hence find  $\int \frac{5}{(x^2 + 4)(x + 1)} \, dx$ . 3

e) i) If  $I_n = \int_1^e x(\ln x)^n \, dx$  for  $n = 0, 1, 2, 3, \dots$  use integration by parts 2

to show that  $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$  for  $n = 1, 2, 3, \dots$

ii) Hence find the value of  $I_2$ . 2

**Question 2** (15 marks) Start a NEW page.

**Marks**

a) If  $z = 1 - i$ , find

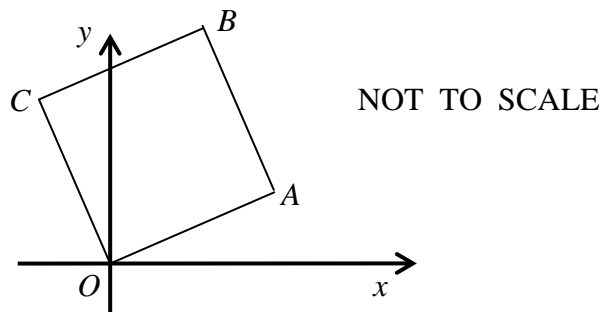
- |      |                         |   |
|------|-------------------------|---|
| i)   | $\bar{z}$               | 1 |
| ii)  | $ z $                   | 1 |
| iii) | $\arg z$                | 1 |
| iv)  | $\arg iz$               | 1 |
| v)   | $z^6$ in $a + ib$ form. | 2 |

b) Express  $\frac{i^5(1-i)}{2+i}$  in the form  $a + ib$  where  $a$  and  $b$  are real. 3

c) Graph the region in the Argand diagram which simultaneously satisfies 3  
 $1 \leq |z - i| \leq 2$  and  $\text{Im}z \geq 0$ .

d) In the Argand diagram  $A$  represents the point  $z_1 = \sqrt{3} + i$ , and  $O$  is the origin.

Given that  $OABC$  is a square:



- |     |  |   |
|-----|--|---|
| i)  | Find the complex number ( $z_3$ ) represented by $C$ . | 1 |
| ii) | Find the complex number ( $z_2$ ) represented by $B$ . | 2 |

- Question 3** (15 marks) Start a NEW page. **Marks**
- a) i) Without using calculus, draw a good size neat sketch of 2  
 $y = (x+1)^2(1-x)$ .
- ii) On a separate diagram using the same scale as above, and also without 2  
calculus, sketch  $y^2 = (x+1)^2(1-x)$ , paying close attention to the shape  
of the curve as 'y' approaches zero.
- b) Neatly sketch each of the following on separate axes for  $0 \leq x \leq 2\pi$ .
- i)  $y = \sin^2 x$ . 1
- ii)  $y = |\sin x|$ . 1
- iii)  $y = \sqrt{\sin x}$ . 1
- iv)  $y = \frac{1}{\sin x}$ . 1
- v)  $y = \frac{|\sin x|}{\sin x}$ . 1
- vi)  $y = e^{\sin x}$ . 2
- c) A plane curve is defined by the equation  $x^2 + 2xy + y^5 = 4$ . 4  
The curve has a horizontal tangent at the point  $P(X, Y)$ .  
By using implicit differentiation or otherwise, show that  $X$  is the  
unique solution to the equation  $X^5 + X^2 + 4 = 0$ .  
[ Do not solve this equation]

**Question 4** (15 marks) Start a NEW page.

**Marks**

- a) Given the hyperbola  $16x^2 - 9y^2 = 144$ , find:
- i) the length of the transverse axis 1
  - ii) the eccentricity 1
  - iii) the coordinates of the foci 1
  - iv) the equations of the directrices 1
  - v) the equations of the asymptotes. 1
- b) If  $\alpha, \beta$  and  $\gamma$  are the roots of the equation  $x^3 + 6x + 1 = 0$ , find the polynomial equation whose roots are  $\alpha\beta, \beta\gamma$  and  $\alpha\gamma$ . 3
- c) Consider the equation  $x^4 - 5x^3 + 7x^2 + 3x - 10 = 0$ .
- i) Given that  $2 - i$  is a root of the equation explain why  $2 + i$  is also a root. 1
  - ii) Find the other roots of the equation. 3
- d) The area enclosed between the curves  $y = \sqrt{x}$  and  $y = x^2$  is rotated about  $y$ -axis through one complete revolution. Use the method of cylindrical shells to find the volume of the solid that is generated. 3

