**TERMINOLOGY**

**Congruent:** Two figures are congruent if they have the same size and shape. They are identical in every way.

**Polygon:** A polygon is a closed plane figure with straight sides.

**Similar:** Two figures are similar if they have the same shape but a different size. Corresponding angles are equal and corresponding sides are in the same ratio.

**Vertex:** A vertex is a corner of a figure (vertices is plural, meaning more than one vertex).
Chapter 1  Geometry 2

INTRODUCTION

YOU STUDIED GEOMETRY IN the Preliminary Course. In this chapter, you will revise this work and extend it to include some more general applications of geometrical properties involving polygons.

You will also use the Preliminary topic on straight-line graphs to explore coordinate methods in geometry.

Plane Figure Geometry

Here is a summary of the geometry you studied in the Preliminary Course.

Vertically opposite angles

\[ \angle AEC \text{ and } \angle DEB \text{ are called vertically opposite angles. } \angle AED \text{ and } \angle CEB \text{ are also vertically opposite angles.} \]

Vertically opposite angles are equal.

Parallel lines

If the lines are parallel, then alternate angles are equal.

If the lines are parallel, then corresponding angles are equal.
If the lines are parallel, **cointerior angles** are supplementary (i.e. their sum is $180^\circ$).

**TESTS FOR PARALLEL LINES**

If **alternate angles** are equal, then the lines are parallel.

If **corresponding angles** are equal, then the lines are parallel.

If **cointerior angles** are supplementary, then the lines are parallel.

**Angle sum of a triangle**

The sum of the interior angles in any triangle is $180^\circ$, that is, $a + b + c = 180$
Exterior angle of a triangle

The exterior angle in any triangle is equal to the sum of the two opposite interior angles. That is,

\[ x + y = z \]

Congruent triangles

Two triangles are **congruent** if they are the same shape and size. All pairs of corresponding sides and angles are equal.

For example:

Two triangles are congruent if

- **SSS**: all three pairs of corresponding sides are equal
- **SAS**: two pairs of corresponding sides and their **included angles** are equal
- **AAS**: two pairs of angles and one pair of corresponding sides are equal
- **RHS**: both have a right angle, their hypotenuses are equal and one other pair of corresponding sides are equal

Similar triangles

Triangles, for example \(ABC\) and \(XYZ\), are similar if they are the same **shape** but different **sizes**.

As in the example, all three pairs of corresponding **angles** are equal.

All three pairs of corresponding **sides** are in proportion (in the same ratio).
We write: \( \triangle ABC \parallel \triangle XYZ \)

\( \triangle XYZ \) is three times larger than \( \triangle ABC \)

\[
\begin{align*}
\frac{XY}{AB} &= \frac{6}{2} = 3 \\
\frac{XZ}{AC} &= \frac{12}{4} = 3 \\
\frac{YZ}{BC} &= \frac{15}{5} = 3
\end{align*}
\]

\[
\therefore \frac{XY}{AB} = \frac{XZ}{AC} = \frac{YZ}{BC}
\]

**TESTS**

There are three tests for similar triangles.

Two triangles are similar if:

- three pairs of corresponding angles are equal
- three pairs of corresponding sides are in proportion
- two pairs of sides are in proportion and their included angles are equal

**Ratios of intercepts**

When two (or more) transversals cut a series of parallel lines, the ratios of their intercepts are equal.

That is, \( AB:BC = DE:EF \)

or \[
\frac{AB}{BC} = \frac{DE}{EF}
\]

**Pythagoras’ theorem**

The square on the hypotenuse in any right angled triangle is equal to the sum of the squares on the other two sides.

