FORT STREET HIGH SCHOOL

Name: ____________________

Teacher: ____________________

Class: ____________________

2016

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics

Time allowed: 3 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
	Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-10
H2, H3, H4, H5	Manipulates algebraic expressions to solve problems from topic areas such as geometry, co-ordinate geometry, quadratics, trigonometry, probability and logarithms	12, 14
H6, H7, H8	Demonstrates skills in the processes of differential and integral calculus and applies them appropriately	11, 13, 15
H9	Synthesises mathematical solutions to harder problems and communicates them in appropriate form	16

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions,
Allow about 15 minutes for this section
Section II 90 Marks
Attempt Questions 11-16,
Allow about 2 hours 45 minutes for this section

General Instructions:

- Questions 11-16 are to be started in a new booklet
- The marks allocated for each question are indicated
- In Questions 11 – 16, show relevant mathematical reas[oning] and/or calculations.
- Marks may be deducted for careless or badly arranged
- Board – approved calculators may be used

Section I	Total 10	Marks
Q1-Q10		
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

Section I

10 marks

Attempt questions 1-10

Allow about 15 minutes for this section

Use the multiple choice answer sheet for questions 1-10.

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1 The mass of 1 atom of oxygen is $2{\cdot}7 \times 10^{-23}$ grams. What is the mass of 8×10^{27} atoms of oxygen?

(A) 21600

(B) 2160000

(C) $2{\cdot}16\times10^{5}$

(D) $2{\cdot}16\times10^{51}$

2 What is the gradient of a line normal to $2x-4y+3=0$

(A) -2

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) 2

3 An arithmetic series is given as $\sum_{1}^{k} 4n-1$. What is the third term of this series?

(A) $4k-1$

(B) 3

(C) 11

(D) 21

4 What is the exact value of $\sec\left(\frac{5\pi}{6}\right)$?

(A) $\frac{1}{2}$

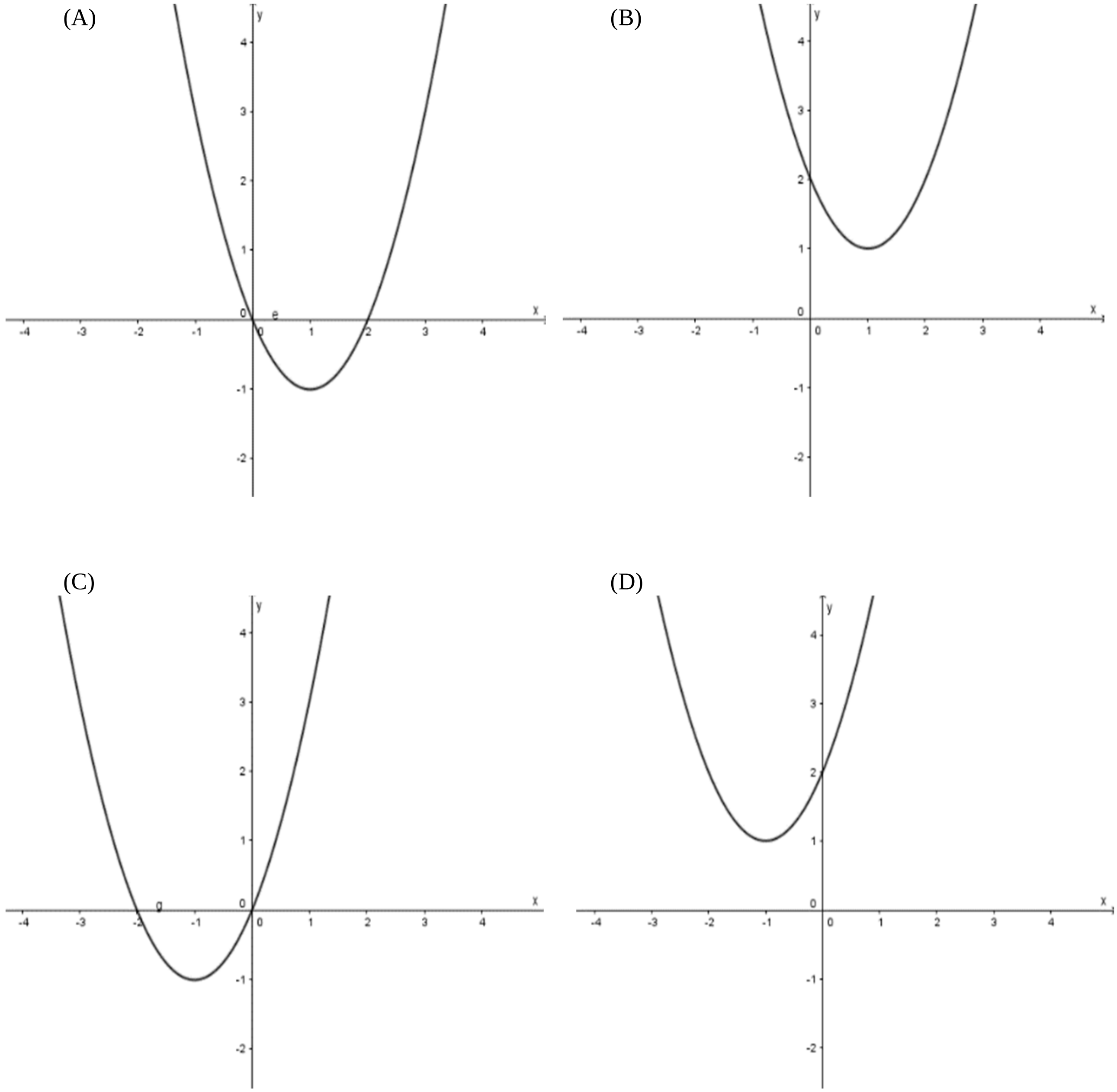
(B) -2

(C) $\frac{2\sqrt{3}}{3}$

(D) $\frac{-2\sqrt{3}}{3}$

5 Which graph best depicts $f(x) = (x-1)^2 + 1$?

(A) (B) (C) (D)



6 For what values of x is the curve $f(x) = x(x-2)^2$ increasing?

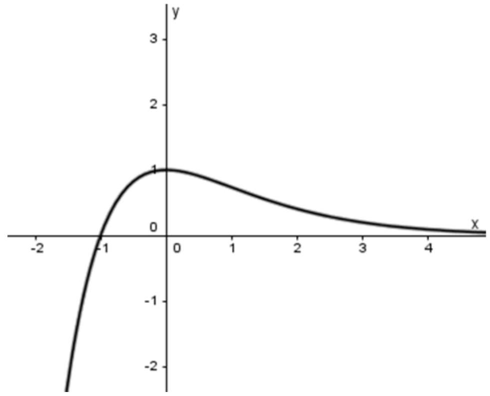
(A) $x > 2$

(B) $x < 0 \text{ or } x > 2$

(C) $\frac{2}{3} < x < 2$

(D) $x < \frac{2}{3} \text{ or } x > 2$

7 The diagram shows the graph of $y = e^{-x}(1+x)$.



How many solutions are there to the equation $e^{-x}(1+x) = 1 - \ln x$?

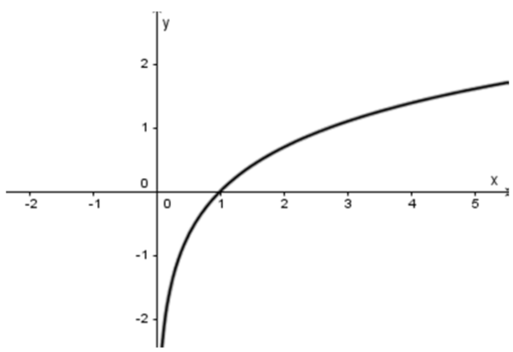
(A) 0

(B) 1

(C) 2

(D) 3

8 The diagram shows the graph of $y = \ln x$.



If the area between the axes, the line $y = 1$ and the curve $y = \ln x$ is rotated about the y-axis, which expression would give the volume of the solid formed?

(A) $\pi\int_1^2 \ln x \; dx$

(B) $\pi\int_1^2 (\ln x)^2 \; dx$

(C) $\pi\int_0^1 (\ln x)^2 \; dy$

(D) $\pi\int_0^1 e^{2y} \; dy$

9 Which expression is the factorisation of $27x^3 - y^3$

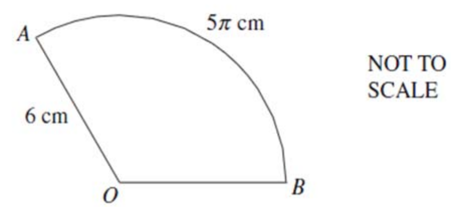
(A) $(3x - y)(9x^2 + 3xy + y^2)$

(B) $(3x + y)(9x^2 - 3xy - y^2)$

(C) $(x + y)(27x^2 - y^2)$

(D) $(x - y)(27x^2 + y^2)$

10 The diagram shows sector *AOB*.



What is the exact area of the minor segment cut off by a chord *AB*?