**Question 5** (15 marks) Start a NEW page.

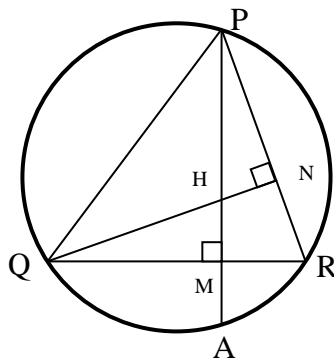
**Marks**

a) Solve for  $x$ :  $2^{3x+1} = 5^{x+1}$ , correct to 3 significant figures. 2

b) i) Prove the result  $\int_{-a}^a f(x) dx = \int_0^a [f(x) + f(-x)] dx$  2

ii) Hence use the above result to evaluate  $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1 + \sin x} dx$  3

c) The altitudes  $PM$  and  $QN$  of an acute angled triangle  $PQR$  meet at  $H$ .  $PM$  produced cuts the circle  $PQR$  at  $A$ . [A larger diagram is included, use it and submit it with your solutions]



i) Explain why PQMN is a cyclic quadrilateral. 1

ii) Prove that  $HM = MA$ . 3

d) The point  $P(a\cos\theta, b\sin\theta)$  lies on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . The tangent at  $P$  cuts the  $y$ -axis at  $B$  and  $M$  is the foot of the perpendicular from  $P$  to the  $y$ -axis.

i) Show that the equation of the tangent to the ellipse at the point  $P$  is 2

given by  $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

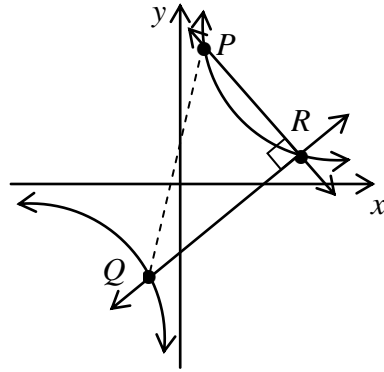
ii) Show that  $OM \cdot OB = b^2$ , where  $O$  is the origin. 2

**Question 6** (15 marks) Start a NEW page.

**Marks**

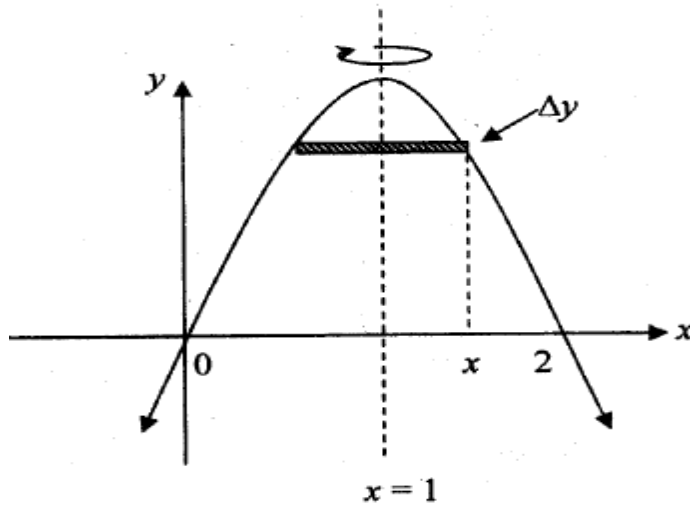
- a) The points  $P\left(cp, \frac{c}{p}\right)$  and  $Q\left cq, \frac{c}{q}\right)$  lie on the rectangular hyperbola  $xy = c^2$ . 4

The chord  $PQ$  subtends a right angle at another point  $R\left(cr, \frac{c}{r}\right)$  on the hyperbola.



Show that the normal at  $R$  is parallel to  $PQ$ .

- b) The area bounded by the curve  $y = 2x - x^2$  and the  $x$ -axis is rotated through  $180^\circ$  about the line  $x = 1$ .



- i) Show that the volume,  $\Delta V$ , of a representative horizontal slice of width  $\Delta y$  is given by  $\Delta V = \pi(x - 1)^2 \Delta y$ . 2
- ii) Hence show that the volume of the solid of revolution is given by 2

$$V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^1 \pi(1 - y) \Delta y$$

- iii) Hence, find the volume of the solid of revolution. 2

Question 6 continued on page 8

**Question 6 continued**

**Marks**

- c) i) Show that  $a^2 + b^2 > 2ab$ , where  $a$  and  $b$  are distinct positive real numbers. 1
- ii) Hence show that  $a^2 + b^2 + c^2 > ab + bc + ca$ , where  $a$ ,  $b$  and  $c$  are distinct positive real numbers. 2
- iii) Hence, or otherwise prove that  $\frac{a^2b^2 + b^2c^2 + c^2a^2}{a + b + c} > abc$ , 2  
where  $a$ ,  $b$  and  $c$  are distinct positive real numbers.