That is, \( c^2 = a^2 + b^2 \)

or \[
c = \sqrt{a^2 + b^2}
\]
Quadrilaterals

A quadrilateral is any **four-sided** figure

In any quadrilateral the sum of the interior angles is 360°

**PARALLELOGRAM**

A parallelogram is a quadrilateral with opposite sides parallel

Properties of a parallelogram:
- **opposite sides** of a parallelogram are equal
- **opposite angles** of a parallelogram are equal
- **diagonals** in a parallelogram bisect each other
- each diagonal bisects the parallelogram into two **congruent triangles**

**TESTS**

A **quadrilateral** is a **parallelogram** if:
- both pairs of **opposite sides** are equal
- both pairs of **opposite angles** are equal
- **one pair of sides** is both equal and parallel
- the **diagonals** bisect each other
A rectangle is a parallelogram with one angle a right angle

Properties of a rectangle:
• the same as for a parallelogram, and also
• diagonals are equal

TEST
A quadrilateral is a rectangle if its diagonals are equal

Application
Builders use the property of equal diagonals to check if a rectangle is accurate. For example, a timber frame may look rectangular, but may be slightly slanting. Checking the diagonals makes sure that a building does not end up like the Leaning Tower of Pisa!

A rhombus is a parallelogram with a pair of adjacent sides equal

Properties of a rhombus:
• the same as for parallelogram, and also
• diagonals bisect at right angles
• diagonals bisect the angles of the rhombus
**TESTS**

A quadrilateral is a rhombus if:
- all sides are equal
- diagonals bisect each other at right angles

**SQUARE**

A square is a rectangle with a pair of adjacent sides equal

Properties of a square:
- the same as for rectangle, and also
- diagonals are perpendicular
- diagonals make angles of 45° with the sides

**TRAPEZIUM**

A trapezium is a quadrilateral with one pair of sides parallel

**KITE**

A kite is a quadrilateral with two pairs of adjacent sides equal
**Polygons**

A polygon is a plane figure with straight sides.

A regular polygon has all sides and all interior angles equal.

The sum of the interior angles of an \( n \)-sided polygon is given by

\[
S = (n - 2) \times 180^\circ
\]

The sum of the exterior angles of any polygon is \( 360^\circ \).

**Areas**

Most areas of plane figures come from the area of a rectangle.

**RECTANGLE**

\[
A = lb
\]

**SQUARE**

\[
A = x^2
\]

A square is a special rectangle.

**TRIANGLE**

\[
A = \frac{1}{2}bh
\]

The area of a triangle is half the area of a rectangle.
The area of a parallelogram is the same as the area of two triangles.

**PARALLELOGRAM**

![Parallelogram diagram]

\[ A = bh \]

**RHOMBUS**

![Rhombus diagram]

\[ A = \frac{1}{2}xy \]

*(x and y are lengths of diagonals)*

**TRAPEZIUM**

![Trapezium diagram]

\[ A = \frac{1}{2}h(a + b) \]

**CIRCLE**

![Circle diagram]

\[ A = \pi r^2 \]

The following examples and exercises use these results to prove properties of plane figures.

You will study the circle in more detail. See Chapter 5.
EXAMPLES

1. Prove $\angle A = \angle C$ in parallelogram $ABCD$.

![Parallelogram Diagram]

**Solution**

Let $\angle A = x^\circ$

Then $\angle B = 180^\circ - x^\circ$  \(\angle A, \angle B \text{ cointerior angles, } AD \parallel BC\)

$\angle C = 180^\circ - (180^\circ - x^\circ)$  \(\angle B, \angle C \text{ cointerior angles, } AB \parallel DC\)

$\angle C = 180^\circ - 180^\circ + x^\circ$

$\angle C = x^\circ$

$\therefore \angle A = \angle C$

2. Triangle $ABC$ below is isosceles with $AB = AC$. Prove that the base angles of $\triangle ABC$ are equal by showing that $\triangle ABD$ and $\triangle ACD$ are congruent.

![Isosceles Triangle Diagram]

**Solution**

$\angle ADB = \angle ADC = 90^\circ$  \(\text{given}\)

$AB = AC$  \(\text{given}\)

$AD$ is common

$\therefore \triangle ABD \cong \triangle ACD$  \(\text{RHS}\)

So $\angle ABD = \angle ACD$  \(\text{corresponding angles in congruent } \triangle s\)

$\therefore$ base angles are equal
3. Prove that opposite sides in a parallelogram are equal.

