Chapter 9: Derivatives: Exercise

1. Evaluate: (a) \( \lim_{x \to 3} \frac{9 - x^2}{x + 3} \)
   \( \lim_{x \to \infty} \frac{x^2 + x + 1}{2x^2 - x + 3} \)

2. Using the definition
   \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
   Find \( f'(x) \) if
   (a) \( f(x) = x^3 - 3x \)  \( (b) f(x) = \frac{2}{x} \)

3. Find \( \frac{dy}{dx} \) or \( f'(x) \) for
   (a) \( y = 2x^2 + \frac{x}{3} - 2 \)
   (b) \( y = x - \frac{2}{x} \)
   (c) \( f(x) = \frac{x^2 - 2x + 1}{x} \)
   (d) \( y = \frac{x - 1}{\sqrt{x}} \)

4. Find the gradient \( \frac{dy}{dx} = f'(x) \) at the given point.
   (a) \( y = x^2 - 4x \) at \( x = 2 \)
   (b) \( f(x) = \frac{4}{\sqrt{x}} \) at \( x = 4 \)
   (c) \( y = x - \frac{2}{x} \) at \( x = \frac{1}{2} \)

5. Find the equation of the tangent and normal:
   (a) \( y = x^2 + 3x - 1 \) at \( x = -1 \)
   (b) \( y = 2\sqrt{x} \) at \( x = 4 \)

6. Find the derivative:
   (a) \( y = \frac{2x}{x-2} \)  \( (b) y = (x^2 - 1)^4 \)
   (c) \( y = \frac{-1}{\sqrt{1-x^2}} \)  \( (d) y = \frac{1}{x+1} + \frac{1}{2x} \)
   (e) \( y = x\sqrt{x^2 + 1} \)  \( (f) y = \frac{x}{\sqrt{x^2 + 4}} \)
   (g) \( y = x\sqrt{x} \)

7. A curve is defined by the equation
   \( y = f(x) = x^3 - 6x + 4 \)
   (a) Find \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \)
   (b) Find the coordinates of the turning points of the curve and identify them.
   (c) Find the point of inflection, if any.
   (d) Sketch the curve for \(-3 \leq x \leq 3\) and find the maximum and minimum values of \( f(x) \).

8. Let \( f(x) = 7 + 4x^3 - 3x^4 \).
   (a) find the coordinates of stationary points
   (b) Find \( \frac{d^2y}{dx^2} \) and hence the point(s) of inflection.
   (c) Determine the nature of stationary points.
   (d) Sketch \( f(x) \) \(-1 \leq x \leq 2\) and determine the maximum and minimum values in the domain.

9. ABCD is a rectangle with the lower base CD on the \( x \) – axis and vertices A and B on the parabola \( y = 12 - x^2 \)

Find the area of the largest (maximum) such rectangle.
10. A rectangular box is open at the top end. The dimensions of box are \( x \), \( 2x \) and \( y \) cm.

(a) Show that the surface area of the box is given by \( A = 2x^2 + 6xy \).

(b) Let \( A = 300 \text{cm}^2 \). Show that the volume of the box is given by \( V = 100x - \frac{2x^3}{3} \).

(c) Show that \( x < 5\sqrt{6} \) \((\text{Hint: } v > 0)\)

(d) Find the value of \( x \) for which \( V \) is the maximum. Find \( V_{\text{max}} \).

11. ADF is a triangle with \( \angle D = 90^\circ \)

BCED is a rectangle where

\( BC = 12\text{cm} \), \( CE = 8\text{cm} \), \( AB = x \), \( EF = y \)

(a) Show that \( \triangle ABC \) III \( \triangle ACEF \) and hence \( xy = 96 \)

(b) Show that area of the \( \triangle ADF \) is given by

\[ A = 96 + 6x + \frac{384}{x} \]

(c) Find the value of \( x \), so that \( A \) is the minimum.

12. A rectangular poster has a margin of 1cm at the top and bottom ends and 3cm at each side. The region inside the margins is to have a print area of 48\( \text{cm}^2 \).

Show that:

(a) \( y = 6 + \frac{48}{x-2} \)

(b) The area of poster is given by

\[ A = 6x + \frac{48x}{x-2} \]

(c) Find the values of \( x \) and \( y \) such that \( A \) is the minimum.