**End of Question 6**



**Question 7** (15 marks) Start a NEW page.

**Marks**

a) Given that  $P(x) = 3x^3 - 11x^2 + 8x + 4$  has a double root, fully factorise  $P(x)$ . 3

b) Show that  $\tan^{-1}x > x - \frac{1}{3}x^3$  for all values of  $x > 0$ . 3

c) The acceleration of a particle which is moving along the  $x$ -axis is given by

$$\frac{d^2x}{dt^2} = 2x^3 - 10x.$$

i) If the particle starts at the origin with velocity  $u$  show that its velocity  $v$  is given by  $v^2 - u^2 = x^4 - 10x^2$ . 2

ii) If  $u = 3$  show that the particle oscillates within the interval  $-1 \leq x \leq 1$ . 4

iii) Is the motion referred to in (ii) an example of simple harmonic motion? 1  
Give a clear reason for your answer.

iv) If  $u = 6$ , carefully describe the motion. 2

**Question 8** (15 marks) Start a NEW page.

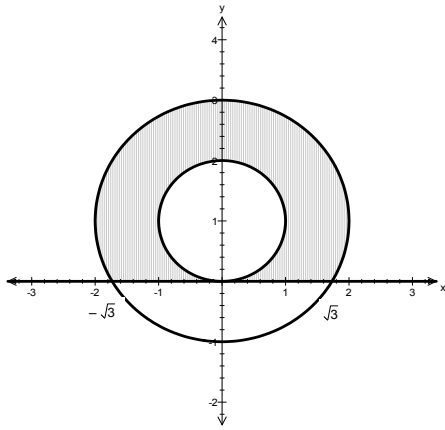
**Marks**

- a) Let  $\omega$  be one of the non-real cube roots of unity.
- i) Show that  $1 + \omega + \omega^2 = 0$ . 1
- ii) Hence find the value of  $(2 - \omega)(2 - \omega^2)(2 - \omega^4)(2 - \omega^5)$ . 2
- b) i) By using the expansion for  $\cos(A + B)$ , show that  $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ . 2
- ii) By using the substitution  $x = 2\cos\theta$ , solve the equation  $x^3 - 3x = \sqrt{2}$ . 3
- iii) Hence explain why  $\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{7\pi}{12}\right) + \cos\left(\frac{9\pi}{12}\right) = 0$  1
- c) The equation  $x^3 - 3px + q = 0$ , where  $p > 0, q \neq 0$  are both real, has three distinct, non-zero real roots.
- i) Show that the graph of  $y = x^3 - 3px + q$  has a relative maximum value of  $q + 2p\sqrt{p}$  and a relative minimum of  $q - 2p\sqrt{p}$ . 3
- ii) Hence show giving reasons that  $q^2 < 4p^3$ . 3

**END OF EXAM**

YEAR 12 EXTENSION 2 CARINGBAH	HIGH SCHOOL TRIAL HSC SOLUTIONS 2009
<p>1a) <math>\int \cos x \sin^4 x \, dx = \frac{\sin^5 x}{5} + C</math> <input checked="" type="checkbox"/></p> <p>b) <math>\int \frac{dx}{x^2 - 4x + 8} = \int \frac{dx}{(x-2)^2 + 2^2}</math> <input checked="" type="checkbox"/>  <math>= \frac{1}{2} \tan^{-1} \left( \frac{x-2}{2} \right) + C</math> <input checked="" type="checkbox"/></p> <p>c) Let <math>u = x - 2 \rightarrow x = u + 2 \rightarrow du = dx</math>  When <math>x = 1, u = -1; x = 3, u = 1</math></p> <p><math>\therefore I = \int_{-1}^1 (u+2) \cdot u^5 \, du</math> <input checked="" type="checkbox"/>  <math>= \int_{-1}^1 u^6 + 2u^5 \, du</math>  <math>= \left[ \frac{u^7}{7} + \frac{2u^6}{6} \right]_{-1}^1</math> <input checked="" type="checkbox"/>  <math>= \left( \frac{1}{7} + \frac{1}{3} \right) - \left( -\frac{1}{7} + \frac{1}{3} \right) = \frac{2}{7}</math> <input checked="" type="checkbox"/></p> <p>d) i) <math>5 \equiv (Ax + B)(x + 1) + C(x^2 + 4)</math>  Let <math>x = -1 : 5 = 5C \rightarrow C = 1</math> <input checked="" type="checkbox"/>  Let <math>x = 0 : 5 = B + 4C \rightarrow B = 1</math>  Let <math>x = 1 : 5 = 2(A + B) + 5C \rightarrow A = -1</math> <input checked="" type="checkbox"/></p> <p>ii) <math>I = \int \frac{-x+1}{x^2+4} + \frac{1}{x+1} \, dx</math>  <math>= \int \frac{-x}{x^2+4} + \frac{1}{x^2+4} + \frac{1}{x+1} \, dx</math> <input checked="" type="checkbox"/>  <math>= -\frac{1}{2} \ln(x^2 + 4) + \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) + \ln(x + 1) + c</math> <input checked="" type="checkbox"/></p> <p>e) <math>I_n = \int_1^e x(\ln x)^n \, dx</math>  Let <math>u = (\ln x)^n, \quad v' = x</math>  <math>u' = \frac{n}{x} (\ln x)^{n-1}, \quad v = \frac{x^2}{2}</math> <input checked="" type="checkbox"/></p>	<p><math>\therefore I_n = \left[ \frac{x^2}{2} (\ln x)^n \right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{n}{x} (\ln x)^{n-1} \, dx</math>  <math>= \frac{e^2}{2} - \frac{1^2}{2} \cdot 0^n - \frac{n}{2} I_{n-1} = \frac{e^2}{2} - \frac{n}{2} I_{n-1}</math> <input checked="" type="checkbox"/></p> <p>e)ii) <math>I_2 = \frac{e^2}{2} - \frac{2}{2} I_1</math>  <math>= \frac{e^2}{2} - \left[ \frac{e^2}{2} - \frac{1}{2} I_0 \right]</math> <input checked="" type="checkbox"/>  <math>= \frac{1}{2} \int_1^e x \, dx = \frac{e^2 - 1}{4}</math> <input checked="" type="checkbox"/></p> <hr/> <p>2a) i) <math>\bar{z} = 1 + i</math> <input checked="" type="checkbox"/>  ii) <math> z  = \sqrt{2}</math> <input checked="" type="checkbox"/>  iii) <math>\arg z = -\frac{\pi}{4}</math> <input checked="" type="checkbox"/>  iv) <math>\arg iz = \arg(i) + \arg z = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}</math> <input checked="" type="checkbox"/></p> <p>v) <math>z^6 = \left( \sqrt{2} \operatorname{cis} \left( -\frac{\pi}{4} \right) \right)^6</math> <input checked="" type="checkbox"/>  <math>= \sqrt{2}^6 \cdot \operatorname{cis} \left( -\frac{6\pi}{4} \right)</math> (DMT)  <math>= 8 \times i = 0 + 8i</math> <input checked="" type="checkbox"/></p> <p>b) <math>\frac{i^5(1-i)}{2+i} = \frac{i \cdot (i^2)^2(1-i)}{2+i} \times \frac{2-i}{2-i}</math> <input checked="" type="checkbox"/>  <math>= \frac{i(1-i)(2-i)}{1+4}</math> <input checked="" type="checkbox"/>  <math>= \frac{i(2-i-2i-1)}{5}</math>  <math>= \frac{3}{5} + \frac{1}{5}i</math> <input checked="" type="checkbox"/></p>