Solution

Let \(ABCD\) be any parallelogram and draw in diagonal \(AC\).

\[\angle DAK = \angle CAK\quad \text{(alternate } \angle \text{s, } AD \parallel BC)\]
\[\angle BAK = \angle ACK\quad \text{(alternate } \angle \text{s, } AB \parallel DC)\]

\(AC\) is common.

\[\therefore \triangle ABC \cong \triangle ADC\quad \text{(AAS)}\]

\[\therefore AB = DC\quad \text{(corresponding sides in congruent } \triangle \text{s)}\]
\[AD = BC\quad \text{(similarly)}\]

\[\therefore \text{opposite sides in a parallelogram are equal}\]

1.1 Exercises

1. \(DE\) bisects acute angle \(\angle ABC\) so that \(\angle ABD = \angle CBD\). Prove that \(DE\) also bisects reflex angle \(\angle ABC\). That is, prove \(\angle ABE = \angle CBE\).

2. Prove that \(CD\) bisects \(\angle AFE\).

3. Prove \(VW \parallel XY\).

4. Given \(x + y = 180^\circ\), prove that \(ABCD\) is a parallelogram.
5. \( BD \) bisects \( \angle ABC \). Prove that \( \triangle ABD \equiv \triangle CBD \).

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C} \\
\hline
\end{array} \]

6. (a) Show that \( \triangle ABC \equiv \triangle AED \).
(b) Hence prove that \( \triangle ACD \) is isosceles.

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C} \\
\text{D} \\
\text{E} \\
\hline
\end{array} \]

7. \( ABCD \) is a square. Lines are drawn from \( C \) to \( M \) and \( N \), the midpoints of \( AD \) and \( AB \) respectively. Prove that \( MC = NC \).

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C} \\
\text{D} \\
\hline
\end{array} \]

8. \( OC \) is drawn so that it is perpendicular to chord \( AB \) and \( O \) is the centre of the circle. Prove that \( \triangle OAC \) and \( \triangle OBC \) are congruent, and hence that \( OC \) bisects \( AB \).

\[ \begin{array}{c}
\text{A} \\
\text{C} \\
\text{B} \\
\text{D} \\
\text{O} \\
\hline
\end{array} \]

9. \( CE \) and \( BD \) are altitudes of \( \triangle ABC \), and \( \triangle ABC \) is isosceles (\( AB = AC \)). Prove that \( CE = BD \).

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C} \\
\text{D} \\
\text{E} \\
\hline
\end{array} \]

10. \( ABCD \) is a kite where \( AB = AD \) and \( BC = DC \). Prove that diagonal \( AC \) bisects both \( \angle DAB \) and \( \angle DCB \).

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C} \\
\text{D} \\
\hline
\end{array} \]

11. \( MNOP \) is a rhombus with \( MN = NO \). Show that
(a) \( \triangle MNO \) is congruent to \( \triangle MPO \)
(b) \( \angle PMQ = \angle NMQ \)
(c) \( \triangle PMQ \) is congruent to \( \triangle NMQ \)
(d) \( \angle MQN = 90^\circ \)

\[ \begin{array}{c}
\text{P} \\
\text{M} \\
\text{Q} \\
\text{N} \\
\text{O} \\
\hline
\end{array} \]

12. Show that a diagonal cuts a parallelogram into two congruent triangles.

13. Prove that opposite angles are equal in any parallelogram.
14. \(ABCD\) is a parallelogram with \(BM = DN\). Prove that \(AMCN\) is also a parallelogram.

15. \(ABCD\) and \(BCEF\) are parallelograms. Show that \(AFED\) is a parallelogram.

16. \(ABCD\) is a parallelogram with \(DE = DC\). Prove that \(CE\) bisects \(\angle BCD\).