(A) 15π

(B) $15\pi - 9$

(C) 12π

(D) $12\pi - 3\sqrt{3}$

Section II

90 marks

Attempt questions 11-16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.
In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a *new* booklet.

(a) Find integers a and b such that $\dfrac{2}{2+\sqrt{5}} = a + b\sqrt{5}$ **2**

(b) Factorise $2x^2 - 17x + 30$ **2**

(c) Differentiate $\dfrac{(x-2)^3}{2x^2}$ **2**

(d) Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 3x \; dx$ **2**

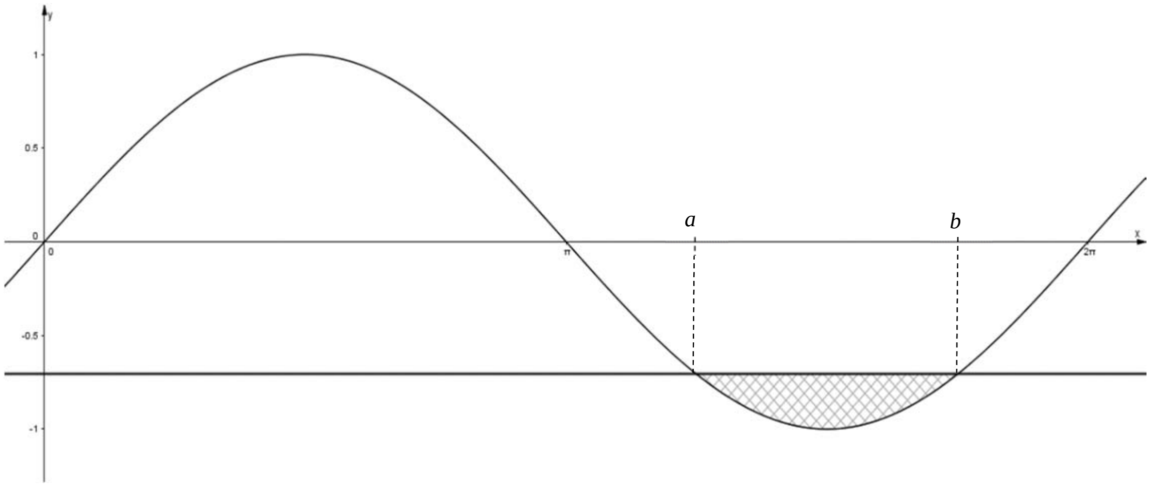
(e) Find $\int \dfrac{x^2}{x^3-1} \, dx$ **2**

(f) (i) Differentiate $x \ln x$ **2**

(ii) Hence, evaluate $\int_1^e \ln x \; dx$ **3**

Question 12 (15 marks) Start a *new* booklet.

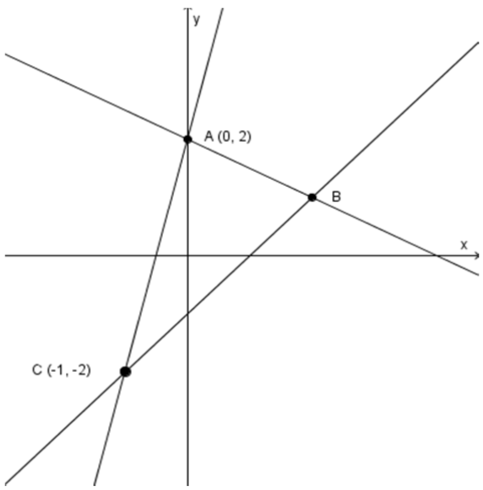
(a) The shaded area shows the region between $y=-\frac{1}{\sqrt{2}}$ and $y=\sin x$ where $a\le x\le b$



(i) Find the values of a and b by solving $\sin x=-\frac{1}{\sqrt{2}}$ for $0<x<2\pi$ **2**

(ii) Hence, show the area of the shaded region is $\frac{\sqrt{2}(4-\pi)}{4}$ units^2. **4**

(b) The diagram shows three lines forming a triangle ABC. The points A and C have the coordinates $(0,2)$ and $(-1,-2)$ respectively.



The line passing through points B and C make an angle of 45° with the horizontal.

(i) Show the equation of the line passing through BC is given by $x-y-1=0$. **2**

(ii) The equation of the line passing through AB is given by $x+2y-4=0$.

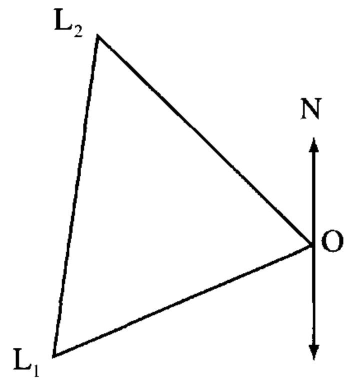
Find the point of intersection B. **2**

(iii) Find the exact distance BC **2**

(iv) Hence, find the exact area of triangle ABC. **3**

Question 13 (15 marks) Start a *new* booklet.

(a) From an observation tower O on a ship, a sailor determines the lighthouse L_1 is on a bearing of 255° T and a different lighthouse L_2 is at a bearing of 315°T.



L_1 and L_2 are a distance of 70 and 75 kilometres respectively from O.

(i) Show that angle L_1OL_2 is 60° **1**

(ii) Find, correct to the nearest *km*, the distance between the two lighthouses L_1 and L_2. **2**

(iii) Find the bearing of L_2 from L_1 to the nearest degree. **3**

(b) Consider the curve $y = x^5 - 5x^4 + 5x^3$.

(i) Find the stationary points and determine their nature. **4**

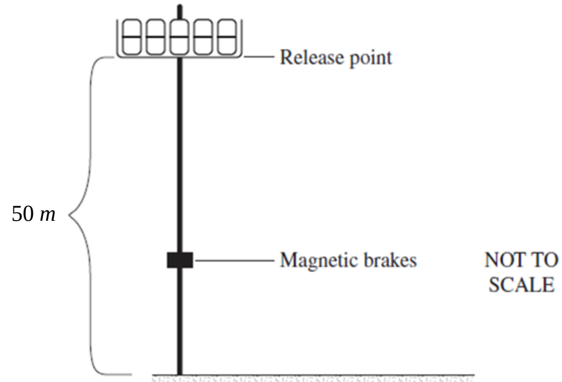
(ii) Show that there are inflexions at $x = 0$, $x = \dfrac{3-\sqrt{3}}{2}$ and $x = \dfrac{3+\sqrt{3}}{2}$ **3**

(You do not need to find their corresponding y value)

(iii) Sketch the curve labelling the *stationary points only*. **2**

Question 14 (15 marks) Start a *new* booklet.

(a) At Luna Park, a chair is released from a height of 50 metres and falls vertically. Magnetic brakes are applied to stop the fall.



The height of the chair at time t seconds is x metres. The acceleration of the chair is given by $\ddot{x} = -9 \cdot 8$.
At the release point, $t = 0,\ x = 50$ and $\dot{x} = 0$.

(i) Show that the chair's displacement x at time t is given by $x = -4 \cdot 9t^2 + 50$ **3**

(ii) If it takes half a second for the brakes to stop the fall, find the latest time the brakes could be applied? (Correct your answer to two decimal places) **2**

(iii) How far has the chair fallen and what is its speed when the brakes are applied? (correct your answer to two decimal places) **2**

(b) For what values of k is the quadratic $(k+1)x^2 + kx + k + 1$ a positive definite? **3**

(c) Jasmine sets aside savings each week for a vacation. In the first week she saves \$36 and each subsequent week she is able to save \$4 more than the previous week.

The maximum Jasmine is able to save is \$180 per week.

(i) How many weeks would it take for Jasmine to be able to save \$180 per week? **2**

(ii) Jasmine needs to save \$7356 for her vacation.
How many weeks would it take for Jasmine to save the amount of her vacation? **3**

Question 15 (15 marks) Start a *new* booklet.

(a) The population, P, of a Dodo colony was decreasing to the differential equation

$$\frac{dP}{dt} = -kP$$

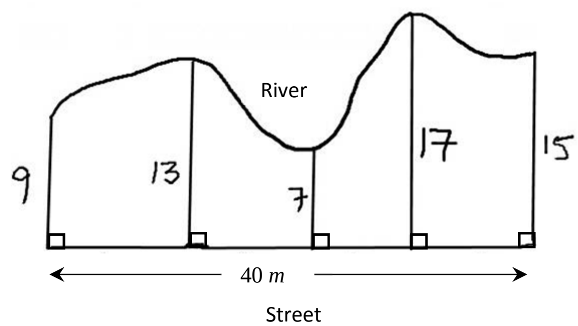
where t is the time measured in years from when recordings were first kept in January, 1684.