2c)

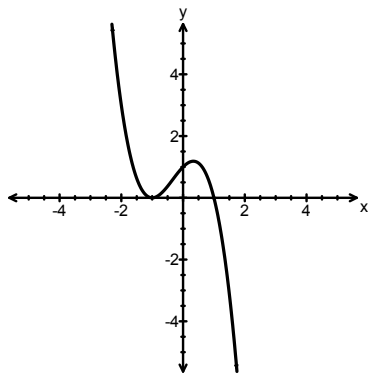


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d) i)  $C$  is  $z_3 = i(\sqrt{3} + i) = -1 + i\sqrt{3}$  ☑

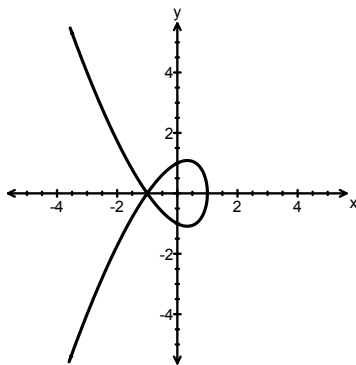
ii)  $B$  is  $z_2 = \overrightarrow{OA} + \overrightarrow{AB}$   
 $= \overrightarrow{OA} + \overrightarrow{OC}$  ☑  
 $= z_1 + z_3$   
 $= (\sqrt{3} - 1) + i(1 + \sqrt{3})$  ☑

3a) i)



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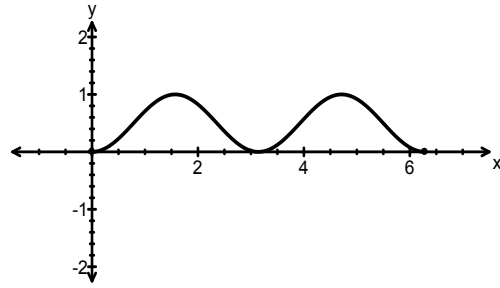
ii)



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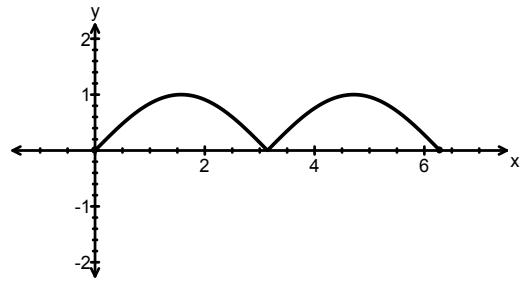
3b) i)

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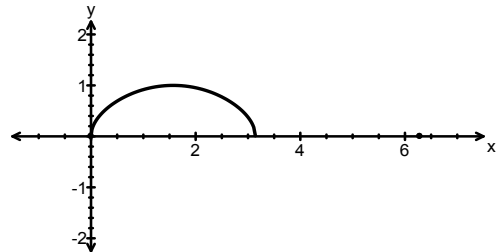
ii)