17. In quadrilateral \(ABCD\), \(AB = CD\) and \(\angle BAC = \angle DCA\). Prove \(ABCD\) is a parallelogram.

18. \(ABCD\) is a parallelogram with \(\angle AEB = 90^\circ\). Prove
   (a) \(AB = BC\)
   (b) \(\angle ABE = \angle CBE\)

19. Prove that the diagonals in any rectangle are equal.

20. Prove that if one angle in a rectangle is \(90^\circ\) then all the angles are \(90^\circ\).

21. \(ABCD\) is a rhombus with \(AD = CD\). Prove that all sides of the rhombus are equal.

22. \(ABCD\) is an isosceles trapezium. Prove the base angles \(\angle ADC\) and \(\angle BCD\) are equal.

23. Prove that \(\angle ADC = \angle ABC\) in kite \(ABCD\).

24. In rectangle \(ABCD\), \(E\) is the midpoint of \(CD\). Prove \(AE = BE\).
25. \(ABCD\) is a rhombus.
(a) Prove \(\triangle ADB\) and \(\triangle BCD\) are congruent.
(b) Hence show \(\angle ABE = \angle CBE\).
(c) Prove \(\triangle ABE\) and \(\triangle CBE\) are congruent.
(d) Prove \(\angle AEB = 90^\circ\).

Surface Areas and Volumes

Areas are used in finding the surface area and volume of solids. Here is a summary of some of the most common ones.

<table>
<thead>
<tr>
<th>Surface Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S = 2(lb + bh + lh))</td>
<td>(V = lbh)</td>
</tr>
<tr>
<td>(S = 6x^2)</td>
<td>(V = x^3)</td>
</tr>
<tr>
<td>(S = 2\pi(r + h))</td>
<td>(V = \pi r^2 h)</td>
</tr>
<tr>
<td>(S = 4\pi r^2)</td>
<td>(V = \frac{4}{3}\pi r^3)</td>
</tr>
</tbody>
</table>

You will need some of these formulae when you study maxima and minima problems in Chapter 2.
In general, the volume of any prism is given by

\[ V = Ah \]

where \( A \) is the area of the base and \( h \) is its height.

In general, the volume of any pyramid is given by

\[ V = \frac{1}{3} Ah \]

Where \( A \) is the area of the base and \( h \) is its height.
While surface area and volume is not a part of the geometry in the HSC syllabus, the topic in Chapter 2 uses calculus to find maximum or minimum areas, perimeters, surface areas or volumes. So you will need to know these formulae in order to answer the questions in the next chapter. Here are some questions to get you started.

**EXAMPLE**

Find the surface area of a cone whose height is twice the radius, in terms of $r$.

**Solution**

\[
\begin{align*}
h &= 2r \\
l^2 &= r^2 + h^2 \\
&= r^2 + (2r)^2 \\
&= r^2 + 4r^2 \\
&= 5r^2 \\
\therefore l &= \sqrt{5r^2} \\
&= \sqrt{5} r
\end{align*}
\]

Surface area \( S = \pi r (r + l) \) where \( l \) is slant height

\[ = \pi r (r + \sqrt{5} r) \]

1. A rectangular prism has dimensions 12.5 mm, 84 mm and 64 mm. Find its
   (a) surface area and
   (b) volume.

2. A sphere has a volume of $120\pi$ m$^3$. Find the exact value of \( r \).

3. A rectangular prism has dimensions \( x, x + 2 \) and \( 2x - 1 \). Find its volume in terms of \( x \).
4. A cylinder has a volume of 250 cm\(^3\). If its base has radius \(r\) and its height is \(h\), show that \(r = \sqrt{\frac{250}{\pi h}}\).

5. Find the volume of a cylinder in terms of \(r\) if its height is five times the size of its radius.

6. The ratio of the length to the breadth of a certain rectangle is 3:2. If the breadth of this rectangle is \(b\) units, find a formula for the area of the rectangle in terms of \(b\).

7. Find the volume of a cube with sides \((x + 2)\) cm.

8. What would the surface area of a cylinder be in terms of \(h\) if its height is a third of its radius?

9. A square piece of metal with sides 3 m has a square of side \(x\) cut out of each corner. The metal is then folded up to form a rectangular prism. Find its volume in terms of \(x\).