13. Farmer Smith wishes to fence off a triangular property \( \triangle ABC \). On one side there is a river bank where no fence is needed. The fence \( BC \) is common with the farmer Andrew. The area of Smith property is 3600\( \text{m}^2 \), where \( \angle BCA = 90^\circ \). Fencing costs \$60 per metre and Andrew has agreed to pay half the cost of fencing the side \( BC \).

(a) Show that cost of fencing is given by

\[ C = 60x + \frac{21600}{x} \] \$\text{dollars.}

(b) Find the value of \( x \), so that \( C \) is the minimum.
A cylinder of radius $r$ and height $h$ is inscribed in a cone of base radius 6 cm and height 20 cm.

(a) Using similar triangles, show that
\[ h = \frac{10}{3} (6 - r) \]

(b) Show that volume of the cylinder is given by
\[ V = \frac{10\pi r^2 (6 - r)}{3} \text{ cm}^3 \]

(c) Find $r$ and $h$ for which $V$ is the maximum and find $V_{\text{max}}$. 

1. (a) \[ \lim_{x \to -3} \frac{9-x^2}{x+3} = \lim_{x \to -3} \frac{(3-x)(3+x)}{3+x} \]
\[ = \lim_{x \to -3} (3-x) \]
\[ = 6 \]

(b) \[ \lim_{x \to \infty} \frac{x^2 + x + 1}{2x^2 - x + 3} \text{ divide by } x^2 \]
\[ = \lim_{x \to \infty} \left( \frac{1 + \frac{1}{x} + \frac{1}{x^2}}{2 - \frac{1}{x} + \frac{3}{x^2}} \right) \]
\[ = \frac{1}{2} \text{ as } \lim_{x \to \infty} \frac{1}{x} = 0 \]

2. (a) \[ f(x) = x^2 - 3x \]
\[ f(x + h) = (x + h)^2 - 3(x + h) \]
\[ = x^2 + 2xh + h^2 - 3x - 3h \]
\[ \therefore f(x + h) - f(x) = 2xh + h^2 - 3h \]
\[ \therefore f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \frac{2x + h - 3}{h} \]
\[ = \lim_{h \to 0} \frac{2x + h - 3}{h} \]
\[ = 2x - 3 \]

(b) \[ f(x) = \frac{2}{x}, f(x + h) = \frac{2}{x + h} \]
\[ f(x + h) - f(x) = \frac{2}{x + h} - \frac{2}{x} \]
\[ = \frac{2x - x - 2xh}{x(x + h)} \]
\[ f'(x) = \lim_{h \to 0} \frac{-2x}{x(x + h)} + h \]
\[ = \lim_{h \to 0} \frac{-2}{x + h}, h \neq 0 \]
\[ = -\frac{2}{x} \]

3. (a) \[ y = 2x^2 + \frac{1}{3}x - 2 \]
\[ \frac{dy}{dx} = 4x + \frac{1}{3} \]

(b) \[ y = x - 2x^1, \frac{dy}{dx} = 1 + 2x^2 \]
\[ = 1 + \frac{2}{x^2} \]

4. (a) \[ y = x^2 - 4x, \]
\[ \frac{dy}{dx} = 2x - 4 \]
\[ \text{At } x = 2, \frac{dy}{dx} = 0 \]

(b) \[ f(x) = 4x - \frac{1}{2}, x = 4 \]
\[ f(x) = -2x - \frac{1}{2} \]
\[ f'(4) = -2(4)^{-\frac{1}{2}} = -\frac{1}{4} \]

(c) \[ y = x - \frac{2}{x} \]
\[ x = 2x^1 \]
\[ \frac{dy}{dx} = 1 + \frac{2}{x^2} \]
\[ = 1 + \frac{2}{x^2} \]

5. (a) \[ y = x^2 + 3x - 1, x = -1 \]
\[ \frac{dy}{dx} = 2x + 3, y = 1 - 3 - 1 = -3 \]
\[ = -2 + 3, x = -1 \]
\[ \therefore m = 1 \text{ (gradient), } P(-1, -3) \]

Using \( y - y_1 = m(x-x_1) \)
\[ y + 3 = 1(x - 1) \]
\[ y = x - 2 \text{ is the equation of tangent.} \]

(b) \[ y = 2x^{1/2}, x = 4, y = 4 \]
\[ \frac{dy}{dx} = 2x^{1/2}, x^{1/2} = \frac{1}{\sqrt{x}} \]
\[ x = 4, m = \frac{1}{\sqrt{4}} = \frac{1}{2} \text{ (grad)} \]