(i) Show that $P = P_0 e^{-kt}$ is a solution to the differential equation, where P_0 is the initial population of the colony on January 1st, 1684. **1**

(ii) Find, correct to 4 significant figures, the value of k if records showed only half the population of Dodos remained after 30 years. **2**

(iii) During what year did the Dodo population become less than 1% of the initial population P_0? **3**

(b) Neil has just purchased land. The street side is 40 metres wide and the property runs to a river as shown in the diagram. Neil has surveyed how far it is to the river from regular intervals.



Use Simpson's rule with 5 function values to approximate the area of Neil's land. **3**

(c) Ava and Owen are playing a game by taking turns throwing a 20 sided die. Ava wins if she throws a number divisible by 5 or the number 13 and Owen wins if he throws a number divisible by 3.

Ava throws first.

(i) Find the probability that Ava wins the game on her first throw. **1**

(ii) Find the probability that Ava wins the game on her first or second throw. **2**

(iii) With how many of Ava's throws will she have had a better than a 50% chance of winning the game? **3**

Question 16 (15 marks) Start a *new* booklet.

(a) Samantha has retired with savings of $\$400000$. She intends to live off this money by taking equal monthly instalments of $\$M$ at the *start* of each month, starting immediately.

Samantha receives 6% p.a. interest on her account balance at the end of each month.

Let $\$A_n$ be her account balance at the end of each month.

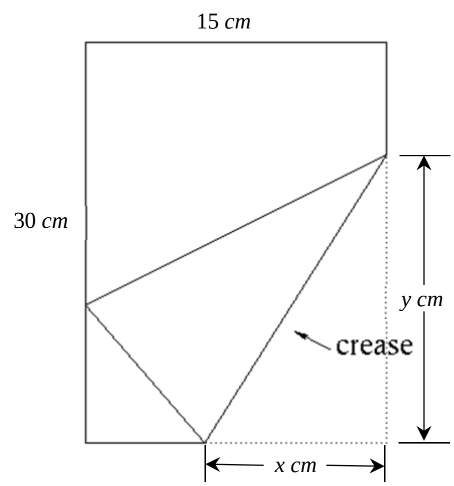
(i) Show that $A_2 = 400000(1\cdot005)^2 - M(1\cdot005)^2 - M(1\cdot005)$ **1**

(ii) Show that $A_n = 400000(1\cdot005)^n - M\left[201(1\cdot005^n - 1)\right]$ **3**

(iii) Samantha would like her money to last 25 years. What monthly instalment does Samantha take? **2**

(b) A rectangular piece of paper is 30 centimetres high and 15 centimetres wide. The lower right-hand corner is folded over so as to reach the leftmost edge of the paper.

Let x be the horizontal distance folded and y be the vertical distance folded as shown in the diagram.



(i) By considering the areas of three triangles and trapezium that make up the total area of the paper, show **3**

$$y = \frac{x\sqrt{15(2x-15)}}{2x-15}$$

(ii) Show that the crease, C, is found by the expression **3**

$$C = \sqrt{\frac{2x^3}{2x-15}}$$

(iii) Hence, find the minimum length of C. **3**

End of paper

Section I

1 The mass of 1 atom of oxygen is $2{\cdot}7 \times 10^{-23}$ grams. What is the mass of 8×10^{27} atoms of oxygen?

(C) $2{\cdot}16 \times 10^{5}$

2 What is the gradient of a line normal to $2x - 4y + 3 = 0$

(A) -2

3 An arithmetic series is given as $\sum_{1}^{k} 4n - 1$. What is the third term of this series?

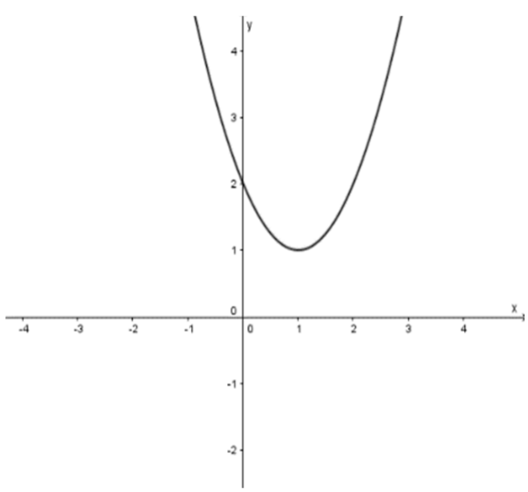
(C) 11

4 What is the exact value of $\sec\left(\frac{5\pi}{6}\right)$?

(D) $\frac{-2\sqrt{3}}{3}$

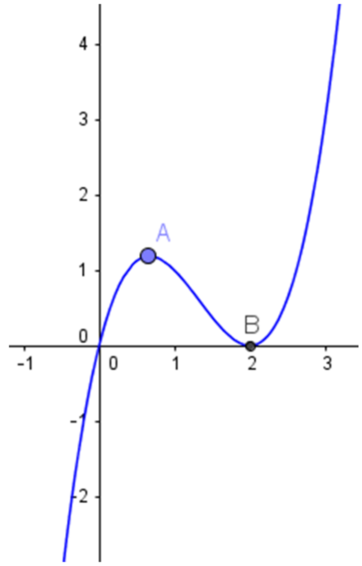
5 Which graph best depicts $f(x) = (x-1)^2 + 1$?

(B)

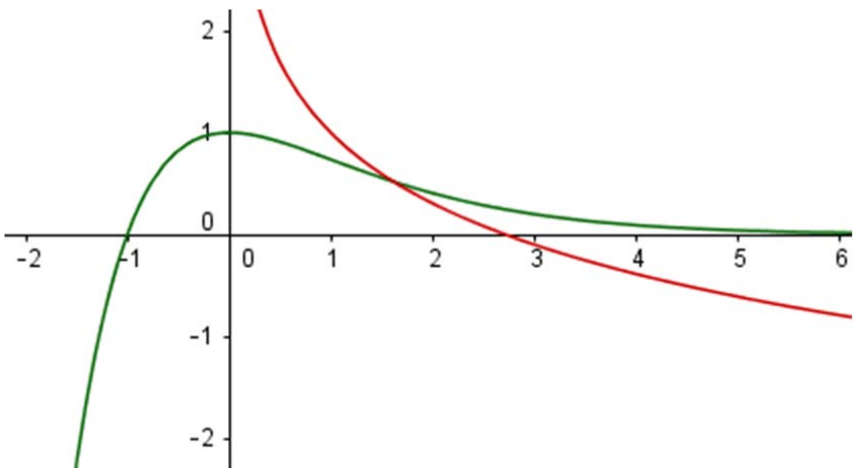


6 For what values of x is the curve $f(x)=x(x-2)^2$ increasing?

(D) $x<\frac{2}{3}$ *or* $x>2$



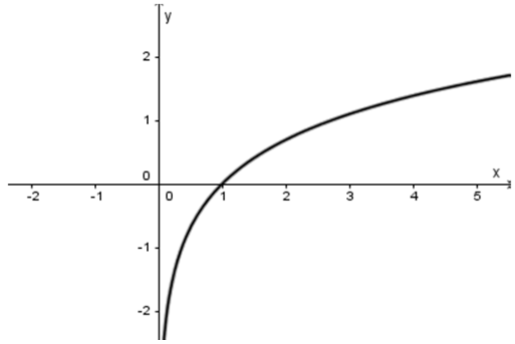
7 The diagram shows the graph of $y=e^{-x}(1+x)$.



How many solutions are there to the equation $e^{-x}(1+x)=1-\ln x$?

(B) 1

8 The diagram shows the graph of $y = \ln x$.



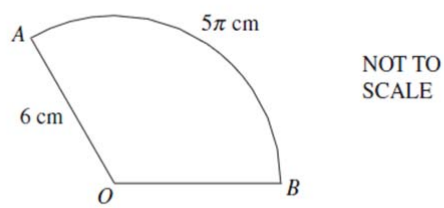
If the area between the axes, the line $y = 1$ and the curve $y = \ln x$ is rotated about the y-axis, which expression would give the volume of the solid formed?

(D) $\pi\int_0^1 e^{2y} \ dy$

9 Which expression is the factorisation of $27x^3 - y^3$

(A) $(3x - y)(9x^2 + 3xy + y^2)$

10 The diagram shows sector *AOB*.



What is the exact area of the minor segment cut off by a chord *AB*?

(B) $15\pi - 9$

Section II

Question 11 (15 marks) Start a *new* booklet.

(a) Find integers a and b such that $\frac{2}{2+\sqrt{5}} = a + b\sqrt{5}$

Solution

$$= \frac{2}{2+\sqrt{5}} \times \frac{2-\sqrt{5}}{2-\sqrt{5}}$$

$$= \frac{4-2\sqrt{5}}{-1}$$

$$= -4 + 2\sqrt{5}$$

$a = -4,\ b = 2$

Marker's guidelines

2 Correct response
1 Correct procedure with an error

Marker's Comments

Some students did not know to rationalise the denominator.