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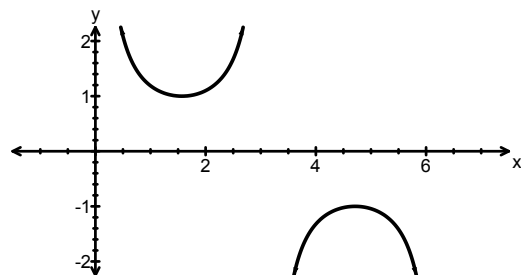
iii)

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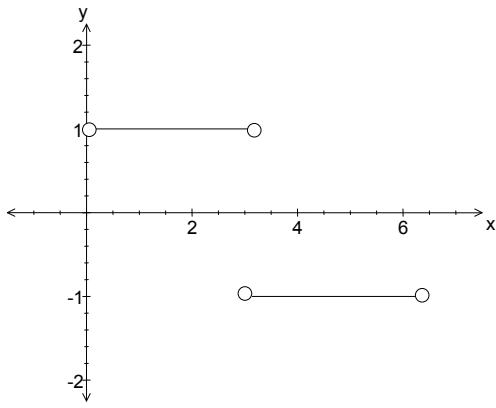


iv)

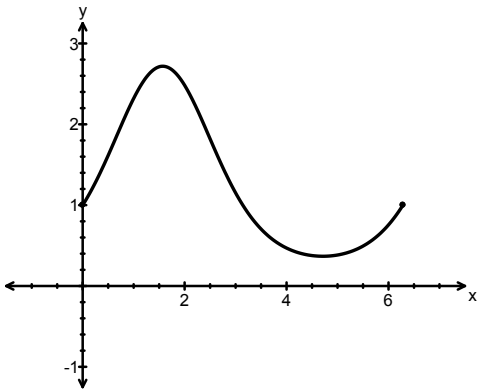
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3b) v)



vi)



c) Horizontal tangent at  $x = X \rightarrow \frac{dy}{dx} = 0$

$$\frac{d}{dx}(x^2 + 2xy + y^5) = \frac{d}{dx}(4)$$

$$2x + 2\left(y \frac{dy}{dx} + \frac{dy}{dx} y\right) = \frac{d}{dx}(4) \quad \checkmark$$

$$(2x + 5y^4) \frac{dy}{dx} = -2(x + y)$$

$$\therefore \frac{dy}{dx} = \frac{-2(x + y)}{(2x + 5y^4)} \quad \checkmark$$

$$\therefore -2(x + y) = 0 \rightarrow y = -x \text{ where } x = X \quad \checkmark$$

$$\therefore X^2 + 2X(-X) + (-X)^5 = 4$$

$$X^2 - 2X^2 - X^5 = 4 \quad \checkmark$$

$$\therefore X^5 + X^2 + 4 = 0$$

$$4a) \quad \frac{x^2}{9} - \frac{y^2}{16} = 1 \rightarrow a = 3, b = 4$$

i) transverse axis = 6

ii)  $e^2 = 1 + \frac{b^2}{a^2} = \frac{25}{9}$   
 $\therefore e = \frac{5}{3}$

iii)  $ae = 3 \times \frac{5}{3} = 5$   
 $\therefore$  Foci are  $S(5,0)$  and  $S'(-5,0)$

iv)  $\frac{a}{e} = \frac{3}{5/3} = \frac{9}{5}$   
 $\therefore$  directrices are  $x = \pm \frac{9}{5}$

v)  $\frac{b}{a} = \frac{4}{3} \rightarrow$  asymptotes  $y = \pm \frac{4}{3}x$

b) Let the roots be:  $\alpha\beta, \beta\gamma, \alpha\gamma$

which is the equivalent of:  $\frac{\alpha\beta\gamma}{\gamma}, \frac{\beta\gamma\alpha}{\alpha}, \frac{\alpha\gamma\beta}{\beta}$

and since  $\alpha\beta\gamma = -\frac{d}{a} = -1$

then the roots are  $-\frac{1}{\alpha}, -\frac{1}{\beta}, -\frac{1}{\gamma}$ .

Let  $x = -\frac{1}{\alpha} \rightarrow \alpha = -\frac{1}{x}$

and since  $\alpha$  satisfies the original equation then we get

$$\left(-\frac{1}{x}\right)^3 + 6\left(-\frac{1}{x}\right) + 1 = 0 \quad \checkmark$$

which gives  $x^3 - 6x^2 - 1 = 0$ .

c) i) Since the coefficients are real, the complex roots occur in conjugate pairs.

ii) Let  $P(x) = (x - (2 - i))(x - (2 + i))Q(x)$

$$= (x^2 - 4x + 5)(x^2 + ax + b) \quad \checkmark$$

$$\therefore ax^3 - 4x^3 = -5x^3 \rightarrow a = -1$$

also  $5b = -10 \rightarrow b = -2$

$$\therefore P(x) = (x^2 - 4x + 5)(x^2 - x - 2)$$

$$= (x^2 - 4x + 5)(x - 2)(x + 1)$$

Hence the other roots are  $x = 2$  and  $x = -1$ .