10. A cone-shaped vessel has a height of twice its radius. If I fill the vessel with water to a depth of 10 cm, find the volume of water to the nearest cm\(^3\).

11. The area of the base of a prism is given by \(3h + 2\), where \(h\) is the height of the prism. Find a formula for the volume of the prism.

12. The area of the base of a pyramid is \(6h + 15\) where \(h\) is the height of the pyramid. Find the volume of the pyramid in terms of \(h\).

13. A rectangular pyramid has base dimensions \(x - 3\) and \(3x + 5\), and a height of \(2x + 1\). Write a formula for the volume of the pyramid in terms of \(x\).

14. The height of a rectangular prism is twice the length of its base. If the width of the base is \(x\) and the length is \(3x - 1\), find an expression for the (a) volume and (b) surface area of the prism.

15. Find a formula for the slant height of a cone in terms of its radius \(r\) and height \(h\).

16. A page measuring \(x\) by \(y\) is curved around to make an open cylinder with height \(y\). Find the volume of the cylinder in terms of \(x\) and \(y\).

17. The volume of a cylinder is 400 cm\(^3\). Find the height of the cylinder in terms of its radius \(r\).

18. A cylinder has a surface area of 1500 cm\(^2\). Find a formula for its height \(h\) in terms of \(r\).

19. The surface area of a cone is given by \(S = \pi r (r + l)\) where \(l\) is the slant height. Find a formula for the slant height of a cone with surface area 850 cm\(^2\) in terms of \(r\).
20. A rectangular timber post is cut out of a log with diameter $d$ as shown. If the post has length $x$ and breadth $y$, write $y$ in terms of $x$ when $d = 900$ mm.

**DID YOU KNOW?**

**REGULAR SOLIDS**

There are only five solids with each face the same size and shape. These are called platonic solids. Research these on the internet.
The skeletons opposite are those of radiolarians. These are tiny sea animals, with their skeletons shaped like regular solids. A salt crystal is a cube. A diamond crystal is an octahedron.

Diamond crystal

Coordinate Methods in Geometry

Problems in plane geometry can be solved by using the number plane.

You studied straight-line graphs in the Preliminary Course. Some of the main results that you learned will be used in this section. You may need to revise that work before studying this section.

Here is a summary of the main formulae.

**Distance**

The distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

**Midpoint**

The midpoint of two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[
p = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

**Gradient**

The gradient of the line between \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

The gradient of a straight line is given by

\[
m = \tan \theta
\]

where \(\theta\) is the angle the line makes with the x-axis in the positive direction.
The gradient of the line $ax + by + c = 0$ is given by

$$m = -\frac{a}{b}$$

**Equation of a straight line**

The equation of a straight line is given by

$$y - y_1 = m(x - x_1)$$

where $(x_1, y_1)$ lies on the line with gradient $m$.

**Parallel lines**

If two lines are parallel, then they have the same gradient. That is,

$$m_1 = m_2$$

**Perpendicular lines**

If two lines with gradients $m_1$ and $m_2$ respectively are perpendicular, then

$$m_1 m_2 = -1 \text{ or } m_2 = -\frac{1}{m_1}$$

**Perpendicular distance**

The perpendicular distance from $(x_1, y_1)$ to the line $ax + by + c = 0$ is given by

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

**EXAMPLES**

1. Show that triangle $ABC$ is right angled, where $A = (3, 4)$, $B = (-1, -1)$ and $C = (-2, 8)$.

**Solution**

Method 1:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$AB = \sqrt{(-1 - 3)^2 + (-1 - 4)^2}$$

$$= \sqrt{(-4)^2 + (-5)^2}$$

$$= \sqrt{16 + 25}$$

$$= \sqrt{41}$$

$$AC = \sqrt{(-2 - 3)^2 + (8 - 4)^2}$$

$$= \sqrt{(-5)^2 + 4^2}$$

$$= \sqrt{25 + 16}$$

$$= \sqrt{41}$$

$$BC = \sqrt{(-1 - (-2))^2 + (-1 - 8)^2}$$

$$= \sqrt{1^2 + (-9)^2}$$

$$= \sqrt{1 + 81}$$

$$= \sqrt{82}$$
Since Pythagoras’ theorem is true, the triangle \(ABC\) is right angled.