Some students did not explicitly answer the question and left their answer as $-4+2\sqrt{5}$

(b) Factorise $2x^2 - 17x + 30$

Solution

$2x^2 - 17x + 30 = (x-6)(2x-5)$

Marker's guidelines

2 Correct response
1 Correct procedure with an error

Marker's Comments

Some students need to review factorisation

(c) Differentiate $\frac{(x-2)^3}{2x^2}$

Solution

$$\frac{d\left[\frac{(x-2)^3}{2x^2}\right]}{dx} = \frac{2x^2 \times 3(x-2)^2.1 - (x-2)^3.4x}{\left(2x^2\right)^2}$$

$$= \frac{(x-2)^2\left[6x^2 - 4x(x-2)\right]}{4x^4}$$

$$= \frac{(x-2)^2\left(2x^2+8x\right)}{4x^4}$$

$$= \frac{(x+4)(x-2)^2}{2x^3}$$

Marker's guidelines

2 Correct response
1 Correct procedure with an error

Marker's Comments

When using the quotient rule derivatives should be presented in simplest factored form.

(d) Evaluate $\int_0^{\frac{\pi}{4}} \sec^2 3x \; dx$

Solution

$$\int_0^{\frac{\pi}{4}} \sec^2 3x \; dx = \left[\frac{1}{3}\tan(3x)\right]_0^{\frac{\pi}{4}}$$
$$= \left(\frac{1}{3}\tan\left(\frac{3\pi}{4}\right)\right) - \left(\frac{1}{3}\tan(0)\right)$$
$$= -\frac{1}{3}$$

Marker's guidelines

2 Correct response

1 Correct procedure with an error

Marker's Comments

Mostly well done.

Some students did not evaluate $\tan\frac{3\pi}{4}$

Note: from the reference sheet $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$

(e) Find $\int \frac{x^2}{x^3-1}\,dx$

Solution

$$\int \frac{x^2}{x^3-1}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3-1}\,dx$$
$$= \frac{1}{3}\ln\left|x^3-1\right| + C$$

Marker's guidelines

2 Correct response

1 Correct procedure with an error e.g. omitting the constant term

Marker's Comments

Some students $\times 3$ instead of dividing by 3

Some student omitted the + *C*

Note: from the reference sheet $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

(f) (i) Differentiate $x\ln x$

Solution

$$\frac{d\ \ x\ln x}{dx} = x \times \frac{1}{x} + \ln x \times 1$$
$$= 1 + \ln x$$

Marker's guidelines

2 Correct response

1 Correct procedure with an error

Marker's Comments

Generally well done.

(ii) Hence, evaluate $\int_1^e \ln x \ dx$

Solution

From part (i) $\int 1+\ln x \ dx = x\ln x + C \quad \Rightarrow$

$$\int_1^e 1+\ln x \ \ dx = \left[x\ln x\right]_1^e$$

$$\int_1^e 1 \ \ dx \ + \int_1^e \ln x \ \ dx = \left[x\ln x\right]_1^e$$

$$\int_1^e \ln x \ \ dx = \left(e\ln e\right) - \left(1\ln 1\right) - \int_1^e 1 \ \ dx$$

$$= \left(e\ln e\right) - \left(1\ln 1\right) - \left[x\right]_1^e$$

$$= e - \left[e-1\right]$$

$$= 1$$

Marker's guidelines

3 Correct response
2 Correct procedure with an error
1 Partial solution

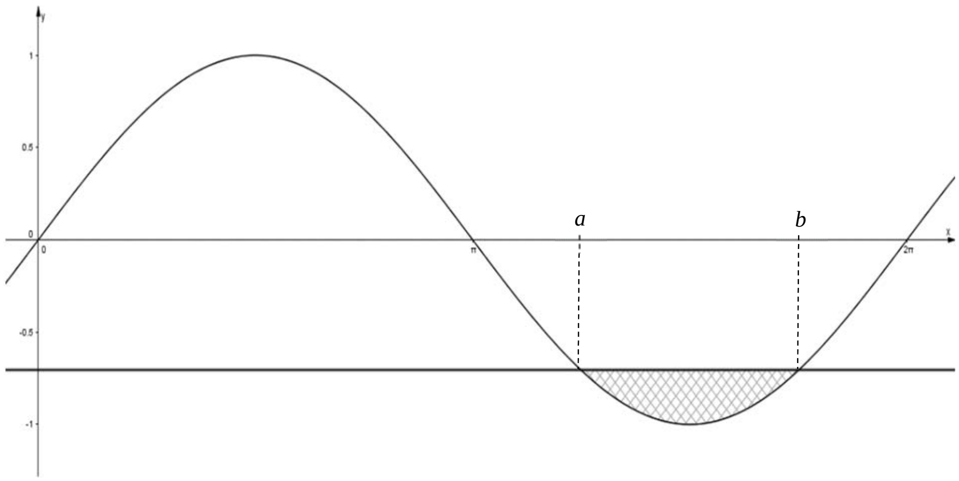
Marker's Comments

Many students could not see the relationship from part (i).

Students are encouraged to set up the integral from part (i) and rearrange to make $\int_1^e \ln x \ dx$ the subject.

Question 12 (15 marks) Start a *new* booklet.

(a) The shaded area shows the region between $y = -\frac{1}{\sqrt{2}}$ and $y = \sin x$ where $a \le x \le b$



(i) Find the values of *a* and *b* by solving $\sin x = -\frac{1}{\sqrt{2}}$ for $0 < x < 2\pi$

Solution

$\sin x = -\frac{1}{\sqrt{2}}$

$$x = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}$$
$$= \frac{5\pi}{4}, \frac{7\pi}{4}$$

$a = \frac{5\pi}{4}$, $b = \frac{7\pi}{4}$

Marker's guidelines

2 Correct response
1 Correct procedure with an error

Marker's Comments

Some students did not answer the question and needed to explicitly state the values for *a* and *b*.

(ii) Hence, show the area of the shaded region is $\dfrac{\sqrt{2}(4-\pi)}{4}$ units^2.

Solution

$$A = \int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}} -\frac{1}{\sqrt{2}} - \sin x \quad dx$$

$$= \left[-\frac{1}{\sqrt{2}} x + \cos x \right]_{\frac{5\pi}{4}}^{\frac{7\pi}{4}}$$

$$= \left(-\frac{1}{\sqrt{2}} \times \frac{7\pi}{4} + \cos \frac{7\pi}{4} \right) - \left(-\frac{1}{\sqrt{2}} \times \frac{5\pi}{4} + \cos \frac{5\pi}{4} \right)$$

$$= \left(-\frac{7\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - \left(-\frac{5\pi}{4\sqrt{2}} - \frac{1}{\sqrt{2}} \right)$$

$$= \frac{-2\pi}{4\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$= \frac{-2\sqrt{2}\pi + 8\sqrt{2}}{8}$$

$$= \frac{4\sqrt{2} - \sqrt{2}\pi}{4}$$

$$= \frac{\sqrt{2}(4-\pi)}{4}$$

Marker's guidelines

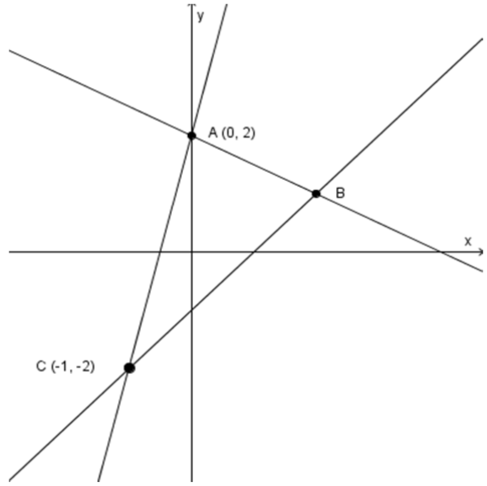
4	Correct response
3	Correct procedure with an error
2	Correct to line 3
1	For finding $-\frac{1}{\sqrt{2}} x + \cos x$

Marker's Comments

Some students could not find the correct equation to integrate.

Many careless errors involving signs and absolute values.

(b) The diagram shows three lines forming a triangle *ABC*. The points *A* and *C* have the coordinates $(0,2)$ and $(-1,-2)$ respectively.



The line passing through points *B* and *C* make an angle of 45° with the horizontal.

(i) Show the equation of the line passing through *BC* is given by $x-y-1=0$.

Solution

$$m_{BC}=\tan 45^\circ$$
$$=1$$

$$y-y_1=m(x-x_1)$$
$$y+2=1(x+1)$$
$$x-y-1=0$$

Marker's guidelines

2 Correct response
1 For finding the gradient

Marker's Comments

Students need to show how they ascertained the gradient was equal to 1.