Using \( y - y_1 = m(x-x_1) \)
\[ y - 4 = \frac{1}{2}(x - 4) \]
\[ x - 2y + 4 = 0 \]
Is the equation of tangent.
6. \( a \) \( y = \frac{2x}{x-2}, \ u = 2x, \ v = x-2 \)

\[
u' = 2, \ v' = 1
\]

\[
\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{2(x-2) - 2x}{(x-2)^2} = \frac{-4}{(x-2)^2}
\]

\( b \) \( y = (x^2 - 1)^4 \), chain rule

\[
\frac{dy}{dx} = 4x \cdot 2x(x^2 - 1)^3
\]

\[
= 8x(x^2 - 1)^3
\]

\( c \) \( y = -\sqrt{1-x^2} \)

\[
\frac{dy}{dx} = \frac{1}{2} \cdot \frac{-2x}{\sqrt{1-x^2}} - 2x
\]

\[
= x(1-x^2)^{3/2}
\]

\( d \) \( y = 1 + \frac{1}{x} \)

\[
\frac{dy}{dx} = -1 \cdot x^{-2} - \frac{1}{2} \cdot x^{-3}
\]

\[
= \frac{1}{x} \frac{1}{1+x^2} - \frac{1}{2} \cdot \frac{1}{x}
\]

\[
= -3x^2 - 2x - 1
\]

\[
= 2x^2 (1 + x)^2
\]

\( e \) \( y = (x^2 + 1)^2, u = x, v = (x^2 + 1) \)

\[
u' = 1, v' = \frac{x}{\sqrt{x^2 + 1}}
\]

\[
\frac{dy}{dx} = -(uv' + vu')
\]

\[
= x \cdot \frac{x}{\sqrt{x^2 + 1}} + 1 \cdot \sqrt{x^2 + 1}
\]

\[
= \frac{x^3 + x^2 + 1}{\sqrt{x^2 + 1}}
\]

\( f \)

\[
y = \sqrt{x}, x = \sqrt{x}, y' = x^{\frac{1}{2}}
\]

\[
\frac{dy}{dx} = \frac{3}{2} \cdot \sqrt{x} = \frac{3}{2} \cdot \sqrt{x}
\]

7. \( f(x) = x^3 - 6x + 4 \) \( \ldots (1) \)

\( a \) \( \frac{dy}{dx} = 3x^2 - 6 \equiv 3(x^2 - 2) \) \( \ldots (2) \)

\( b \) for the stationary points,

\[
\frac{dy}{dx} = 0 \Rightarrow x^2 = 2 = 0, \ x = \pm \sqrt{2}
\]

\[
x = \sqrt{2}, y = x(x^2 - 6) + 4 = 4 - 4 \sqrt{2}
\]

\[
x = -\sqrt{2}, y = -\sqrt{2} (2-6) + 4 = 4 + 4 \sqrt{2}
\]

\[. \] The stationary points are

\[ A(\sqrt{2}, 4-4 \sqrt{2} = -1.6) \]

\[ B(-\sqrt{2}, 4 + 4 \sqrt{2} = 9.6) \]

At \( A, x = \sqrt{2}, \ \frac{d^2y}{dx^2} = 6 \sqrt{2} > 0 \)

At \( B, x = -\sqrt{2}, \ \frac{d^2y}{dx^2} = -6 \sqrt{2} < 0 \)

\[. \] : A is the minimum point,

\[ B \] is the maximum point.

\( c \) for the point of inflection,

\[ f''(x) = 0 \Rightarrow 6x = 0 \]

\[ x = 0, y = 4 \Rightarrow C(0, 4) \]

Test:

\[
\begin{array}{c|ccc}
\hline
x & -1 & 0 & 1 \\
\hline
f''(x) & - & 0 & + \\
\hline
\end{array}
\]

\[. \] : \( C(0, 4) \) is the point of inflection.

8. \( y = f(x) = -3x^4 + 4x^3 + 7 \ldots (1) \)

\( a \) \( \frac{dy}{dx} = f'(x) = -12x^3 + 12x^2 \) \( \ldots (2) \)

\[
\frac{d^2y}{dx^2} = -36x^2 + 24x = -12x(x^2 - 2) \ldots (3)
\]

For the stationary points,

\[
f'(x) = 0 \Rightarrow x^2 = (x-1) = 0 \]

\[
x = 0, 1 \Rightarrow A(0, 7), B(1, 8) \]
\( (b) \ f''(x) = 0 \Rightarrow x(3x - 2) = 0 \)
\[ x = 0 \Rightarrow A(0, 7) \]
\[ x = \frac{2}{3} \Rightarrow C \left( \frac{2}{3}, \frac{16}{17} \right) \]
are the possible points of inflection.