$$4d) V_s = 2\pi xy\delta x$$

$$\therefore V = \lim_{\delta x \rightarrow 0} \sum_{x=0}^1 2\pi xy\delta x \quad \checkmark$$

$$= 2\pi \int_0^1 x(\sqrt{x} - x^2) dx \quad \checkmark$$

$$= 2\pi \int_0^1 x^{\frac{3}{2}} - x^3 dx$$

$$= 2\pi \left[ \frac{2x^{\frac{5}{2}}}{5} - \frac{x^4}{4} \right]_0^1 \quad \checkmark$$

$$= \frac{3\pi}{10} u^3 \quad \checkmark$$

$$5a) \ln(2^{3x+1}) = \ln(5^{x+1})$$

$$\therefore (3x+1)\ln 2 = (x+1)\ln 5 \quad \checkmark$$

$$3x\ln 2 + \ln 2 = x\ln 5 + \ln 5$$

$$x(3\ln 2 - \ln 5) = \ln 5 - \ln 2$$

$$x = \frac{\ln 5 - \ln 2}{3\ln 2 - \ln 5} \approx 1.95 \quad \checkmark$$

$$b) i) I = \int_{-a}^a f(x) dx \\ = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$$

$$\text{Let } x = -u \rightarrow dx = -du$$

$$\text{When } x = -a, u = a; x = 0, u = 0 \quad \checkmark$$

$$\therefore I = -\int_a^0 f(-u) du + \int_0^a f(x) dx$$

$$= \int_0^a f(-u) du + \int_0^a f(x) dx$$

$$= \int_0^a f(-x) dx + \int_0^a f(x) dx \quad \checkmark$$

$$= \int_0^a [f(x) + f(-x)] dx$$

ii)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x} dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} + \frac{1}{1+\sin(-x)} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} + \frac{1}{1-\sin x} dx \quad \checkmark$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin x + 1 + \sin x}{1 - \sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2}{\cos^2 x} dx \quad \checkmark$$

$$= 2 \int_0^{\frac{\pi}{4}} \sec^2 x dx$$

$$= 2 [\tan x]_0^{\frac{\pi}{4}} = 2 \quad \checkmark$$

c) i)  $\angle PNQ = \angle PMQ = 90^\circ$  [ $\angle$ 's in same segment]  $\checkmark$

ii) Join QA and AR.

Let  $\angle RPA = \alpha = \angle RQA$

[ $\angle$ 's in same segment in cyclic quad PQAR]  $\checkmark$

Also  $\angle NPM = \alpha = \angle NQM$

[ $\angle$ 's in same segment in cyclic quad PQMN]  $\checkmark$

Also since  $\angle HMQ = \angle AMQ = 90^\circ$

and since QM is a common side

then  $\triangle QHM \equiv \triangle QAM$  [AAS]  $\checkmark$

Hence  $HM = MA$  [corres sides in cong  $\Delta$ 's]

$$d) i) \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{1}{\frac{dx}{d\theta}}$$

$$\therefore m = \frac{dy}{dx} = b \cos \theta \times \frac{1}{-a \sin \theta} = -\frac{b \cos \theta}{a \sin \theta} \quad \checkmark$$

$$\therefore y - b \sin \theta = -\frac{b \cos \theta}{a \sin \theta} (x - a \cos \theta)$$

$$ay \sin \theta - ab \sin^2 \theta = -bxcos\theta + abcos^2\theta$$

$$\therefore bxcos\theta + ay \sin \theta = ab(\sin^2 \theta + \cos^2 \theta)$$

$$bxcos\theta + ay \sin \theta = ab \quad [\div ab] \text{ to get } \quad \checkmark$$

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1.$$

ii) M has coordinates  $(0, b \sin \theta)$

In the equation in (i) let  $x = 0$  to give

the coordinates of B  $(0, \frac{b}{\sin \theta})$ .  $\checkmark$

$$\therefore OM \cdot OB = |b \sin \theta| \times \frac{b}{|\sin \theta|} = b^2 \quad \checkmark$$

$$6a) m_{PQ} = \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \frac{c(p-q)}{pq} \times \frac{1}{c(q-p)} = \frac{-1}{pq}$$

Since  $\angle PRQ = 90^\circ$  then  $m_{PR} \times m_{QR} = -1$  ✓

$$\therefore \frac{-1}{pr} \times \frac{-1}{qr} = -1 \rightarrow r^2 = -\frac{1}{pq}$$
 ✓

$$\text{Now } xy = c^2 \rightarrow y' = -\frac{c^2}{x^2}$$

Hence at  $R$  the gradient of the tangent =  $\frac{-1}{r^2}$

$\therefore$  at  $R$  the gradient of the normal =  $r^2$  ✓

$$\text{but } r^2 = -\frac{1}{pq} = m_{PQ}$$
 ✓

$\therefore PQ$  parallel to the normal at  $R$ .