(ii) The equation of the line passing through *AB* is given by $x+2y-4=0$.

Find the point of intersection *B*.

Solution

$$x+2y-4=0$$
$$x-y-1=0$$

$$3y-3=0$$
$$y=1$$

$$x-(1)-1=0$$
$$x=2$$

$$B(2,1)$$

Marker's guidelines

2 Correct response
1 For correct procedure with an error

Marker's Comments

Generally well done.

(iii) Find the exact distance *BC*

Solution

$$BC^2 = (2+1)^2 + (1+2)^2$$
$$BC^2 = 18$$

$$BC = 3\sqrt{2}$$

Marker's guidelines

2 Correct response
1 For correctly substituting into formula

Marker's Comments

Generally well done.

(iv) Hence, find the exact area of triangle *ABC*.

Solution

$$A = \frac{1}{2}bh$$

$b = 3\sqrt{2}$ *from part (iii)*
$h =$ perpendicular height of point A to the line BC

$$h = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|0(0) - 1(2) - 1|}{\sqrt{(1)^2 + (-1)^2}}$$

$$= \frac{|-3|}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{2}$$

$$A = \frac{1}{2} \times 3\sqrt{2} \times \frac{3\sqrt{2}}{2}$$

$$= \frac{9}{2} \text{ or } 4\frac{1}{2} \text{ units}^2$$

Marker's guidelines

3 Correct response
2 For correctly finding the perpendicular height
1 For partial solution

Marker's Comments

Not well answered.

Some students incorrectly assumed it was a right triangle and therefore found an incorrect answer.

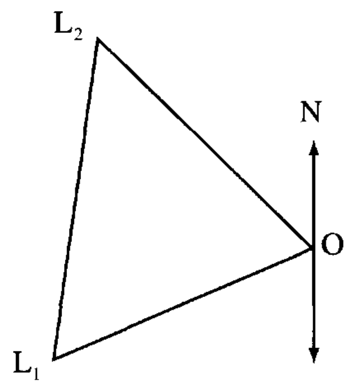
*Some students ignored the *hence* instruction and did not use *BC* as the base or used trigonometric results (often incorrectly).

Many students applied procedures without understanding.

Many students overcomplicated the method used to solve this question.

Question 13 (15 marks) Start a *new* booklet.

(a) From an observation tower O on a ship, a sailor determines the lighthouse L_1 is on a bearing of 255° T and a different lighthouse L_2 is at a bearing of 315°T.



L_1 and L_2 are a distance of 70 and 75 kilometres respectively from O.

(i) Show that angle L_1OL_2 is 60°

Solution

$$\angle L_1OL_2 = \text{Bearing of } L_2 - \text{ Bearing of } L_1$$
$$= 315^\circ - 255^\circ$$
$$= 60^\circ$$

Marker's guidelines

1 Correct response

Marker's Comments

Well done

(ii) Find, correct to the nearest *km*, the distance between the two lighthouses L_1 and L_2.

Solution

$$(L_1L_2)^2 = 70^2 + 75^2 - 2\times70\times75\times\cos 60^\circ$$
$$= 5275$$
$$L_1L_2 = 72\cdot62$$
$$\approx 73\ km \text{ to the nearest } km$$

Marker's guidelines

2 Correct response

1 For correctly substituting into formula

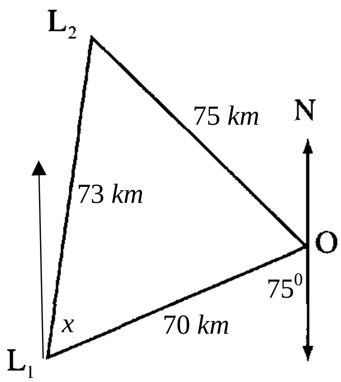
Marker's Comments

Some students incorrectly wrote sine in the cosine rule

Many calculator errors

(iii) Find the bearing of L_2 from L_1 to the nearest degree.

Solution



Marker's guidelines

3 Correct response
2 For correct procedure with an error
1 For partial solution

Marker's Comments

Some students made the question more difficult than it needed to be

Many students did not give reasons

Students referenced added points but did not draw a diagram making it difficult to follow there reasoning.

Let the angle at $L_1 = x$

Finding x :

$$\frac{\sin x}{75} = \frac{\sin 60}{73}$$

$$\sin x = \frac{75 \times \sin 60}{73}$$

$$x = 62^\circ 50'31''$$

Now $\angle L_1 O(South) = 255^\circ - 180^\circ$
$= 75^\circ$

And $\angle OL_1(North) = 75^\circ$ Alternate angles

Bearing of L_2 from L_1 $= 75^\circ - 62^\circ 50'31''$
$= 12^\circ 9'$
$= 012^\circ T$ to the nearest degree

(b) Consider the curve $y = x^5 - 5x^4 + 5x^3$.

(i) Find the stationary points and determine their nature.

Solution

Stationary points occur when $y' = 0$

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2$$

$$5x^4 - 20x^3 + 15x^2 = 0$$
$$5x^2\left(x^2 - 4x + 3\right) = 0$$
$$5x^2\left(x-3\right)\left(x-1\right) = 0$$

$$x = 0, 1, 3$$

Finding the y ordinate:

When $x = 0,\ y = 0$
$x = 1,\ y = 1$
$x = 3,\ y = -27$

Determining the nature

$$\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$$

When $x = 0,\ y'' = 0$ possible inflexion
$x = 1,\ y'' = -10$ maximum turning point
$x = 3,\ y'' = 90$ minimum turning point

Testing for an inflexion at $x = 0$

x	-1	0	0.5
$\frac{d^2y}{dx^2}$	-110	0	$\frac{5}{2}$

Therefore at

$(0,0)$ there is a horizontal point of inflexion

$(1,1)$ there is a maximum turning point

$(3,-27)$ there is a minimum turning point

Marker's scheme

1 mark awarded for finding each of the following

- For finding $x = 0, 1, 3$
- For finding the 2nd derivative
- For showing the horizontal inflexion (must test for)
- For showing the nature of the maximum & minimum points

Marker's Comments

Some students divided by x^2 and so did not find $x = 0$. Subsequently they were unable to show $(0,0)$ as an inflexion

Many students did not test for POI. $y'' = 0$ only indicates a possible inflexion (i.e. not all points where $y'' = 0$ implies a POI).

(ii) Show that there are inflexions at $x = 0$, $x = \frac{3-\sqrt{3}}{2}$ and $x = \frac{3+\sqrt{3}}{2}$

(You do not need to find their corresponding *y* value)

Solution

Possible inflexions occur at $y'' = 0$

From part (i) $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x$

$$20x^3 - 60x^2 + 30x = 0$$
$$10x\left(2x^3 - 6x + 3\right) = 0$$

$$2x^3 - 6x + 3 = 0 \quad or \quad x = 0$$

$$x = \frac{-(-6) \pm \sqrt{36 - 4\times 2\times 3}}{2\times 2}$$
$$= \frac{6 \pm \sqrt{12}}{4}$$
$$= \frac{3 \pm \sqrt{3}}{2}$$

Marker's Scheme

1 mark awarded for finding each of the following

- Equating the correct second derivative to zero
- For showing the x values to be true
- For showing all *x* values to be inflexions by testing concavity.

Marker's Comments

As for pt (ii) Many students did not test for POI.

$y'' = 0$ only indicates a possible inflexion (i.e. not all points where $y'' = 0$ implies a POI).