Test A (0, \( x \) | -1 | 0 | \( \frac{1}{3} \) | 7)
\[ f'''(x) = -1 \quad 0 \quad \frac{1}{3} \quad + \]

\[ C \left( \frac{2}{3}, \frac{16}{17} \right) \]
\[ x \quad -1 \quad 0 \quad \frac{1}{3} \quad + \quad 0 \quad - \]
\[ f''''(x) = \frac{1}{3} \]

\[ \therefore \ A \text{ and } C \text{ are the points of inflection.} \]

Since at A
\[ \frac{dy}{dx} = 0, \ A \text{ is the horizontal point of inflection.} \]

(c) At B(1,8), \[ \frac{d^2y}{dx^2} = -12x(3x - 2) < 0 \]
\[ \therefore B \text{ is the maximum point.} \]

\[ 9. \text{ A = area of the rectangle } ABCD \]
\[ = AB \times AD. \ AB = 2x, \ AD = y = 2xy \]

Now \[ y = 12 - x^2 \]
\[ \therefore A = 2x(12x^2) = 24x - 2x^3 \ldots (1) \]
\[ \frac{dA}{dx} = 24 - 6x^2 = 6(4 - x^2) \ldots (2) \]
\[ \frac{d^2A}{dx^2} = -12x \ldots (3) \]
\[ \frac{dA}{dx} = 0 \text{ for the stationary values} \]
\[ \therefore 4 - x^2 = x = \pm 2 \]

We need not concern with \( x = -2 \) as the length AB > 0.

\[ \frac{d^2A}{dx^2} = -12 < 0 \text{ for } x = 2 \]
\[ \therefore A \text{ is the maximum,} \]
\[ A_{\text{max}} = 2x(12 - x^2), \ x = 2 \]
\[ = 4 \times 8 \]
\[ = 32 \]

\[ 10. (a) \ S = 2x^2 + 2(2x, y + xy) \]
\[ S = 2x^2 + 6xy \]

(b) \[ 2x^2 + 6xy = 300 \]
\[ 6xy = 300 - 2x^2 \]
\[ y = \frac{50 - x}{3} \]
\[ \therefore V = 2x, x, y = 2x^2y \]
\[ = 2x^2 \left( \frac{50 - x}{3} \right) \]
\[ V = 100x - \frac{2x^3}{3} \ldots (1) \]

(c) \[ V > 0 \Rightarrow 2x^2 \left( \frac{50 - x}{3} \right) > 0 \]
Since \( x > 0, 2x^2 \left( 150 - x^2 \right) \)
\[ \therefore 150 - x^2 > 0 \]
\[ x^2 < 150 \Rightarrow x < 5\sqrt{6} \]
(d) \( V = 100x - \frac{2}{3}x^3 \)

\[
\frac{dV}{dx} = 100 - 2x^2 = 2\left(50 - x^2\right)
\]

\[
\frac{d^2V}{dx^2} = -4x
\]

\[
\frac{dV}{dx} = 0 \text{ for the stationary points.}
\]

\[
\therefore 50 - x^2 = 0 \Rightarrow x = \sqrt{50} = 5\sqrt{2} \text{ as the length } x > 0
\]

\[
\frac{d^2V}{dx^2} < 0, 50, v \text{ is the maximum and}
\]

\[
V_{\text{max}} = x\left(100 - \frac{2}{3}x^2\right) = \frac{500\sqrt{2}}{3} \text{ cm}^3
\]

11. (a) From \( \triangle ABC \) and \( \triangle CEF \):

BC/EF (BC and DE are opp sides of a rect)

\[
\therefore \angle C = \angle F \text{ (corres. } \angle s) \\
\angle B = \angle E = 90^\circ
\]

Also AB/CE (BD/CE, a rect)

\[
\therefore \angle A = \angle C \text{ (corres. } \angle S) \\
\triangle ABC \text{ III } \triangle CEF \text{ (AAA)}
\]

\[
x = \frac{12}{8} \frac{y}{y} = \frac{3}{2} (\text{Ratio of corr. Sides})
\]

\[
\therefore xy = 96
\]