bi) Let  $r$  be the radius of a typical slice

$$\therefore r + 1 = x \rightarrow r = x - 1$$
 ✓

$$\text{Now } \Delta V = \pi r^2 h = \pi(x-1)^2 \Delta y$$
 ✓

ii) When  $x = 1, y = 2 - 1 = 1$  ✓

$$\therefore V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^1 \pi(x-1)^2 \Delta y$$

$$\text{Now } -y = x^2 - 2x$$

$$\therefore 1 - y = x^2 - 2x + 1$$
 ✓

$$\therefore 1 - y = (x-1)^2$$

$$\text{hence } \therefore V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^1 \pi(1-y) \Delta y$$

$$\text{iii) } V = \pi \int_0^1 1 - y \, dy$$

$$= \pi \left[ y - \frac{y^2}{2} \right]_0^1$$
 ✓

$$= \frac{\pi}{2} u^3$$
 ✓

$$\text{c) i) } (a-b)^2 > 0 \quad (a \neq b)$$

$$\therefore a^2 - 2ab + b^2 > 0$$
 ✓

$$\therefore a^2 + b^2 > 2ab \text{ ----- [1]}$$

$$\text{ii) Similarly } b^2 + c^2 > 2bc \text{ ----- [2]}$$

$$\text{and } c^2 + a^2 > 2ca \text{ ----- [3]} \quad \checkmark$$

Now [1] + [2] + [3] gives

$$2(a^2 + b^2 + c^2) > 2(ab + bc + ca) \quad \checkmark$$

$$\therefore a^2 + b^2 + c^2 > ab + bc + ca$$

iii) Using the result in (ii)

Let  $a \rightarrow ab; b \rightarrow bc; c \rightarrow ca$ .

$$\therefore a^2 b^2 + b^2 c^2 + c^2 a^2 > ab \cdot bc + bc \cdot ca + ca \cdot ab \quad \checkmark$$

$$a^2 b^2 + b^2 c^2 + c^2 a^2 > abc(a + b + c) \quad \checkmark$$

$$\therefore \frac{a^2 b^2 + b^2 c^2 + c^2 a^2}{a+b+c} > abc$$

$$7a) P(x) = 3x^3 - 11x^2 + 8x + 4$$

$$P'(x) = 9x^2 - 22x + 8$$

$P'(x) = 0$  for stationary points.

$$\therefore (9x-4)(x-2) = 0 \rightarrow x = 2 \text{ or } x = \frac{4}{9} \quad \checkmark$$

$$\text{Since } P(2) = 24 - 44 + 16 + 4 = 0$$

then  $x = 2$  is the double root. ✓

$$\therefore P(x) = (x-2)^2 (3x+1) \quad \checkmark$$

$$\text{b) Let } P(x) = \tan^{-1} x - x + \frac{1}{3} x^3$$

$$\therefore P'(x) = \frac{1}{1+x^2} - 1 + x^2 \quad \checkmark$$

$$= \frac{1 - (1+x^2) + x^2(1+x^2)}{1+x^2}$$

$$= \frac{x^4}{1+x^2} > 0 \text{ for } x > 0 \quad \checkmark$$

$\therefore P(x)$  is an increasing function for  $x > 0$

$$\text{and since } P(0) = 0 \text{ then } \tan^{-1} x > x - \frac{1}{3} x^3 \quad \checkmark$$

$$\text{c) i) } \frac{d}{dx} \left( \frac{1}{2} v^2 \right) = 2x^3 - 10x$$

$$\therefore \frac{1}{2} v^2 = \frac{x^4}{2} - 5x^2 + c \quad \checkmark$$

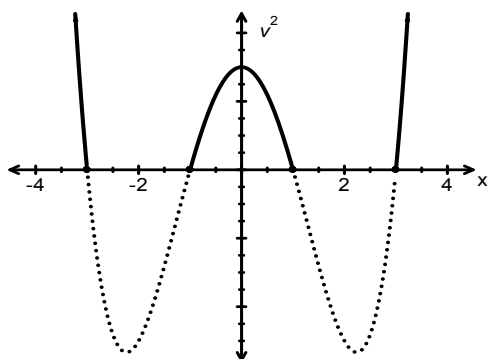
$$\text{when } x = 0, v = u \rightarrow c = \frac{1}{2} u^2$$

$$\therefore \frac{1}{2} v^2 = \frac{x^4}{2} - 5x^2 + \frac{1}{2} u^2 \quad \checkmark$$

$$\therefore v^2 - u^2 = x^4 - 10x^2$$

7cii) If  $u = 3$  then  $v^2 - 9 = x^4 - 10x^2$

$$\begin{aligned} \therefore v^2 &= x^4 - 10x^2 + 9 \\ &= (x^2 - 1)(x^2 - 9) \\ &= (x - 1)(x + 1)(x - 3)(x + 3) \quad \checkmark \end{aligned}$$



Since  $v^2 \geq 0$  for motion to exist then  $(x - 1)(x + 1)(x - 3)(x + 3) \geq 0$  and since the particle starts at  $x = 0$  with  $v = 3$  it is moving to the right.  $\checkmark$