Testing for concavity change

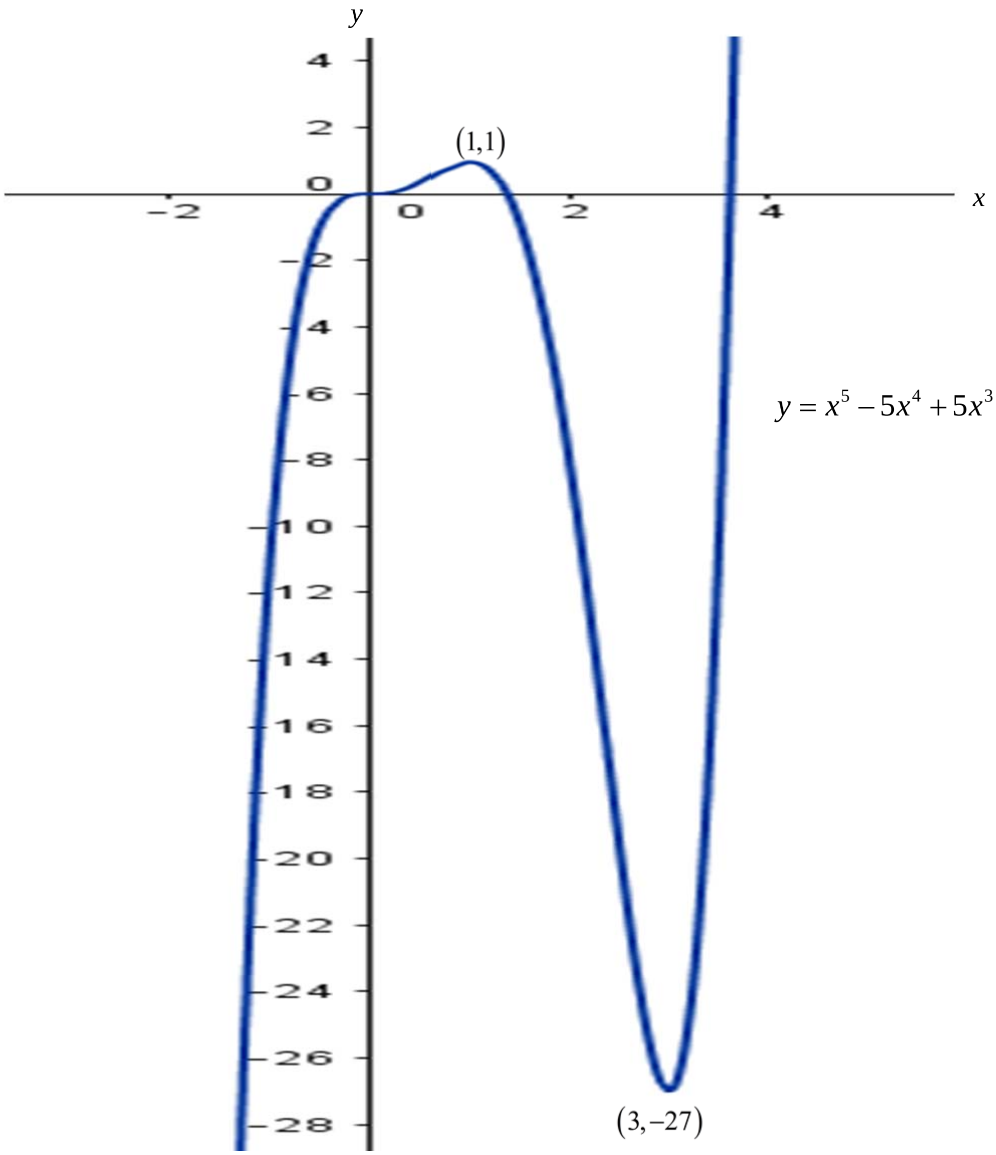
x	-1	0	$\frac{1}{2}$	$\frac{3-\sqrt{3}}{2}$	1	$\frac{3+\sqrt{3}}{2}$	3
$\frac{d^2y}{dx^2}$	-110	0	$\frac{5}{2}$	0	-10	0	90

Therefore there are inflexions at

$x = 0$, $x = \frac{3-\sqrt{3}}{2}$ and $x = \frac{3+\sqrt{3}}{2}$

(iii) Sketch the curve labelling the *stationary points only*.

Solution



Marker's Scheme

1 mark awarded for finding each of the following

- For general shape of curve
- Explicitly sketching the horizontal point of inflexion.

Marker's Comments

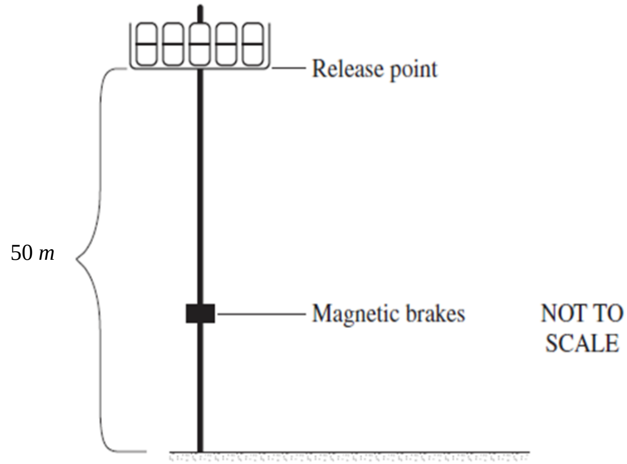
Students had trouble with sketching the POI.

Students did not use the information from pt(iii) to illustrate a change in concavity.

Students did not show understanding of the difference between a horizontal POI & vertical POI.

Question 14 (15 marks) Start a *new* booklet.

(a) At Luna Park, a chair is released from a height of 50 metres and falls vertically. Magnetic brakes are applied to stop the fall.



The height of the chair at time t seconds is x metres. The acceleration of the chair is given by $\ddot{x} = -9\cdot 8$.
At the release point, $t = 0,\ x = 50$ and $\dot{x} = 0$.

(i) Show that the chair's displacement x at time t is given by $x = -4\cdot 9t^2 + 50$

Solution

$\ddot{x} = -9\cdot 8$
$\dot{x} = -9\cdot 8t + C$

When $t = 0,\ \dot{x} = 0 \Rightarrow C = 0$

$\dot{x} = -9\cdot 8t$
$x = -4\cdot 9t^2 + D$

When $t = 0,\ x = 50 \quad \Rightarrow D = 50$

$x = -4\cdot 9t^2 + 50$

Marker's guidelines

3	Correct response
2	For correct procedure with an error
1	For partial solution

Marker's Comments

Mostly well done

(ii) If it takes half a second for the brakes to stop the fall, find the latest time the brakes could be applied? (Correct your answer to two decimal places)

Solution

Chair would hit the ground when $x = 0$

$$-4 \cdot 9t^2 + 50 = 0$$
$$4 \cdot 9t^2 = 50$$
$$t^2 = \frac{50}{4 \cdot 9}$$
$$t \approx 3 \cdot 19438$$

Therefore the brakes must be applied $\frac{1}{2}$ second earlier

So latest time $= 3 \cdot 19438... - 0 \cdot 5$

$= 2 \cdot 69438$

$\approx 2 \cdot 69$ seconds correct to 2 d.p.

Marker's guidelines

2 Correct response
1 For partial solution

Marker's Comments

Many students found the time the chair hits the ground without deducting half a second for the brakes to be applied.

(iii) How far has the chair fallen and what is its speed when the brakes are applied? (correct your answer to two decimal places) **2**

Solution

When $t = 2 \cdot 69$ $\quad x = -4 \cdot 9(2 \cdot 69)^2 + 50$

$= 14 \cdot 54311...$ above the ground

Therefore the chair has fallen $= 50 - 14 \cdot 54311...$

$\approx 35 \cdot 46 \; m$ correct to 2 d.p.

Speed $= |v|$

$= |-9 \cdot 8(2 \cdot 69)|$

$\approx 26 \cdot 36 \; ms^{-1}$ correct to 2 d.p.

Marker's guidelines

2 Correct response
1 For partial solution

Marker's Comments

Many students did not answer the question 'how far has the chair fallen'.

(b) For what values of k is the quadratic $(k+1)x^2+kx+k+1$ a positive definite?

Solution

For a positive definite

$a>0$ and $\Delta<0$

$k+1>0$

$k>-1$

$k^2-4(k+1)(k+1)<0$

$-3k^2-8k-4<0$

$3k^2+8k+4>0$

$(k+2)(3k+2)>0$

$k<-2 \text{ or } k>-\frac{2}{3}$

Therefore $k>-\frac{2}{3}$

Marker's guidelines

3	Correct response
2	For correct procedure with an error
1	For partial solution

Marker's Comments

Many students could not recall the definition of the positive definite.

(c) Jasmine sets aside savings each week for a vacation. In the first week she saves \$36 and each subsequent week she is able to save \$4 more than the previous week.

The maximum Jasmine is able to save is \$180 per week.

(i) How many weeks would it take for Jasmine to be able to save \$180 per week?

Solution

The series is given by $36+40+44+...$

A.P. where $a=36,\ d=4$

$T_n=a+(n-1)d$

$180=36+(n-1)4$

$144=4n-4$

$4n=148$

$n=37$ *weeks*

Marker's guidelines

2	Correct response
1	For partial solution

Marker's Comments

Mostly well done

(ii) Jasmine needs to save \$7356 for her vacation.
How many weeks would it take for Jasmine to save the amount of her vacation? **3**

Solution

Since \$180 is the maximum Jasmine is able to save the series is

$36+40+44+...+180+180+180+180+...$

We want the sum of this series to be \$7356 and finding how many

extra \$180 savings this would take.

$$36+40+44+...+180 \quad + \quad 180n=7356$$

$$\frac{37}{2}\left[36+180\right] \quad +180n=7356$$

$$3996+180n=7356$$

$$180n=3360$$

$$n=19$$

Therefore the total time $=37+19$
$=56 \quad \textit{weeks}$

Marker's guidelines

3	Correct response
2	For correct procedure with an error
1	For partial solution

Marker's Comments

Some students assumed \$7356 was the sum of the arithmetic series with a common difference of 4, and were ignoring the maximum savings series which made up part of the sum.