(b) \( A = \triangle ADF = \frac{1}{2} DF \times AD \)

Now DF = 12 + y, AD = x + 8

\[
\therefore A = \frac{1}{2}(12 + y)(x + 8)
\]

\[
= \frac{1}{2}(12x + xy + 96 + 8y)
\]

\[
= 6x + 96 + 48 + 4y
\]

\[
= 6x + 96 + \frac{4 \times 96}{x}
\]

\[
\therefore A = 6x + 96 + \frac{384}{x} \quad \ldots(1)
\]

(c) \[
\frac{dA}{dx} = 6 - \frac{384}{x^2}
\]

\[
\frac{d^2A}{dx^2} = \frac{768}{x^3}
\]

For the stationary points,

\[
\frac{dA}{dx} = 0 \Rightarrow 6 = \frac{384}{x^2}
\]

\[
x^2 = 64 \Rightarrow x = 8
\]

\( x > 0, \) so A is the minimum and

\[
A_{\text{min}} = 48 + 96 + 64
\]

\[
= 208 \text{ cm}^2
\]

and \( x = 8 \text{ cm} \)

12. Printed area = \((x - 2)(y - 6)\)

(a) \( 48 = (x - 2)(y - 6) \)

\[
\therefore y - 6 = \frac{48}{x - 2}
\]

\[
y = 6 + \frac{48}{x - 2}
\]

(b) \( A = xy \)

\[
= x\left(6 + \frac{48}{x - 2}\right)
\]

\[
\therefore A = 6x + \frac{48x}{x - 2}
\]

(c) \[
\frac{dA}{dx} = 6 + 48 \frac{d}{dx}\left(\frac{x}{x - 2}\right)
\]

\[
= 6 + 48x \frac{x - 2 - x}{(x - 2)^2}
\]

\[
= 6 - \frac{96}{(x - 2)^2}
\]

\[
\frac{d^2A}{dx^2} = -\frac{192}{(x - 2)^3}
\]

\[
\frac{dA}{dx} = 0 \text{ for the stationary points.}
\]

\[
\therefore 6 = \frac{96}{(x - 2)^2}
\]

\[
(x - 2)^2 = 16 \Rightarrow x - 2 = \pm 4
\]

Since \( x > 0, x = 6 \text{ cm} \)

\[
y = 18 \text{ cm}
\]

Also \( \frac{d^2A}{dx^2} = 0 \) for \( x = 6 \)

\( \therefore \) The minimum poster area \( A = 6 \times 18 = 108 \text{ cm}^2 \)

with \( x = 6 \text{ cm}, y = 18 \text{ cm} \)
13. \( A = \text{area of } \triangle ABC = \frac{1}{2} \times xy \)

\[ \frac{xy}{2} = 3600 \Rightarrow y = \frac{7200}{x} \]

(a) \( C = 60x + 60y + 2 \)

\[ = 60x + 30y \]

\[ = 60x + \frac{30 \times 7200}{x} \]

\[ \therefore 6x = \frac{21600}{x} \]

(b) \[ \frac{dc}{dx} = 60 - \frac{21600}{x^2} \]

For the minimum cost \( \frac{dc}{dx} = 0 \)

\[ \therefore x^2 = 3600 \]

\[ x = 60 \]

so \( C \) is the minimum for

\[ \frac{d^2c}{dx^2} = \frac{43200}{x^3} > 0 \]

\[ x = 60 \text{ and} \]

\[ C_{\text{min}} = 60 \times 60 + \frac{21600}{60} \]

\[ = 7200 \text{ dollars} \]

14. From the similar triangles,

(a) \[ \frac{r}{6} = \frac{20 - h}{20} \Rightarrow 20 - h = \frac{10r}{3} \]

\[ h = 20 - \frac{10r}{3} \]

\[ \therefore h = \frac{10}{3} (6 - r) \]

(b) \( V = \pi r^2 h \)

\[ = \pi r^2 \times \frac{10}{3} (6 - r) \]

\[ \therefore V = \frac{10}{3} \pi r^2 (6 - r) \ldots (1) \]

\[ = \frac{10 \pi}{3} (6r - r^3), \text{keeping constants out} \]

\[ \frac{dV}{dr} = \frac{10 \pi}{3} (12r - 3r^2) \]