At  $x = 1$ ,  $v = 0$  and  $\ddot{x} = -8$  and so the particle will move to the left until it reaches  $x = -1$  where  $v = 0$  and  $\ddot{x} = 8$ . This means the particle will then move to the right until  $v = 0$  again at  $x = 1$ .  $\checkmark \checkmark$

Thus it oscillates in the interval  $-1 \leq x \leq 1$ .

iii) Not SHM since  $\ddot{x} \neq -n^2(x - b)$   $\checkmark$

iv) When  $u = 6$ ,  $v^2 = x^4 - 10x^2 + 36 = (x^2 - 5)^2 + 11$

$\therefore v^2 > 0$  for all  $x$  in the domain, hence the particle will never stop.  $\checkmark$

It moves to the right with decreasing velocity until it passes  $x = \sqrt{5}$  (with  $v = \sqrt{11}$ ) and continues to the right with increasing velocity.  $\checkmark$

8 a) i) Let  $\omega$  be a solution to  $z^3 = 1$   
 $\therefore z^3 - 1 = 0 \rightarrow (z - 1)(z^2 + z + 1) = 0$   
 but as  $\omega$  is a non real root, then  $\omega^2 + \omega + 1 = 0$ .  $\checkmark$

ii) Now  $\omega^3 = 1$ , hence  $(2 - \omega)(2 - \omega^2)(2 - \omega^4)(2 - \omega^5)$  reduces to  $(2 - \omega)(2 - \omega^2)(2 - \omega)(2 - \omega^2) = (2 - \omega)^2(2 - \omega^2)^2 = [(2 - \omega)(2 - \omega^2)]^2 = [4 - 2\omega^2 - 2\omega + \omega^3]^2 = [5 - 2(\omega^2 + \omega)]^2 = [5 - 2 \times -1]^2 = 49$   $\checkmark$

8 b) i)  $\cos(3\theta) = \cos(\theta + 2\theta) = \cos \theta \cos 2\theta - \sin \theta \sin 2\theta = \cos \theta (2 \cos^2 \theta - 1) - \sin \theta 2 \sin \theta \cos \theta = 2 \cos^3 \theta - \cos \theta - 2 \cos \theta (1 - \cos^2 \theta) = 2 \cos^3 \theta - \cos \theta - 2 \cos \theta + 2 \cos^3 \theta = 4 \cos^3 \theta - 3 \cos \theta$ .  $\checkmark$

ii)  $x^3 - 3x = \sqrt{2}$  and let  $x = 2 \cos \theta$  to obtain  $8 \cos^3 \theta - 6 \cos \theta = \sqrt{2}$   
 $\therefore 4 \cos^3 \theta - 3 \cos \theta = \frac{1}{\sqrt{2}}$   
 $\therefore \cos 3\theta = \frac{1}{\sqrt{2}}$   $\checkmark$

$\therefore 3\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \dots$   
 $\therefore \theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \dots$   $\checkmark$

hence  $x = 2 \cos \frac{\pi}{12}, 2 \cos \frac{7\pi}{12}, 2 \cos \frac{9\pi}{12}$   $\checkmark$

iii) The sum of the roots of  $x^3 - 3x - \sqrt{2} = 0$  is given by  $-\frac{b}{a} = -\frac{0}{1} = 0$ .

$\therefore 2 \cos \frac{\pi}{12} + 2 \cos \frac{7\pi}{12} + 2 \cos \frac{9\pi}{12} = 0$   $\checkmark$   
 $\therefore \cos \frac{\pi}{12} + \cos \frac{7\pi}{12} + \cos \frac{9\pi}{12} = 0$

c) i)  $y = x^3 - 3px + q$ ,  $y' = 3x^2 - 3p$ ,  $y'' = 6x$

For stat pts  $y' = 0 \rightarrow 3(x^2 - p) = 0$   
 $\therefore x = \pm\sqrt{p}$   $\checkmark$

When  $x = \sqrt{p}$ ,  $y = p\sqrt{p} - 3p\sqrt{p} + q = q - 2p\sqrt{p}$

When  $x = \sqrt{p}$ ,  $y'' = 6\sqrt{p} > 0 \rightarrow$  minimum.  $\checkmark$

When  $x = -\sqrt{p}$ ,  $y = -p\sqrt{p} + 3p\sqrt{p} + q = q + 2p\sqrt{p}$

When  $x = -\sqrt{p}$ ,  $y'' = -6\sqrt{p} < 0 \rightarrow$  maximum.  $\checkmark$

ii) Since  $x^3 - 3px + q = 0$  has 3 distinct non zero real roots then the turning points must be on either side of the  $x$ -axis.  $\checkmark$

$\therefore$  minimum  $\times$  maximum  $< 0$   $\checkmark$

$\therefore (q + 2p\sqrt{p})(q - 2p\sqrt{p}) < 0$   
 $q^2 - 4p^2 \cdot p < 0$   
 hence  $q^2 < 4p^3$   $\checkmark$