Method 2:

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ m_{AB} = \frac{-1 - 4}{-1 - 3} \]
\[ = \frac{-5}{-4} \]
\[ = \frac{5}{4} \]

\[ m_{AC} = \frac{8 - 4}{2 - 3} \]
\[ = \frac{4}{-1} \]
\[ = -4 \]
\[ m_{AC} \times m_{AC} = \frac{5}{4} \times \frac{-4}{5} \]
\[ = -1 \]

So \(AB\) and \(AC\) are perpendicular

So triangle \(ABC\) is right angled at \(A\).

---

2. Prove that points \(A(1, 1), B(-2, -1)\) and \(C(4, 3)\) are collinear.

**Solution**

Collinear points lie on the same straight line, so they will have the same gradient.

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ m_{AB} = \frac{-1 - 1}{-2 - 1} \]
\[ = \frac{-2}{-3} \]
\[ = \frac{2}{3} \]
\[ m_{BC} = \frac{3 - (-1)}{4 - (-2)} \]
\[ = \frac{4}{6} \]
\[ = \frac{2}{3} \]

\[ m_{AB} = m_{BC} \]

So the points are collinear.
1.3 Exercises

1. Show that points $A(-1, 0)$, $B(0, 4)$, $C(7, 0)$ and $D(6, -4)$ are the vertices of a parallelogram.

2. Prove that $A(1, 5)$, $B(4, -6)$ and $C(-3, -2)$ are vertices of a right angled triangle.

3. Given $\triangle ABC$ with vertices $A(0, 8)$, $B(3, 0)$ and $C(-3, 0)$
   (a) show that $\triangle ABC$ is isosceles
   (b) find the length of the altitude from $A$
   (c) find the area of the triangle.

4. Show that the points $X(3, 2)$, $Y(-2, 1)$ and $Z(8, 3)$ are collinear.

5. (a) Show that the points $A(2, 5)$, $B(1, 0)$, $C(-7, -4)$ and $D(-3, 4)$ are the vertices of a kite.
   (b) Prove that the diagonals of the kite are perpendicular.
   (c) Show that $CE = 2AE$ where $E$ is the point of intersection of the diagonals.

6. Find the radius of the circle that has its centre at the origin and a tangent with equation given by $4x - 3y - 5 = 0$.

7. (a) Find the equation of the perpendicular bisector of the line joining $A(3, 2)$ and $B(-1, 8)$.
   (b) Show that the point $C(7, 9)$ lies on the perpendicular bisector.
   (c) What type of triangle is $\triangle ABC$?

8. Show that $\triangle OAB$ and $\triangle OCD$ are similar where $(0, 7), (2, 0), (0, -14)$ and $(-4, 0)$ are the points $A$, $B$, $C$ and $D$ respectively and $O$ is the origin.

9. (a) Prove that $\triangle OAB$ and $\triangle OCB$ are congruent given $A(3, 4)$, $B(5, 0)$, $C(2, -4)$ and $O$ the origin.
   (b) Show that $OABC$ is a parallelogram.

10. The points $A(0, 0)$, $B(2, 0)$, $C(2, 2)$ and $D(0, 2)$ are the vertices of a square. Prove that its diagonals make angles of $45^\circ$ with the sides of the square.

11. Prove that $P(-2, 0)$, $Q(0, 5)$, $R(10, 1)$ and $S(8, -4)$ are the vertices of a rectangle.

12. The points $A(-5, 0)$, $B(1, 4)$ and $C(3, 0)$ form the vertices of a triangle.
   (a) Find $X$ and $Y$, the midpoints of $AB$ and $AC$ respectively.
   (b) Show that $XY$ and $BC$ are parallel.
   (c) Show that $BC = 2XY$.