Alternative solution

$$36+40+44+...+180 \quad + \quad 180n=7356$$

$$\frac{37}{2}\left[2(36)+(37-1)4\right] \quad +180n=7356$$

$$3996+180n=7356$$

$$180n=3360$$

$$n=19$$

Therefore the total time $=37+19$
$=56 \quad \textit{weeks}$

Question 15 (15 marks) Start a *new* booklet.

(a) The population, P, of a Dodo colony was decreasing to the differential equation

$$\frac{dP}{dt} = -kP$$

where t is the time measured in years from when recordings were first kept in January, 1684.

(i) Show that $P = P_0 e^{-kt}$ is a solution to the differential equation, where P_0 is the initial population of the colony on January 1st, 1684.

Solution

$$P = P_0 e^{-kt}$$

$$\frac{dP}{dt} = -kP_0 e^{-kt}$$
$$= -kP \quad as \quad P = P_0 e^{-kt}$$

Marker's guidelines

1 Correct response

Marker's Comments

(ii) Find, correct to 4 significant figures, the value of k if records show only half the population of Dodos remained after 30 years.

Solution

$$P = P_0 e^{-kt}$$
$$\frac{1}{2}P_0 = P_0 e^{-k(30)}$$
$$\frac{1}{2} = e^{-30k}$$
$$\ln e^{-30k} = \ln\frac{1}{2}$$
$$-30k \ln e = \ln\frac{1}{2}$$
$$k = \frac{\ln\frac{1}{2}}{-30}$$
$$\approx 0\cdot 02310$$

Marker's guidelines

2 Correct response

1 For partial solution

Marker's Comments

(iii) During what year did the Dodo population become less than 1% of the initial population P_0? **3**

Solution

$$\begin{aligned} P &= P_0 e^{-kt} \\ 0\cdot 01P_0 &= P_0 e^{-0\cdot 02310t} \\ 0\cdot 01 &= e^{-0\cdot 02310t} \\ \ln e^{-0\cdot 02310t} &= \ln 0\cdot 01 \\ -0\cdot 02310t \ln e &= \ln 0\cdot 01 \\ t &= \frac{\ln 0\cdot 01}{-0\cdot 02310} \\ &\approx 199\cdot 31 \end{aligned}$$

Finding the year:

$$\begin{aligned} \text{Year } &= 1684 + 199\cdot 31 \\ &= 1883\cdot 31 \end{aligned}$$

So during 1883 the Dodo population becomes less than 1% of its 1684 numbers.

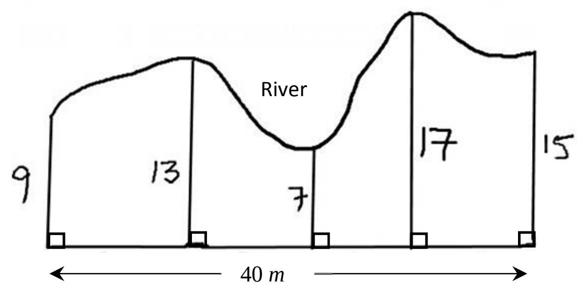
Marker's guidelines

3 Correct response
2 For finding 199 years
1 For partial solution

Marker's Comments

Many students could not correctly state the actual year in which the Dodo numbers dropped below 1%

(b) Neil has just purchased land. The street side is 40 metres wide and the property runs to a river as shown in the diagram. Neil has surveyed how far it is to the river from regular intervals.



Use Simpson's rule with 5 function values to approximate the area of Neil's land. **3**

Solution

x	0	10	20	30	40
$f(x)$	9	13	7	17	15

$$A \approx \frac{h}{3}\left[y_1 + 4(y_2 + y_4) + 2(y_3) + y_5\right]$$
$$\approx \frac{10}{3}\left[9 + 4(13+17) + 2(7) + 15\right]$$
$$\approx 526 \cdot 67 \quad m^2$$

Marker's guidelines

3 Correct response
2 For correct substitution into formula
1 For partial solution

Marker's Comments

Some students could not apply a formula.

(c) Ava and Owen are playing a game by taking turns throwing a 20 sided die. Ava wins if she throws a number divisible by 5 or the number 13 and Owen wins if he throws a number divisible by 3.

Ava throws first.

(i) Find the probability that Ava wins the game on her first throw.

Solution

$$P(\textit{Ava wins on 1st throw}) = \frac{4+1}{20}$$
$$= \frac{1}{4}$$

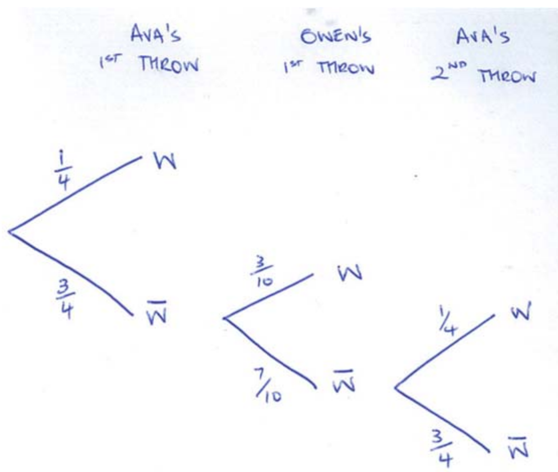
Marker's guidelines

1 Correct response

Marker's Comments

(ii) Find the probability that Ava wins the game on her first or second throw.

Solution



$$P(\text{Ava wins on 1}st \text{ or 2}nd \text{ throw}) = \frac{1}{4} + \frac{3}{4} \times \frac{7}{10} \times \frac{1}{4}$$
$$= \frac{1}{4} + \frac{1}{4} \times \frac{21}{40}$$
$$= \frac{61}{160} \quad or \quad 38 \cdot 125\%$$

Marker's guidelines

2 Correct response
1 For partial solution

Marker's Comments

(iii) With how many of Ava's throws will she have had a better than a 50% chance of winning the game? **3**

Solution

$$P(\text{Ava wins}) = \frac{1}{4} + \frac{1}{4}\left(\frac{21}{40}\right) + \frac{1}{4}\left(\frac{21}{40}\right)^2 + \frac{1}{4}\left(\frac{21}{40}\right)^3 + \ldots + \frac{1}{4}\left(\frac{21}{40}\right)^n > 50\%$$

$$\frac{\frac{1}{4}\left(\left(\frac{21}{40}\right)^n - 1\right)}{\frac{21}{40} - 1} > 0 \cdot 50$$

$$-\frac{10}{19}\left(\left(\frac{21}{40}\right)^n - 1\right) > \frac{1}{2}$$

$$\left(\frac{21}{40}\right)^n - 1 < -\frac{19}{20}$$

$$\left(\frac{21}{40}\right)^n < \frac{1}{20}$$

$$n\ln\left(\frac{21}{40}\right) < \ln\left(\frac{1}{20}\right)$$

$$n > \frac{\ln\left(\frac{1}{20}\right)}{\ln\left(\frac{21}{40}\right)}$$

$$> 4 \cdot 649\ldots$$

Marker's guidelines

3	Correct response
2	For correct procedure with an error
1	For partial solution

Marker's Comments

Some students could not recognise the geometric series in subsequent throws.

So after **5** throws Ava would have had a better than 50% chance of winning the game.

Question 16 (15 marks) Start a *new* booklet.

(a) Samantha has retired with savings of $\$400000$. She intends to live off this money by taking equal monthly instalments of $\$M$ at the *start* of each month, starting immediately.

Samantha receives 6% p.a. interest on her account balance at the end of each month.

Let $\$A_n$ be her account balance at the end of each month.

(i) Show that $A_2 = 400000(1\cdot005)^2 - M(1\cdot005)^2 - M(1\cdot005)$

Solution

Note: the instalment is taken at the start of each month

$$A_1 = (400000 - M)1\cdot005$$
$$= 400000(1\cdot005) - M(1\cdot005)$$

$$A_2 = \left[400000(1\cdot005) - M(1\cdot005) - M\right]1\cdot005$$
$$= 400000(1\cdot005)^2 - M(1\cdot005)^2 - M(1\cdot005)$$

Marker's guidelines

1 Correct response

Marker's Comments

Generally well done.

A few students did not *show* sufficient working.

(ii) Show that $A_n = 400000(1\cdot005)^n - M\left[201\left(1\cdot005^n - 1\right)\right]$

Solution

$$A_2 = 400000(1\cdot005)^2 - M(1\cdot005)^2 - M(1\cdot005)$$
$$A_3 = 400000(1\cdot005)^3 - M(1\cdot005)^3 - M(1\cdot005)^2 - M(1\cdot005)$$

.

.

.

$$A_n = 400000(1\cdot005)^n - M(1\cdot005)^n - M(1\cdot005)^{n-1} - \ \dots\ - M(1\cdot005)$$
$$= 400000(1\cdot005)^n - M\left[1\cdot005^n + 1\cdot005^{n-1} + \dots + 1\cdot005\right]$$
$$= 400000(1\cdot005)^n - M\left[\frac{1\cdot005\left(1\cdot005^n - 1\right)}{1\cdot005 - 1}\right]$$
$$= 400000(1\cdot005)^n - M\left[201\left(1\cdot005^n - 1\right)\right]$$

Marker's guidelines

3 Correct response
2 For correct procedure with an error
1 For finding a simple series for A_n

Marker's Comments

Some students did not *show* the pattern from $A_3 \to A_n$.

Some students did not show the working to summate a geometric series.

(iii) Samantha would like her money to last 25 years. What monthly instalment does Samantha take? **2**

Solution

After 25 years Samantha would have \$0 remaining $\Rightarrow A_{300} = 0$

$$400000(1\cdot005)^{300} - M\left[201\left(1\cdot005^{300} - 1\right)\right] = 0$$

$$M\left[201\left(1\cdot005^{300} - 1\right)\right] = 400000(1\cdot005)^{300}$$

$$M = \frac{400000(1\cdot005)^{300}}{201\left(1\cdot005^{300} - 1\right)}$$

$$= \$2564\cdot38$$

Marker's guidelines

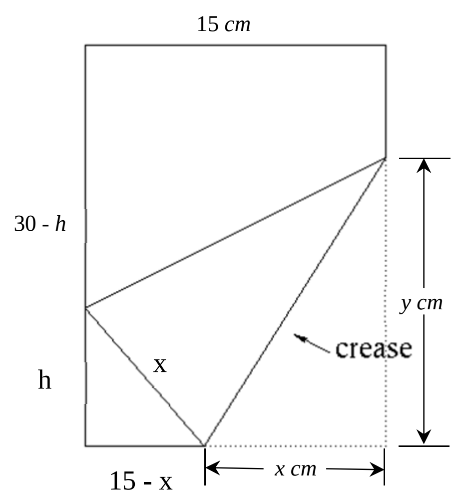
2 Correct working out
1 For partial solution

Marker's Comments

Many students forgot to change the years to months.

(b) A rectangular piece of paper is 30 centimetres high and 15 centimetres wide. The lower right-hand corner is folded over so as to reach the leftmost edge of the paper.

Let x be the horizontal distance folded and y be the vertical distance folded as shown in the diagram.



(i) By considering the areas of three triangles and trapezium that make up the total area of the paper, show **3**

$$y = \frac{x\sqrt{15(2x-15)}}{2x-15}$$

Solution

Labelling the diagram for reference.

$A_1 = A_2 = \frac{1}{2}xy$ (area of triangle)

$A_3 = \frac{1}{2}(15 - x)h$

$A_4 = \frac{15}{2}(30 - y + 30 - h)$
$= \frac{15}{2}(60 - y - h)$

$A_T = A_1 + A_2 + A_3 + A_4$

$15 \times 30 = 2 \times \left(\frac{1}{2}xy\right) + \frac{1}{2}(15 - x)h + \frac{15}{2}(60 - y - h)$

$450 = xy + \frac{15h}{2} - \frac{hx}{2} + 450 - \frac{15y}{2} - \frac{15h}{2}$

$\frac{15y}{2} - xy = -\frac{hx}{2}$

$y\left(\frac{15}{2} - x\right) = -\frac{hx}{2}$

$y\left(\frac{15 - 2x}{2}\right) = -\frac{hx}{2}$

$y = \frac{-hx}{15 - 2x}$

$= \frac{-hx}{-(-15 + 2x)}$

$= \frac{hx}{2x - 15}$

Now from the lower left triangle $h^2 + (15 - x)^2 = x^2$

$h^2 = x^2 - (15 - x)^2$
$= x^2 - (225 - 30x + x^2)$
$= 30x - 225$
$h = \sqrt{30x - 225}$
$= \sqrt{15(2x - 15)}$

Therefore $y = \frac{x\sqrt{15(2x - 15)}}{2x - 15}$

Marker's guidelines

3	Correct response
2	For correct procedure with an error
1	For finding a simple series for A_T

Marker's Comments

Many students redrew the diagram using different labelling than given.

Some students did not recognise the trapezium A_4 and made it more complicated by splitting into a triangle and rectangle.

Many careless errors in the algebra.

(ii) Show that the crease, C, is found by the expression

$$C = \sqrt{\frac{2x^3}{2x-15}}$$

Solution

$$C^2 = x^2 + y^2$$

$$= x^2 + \left[\frac{x\sqrt{15(2x-15)}}{2x-15}\right]^2 \quad \text{from part (i)}$$

$$= x^2 + \frac{x^2 \times 15(2x-15)}{(2x-15)^2}$$

$$= x^2 + \frac{15x^2}{(2x-15)}$$

$$= \frac{x^2(2x-15)+15x^2}{2x-15}$$

$$= \frac{2x^3}{2x-15}$$

$$C = \sqrt{\frac{2x^3}{2x-15}}$$

Marker's guidelines

3 Correct working out
2 For correct procedure with an error
1 For correct procedure with two errors.

Marker's Comments

Many careless errors.

(iii) Hence, find the minimum length of C. **3**

Solution

Finding stationary points

$$C = \left(\frac{2x^3}{2x-15}\right)^{\frac{1}{2}}$$

$$\frac{dC}{dx} = \frac{1}{2}\left(\frac{2x^3}{2x-15}\right)^{-\frac{1}{2}} \times \frac{(2x-15)6x^2 - 2x^3(2)}{(2x-15)^2}$$

Stationary points occur when $\frac{dC}{dx} = 0$

$$\frac{1}{2} \times \frac{1}{\left(\frac{2x^3}{2x-15}\right)^{\frac{1}{2}}} \times \frac{12x^3 - 90x^2 - 4x^3}{(2x-15)^2} = 0$$

$$12x^3 - 90x^2 - 4x^3 = 0$$

$$2x^2(4x - 45) = 0$$

$$x = 0 \ \ or \ \ x = \frac{45}{4}$$

Note: exclude $x = 0$ as $x > \frac{15}{2}$

Determining maximum or minimum

$$\frac{dC}{dx} = \frac{1}{2} \times \frac{1}{\left(\frac{2x^3}{2x-15}\right)^{\frac{1}{2}}} \times \frac{8x^3 - 90x^2}{(2x-15)^2}$$

Marker's Scheme

1 mark each for

- Finding a values for *x*
- Testing for a maximum or minimum
- Finding a value for *C*

Marker's Comments

A few students did not test to confirm the stationary point is a minimum

Some students made errors when differentiating

To determine the sign of $\frac{dC}{dx}$ we need only consider the sign of $8x^3 - 90x^2$ as for all other terms $\frac{dC}{dx}$ would be positive for all of the domain $\frac{15}{2} < x < 15$.

x	8	$\frac{45}{4}$	12
$8x^3 - 90x^2$	-1664	0	864
$\frac{dC}{dx}$	Negative	0	Positive

Therefore when $x = \frac{45}{4}$ C is a minimum.

$$C = \sqrt{\frac{2x^3}{2x-15}}$$

$$= \sqrt{\frac{2\left(\frac{45}{4}\right)^3}{2\left(\frac{45}{4}\right) - 15}}$$

$$\approx 19 \cdot 5 \quad cm$$

Alternatively

You can attain a minimum for C by finding the minimum for C^2

$$C = \sqrt{\frac{2x^3}{2x-15}}$$

$$C^2 = \frac{2x^3}{2x-15}$$

$$\frac{dC^2}{dx} = \frac{(2x-15).6x^2 - 2x^3.2}{(2x-15)^2}$$

$$\frac{8x^3 - 90x^2}{(2x-15)^2} = 0$$

$$8x^3 - 90x^2 = 0$$

$$2x^2(4x-45) = 0$$

$$x = 0 \;\; or \;\; x = \frac{45}{4}$$

Showing a minimum we need only consider the sign of the numerator as the denominator would be positive for all of the domain $\frac{15}{2} < x < 15$.

positive for all of the domain $\frac{15}{2} < x < 15$.

x	8	$\frac{45}{4}$	12
$8x^3 - 90x^2$	-1664	0	864
$\frac{dC^2}{dx}$	Negative	0	Positive

Therefore when $x = \frac{45}{4}$ C^2 is a minimum $\Rightarrow C$ is also minimum.

$$C = \sqrt{\frac{2x^3}{2x-15}}$$

$$= \sqrt{\frac{2\left(\frac{45}{4}\right)^3}{2\left(\frac{45}{4}\right) - 15}}$$

$$= 19 \cdot 5 \quad cm$$