13. Show that the diagonals of a square are perpendicular bisectors, given the vertices of square $ABCD$ where $A = (-a, 0)$, $B = (-a, a)$, $C = (0, a)$ and $D = (0, 0)$.

14. (a) Show that points $X(3, 2)$ and $Y(-1, 0)$ are equal distances from the line $4x - 3y - 1 = 0$.
   (b) Find $Z$, the $x$-intercept of the line.
   (c) What is the area of triangle $XYZ$?

15. $ABCD$ is a quadrilateral with $A(3, 1)$, $B(1, -4)$, $C(-5, -2)$ and $D(-4, 3)$. Show that the midpoints of each side are the vertices of a parallelogram.
Test Yourself 1

1. Triangle $ABC$ is isosceles, with $AB = AC$. $D$ is the midpoint of $AB$ and $E$ is the midpoint of $AC$.

(a) Prove that $\triangle BEC$ is congruent to $\triangle BDC$.
(b) Prove $BE = DC$.

2. If the diagonals of a rhombus are $x$ and $y$, show that the length of its side is $\sqrt{x^2 + y^2}$.

3. If $A = (4, -1), B = (7, -5)$ and $C = (1, 3)$, prove that triangle $ABC$ is isosceles.

4. The surface area of a closed cylinder is 100 m$^2$. Write the height $h$ of the cylinder in terms of its radius $r$.

5. $ABCD$ is a parallelogram with $\angle C = 45^\circ$, $AE$ perpendicular to $ED$ and $CD = DE$.

(a) Show that $\triangle ADE$ is isosceles.
(b) If $AE = y$, show that the area of $ABCE$ is $\frac{3y^2}{2}$.

6. In the figure $AEF$
   (a) prove $\frac{CB}{BA} = \frac{CD}{DE}$
   (b) find the length of $AE$.

7. Given $A = (-1, 3), B = (-2, -4), C = (5, -4)$ and $D = (6, 3)$, prove $ABCD$ is a parallelogram.

8. A parallelogram has sides 5 cm and 12 cm, with diagonal 13 cm.

Show that the parallelogram is a rectangle.

9. Prove that $\triangle PQR$ and $\triangle WXY$ are similar.

10. In quadrilateral $ABCE$, $AD = ED = DC$ and $\angle ACB = 90^\circ$. Also, $AC$ bisects $\angle BAD$. 

Prove $ABCE$ is a parallelogram.

11. If $A = (1, 5)$, $B = (4, 2)$ and $C = (2, -3)$, find the coordinates of $D$ such that $ABCD$ is a parallelogram.

12. (a) Find the equation of $AB$ if $A = (-2, -3)$ and $B = (4, 5)$.
   (b) Find the perpendicular distance from $C(1, -3)$ to line $AB$.
   (c) Find the area of $\triangle ABC$.

13. $ABCD$ is a kite.

14. $A(1, 2), B(3, 3)$ and $C(5, -1)$ are points on a number plane.
   (a) Prove $\triangle ABC$ and $\triangle ADC$ are congruent.
   (b) Prove $\triangle ABE$ and $\triangle ADE$ are congruent.
   (c) Prove $AC$ is the perpendicular bisector of $BD$.

15. The surface area of a box is 500 cm$^2$. Its length is twice its breadth $x$.
   (a) Show that $h$ of the box is given by $h = \frac{250 - 2x^2}{3x}$.
   (b) Show that the volume of the box is $V = \frac{500x - 4x^3}{3}$.

---

**Challenge Exercise 1**

1. In the figure, $BD$ is the perpendicular bisector of $AC$. Prove that $\triangle ABC$ is isosceles.

2. Given $E$ and $D$ are midpoints of $AC$ and $AB$ respectively, prove that $DE$ is parallel to $BC$ and $DE = \frac{1}{2} BC$.

3. Prove that the diagonals in a rhombus bisect the angles they make with the sides.
4. Paper comes in different sizes, called A0, A1, A2, A3, A4 and so on. The largest size is A0, which has an area of one square metre. If the ratio of its length to breadth is $\sqrt{2}:1$, find the dimensions of its sides in millimetres, to the nearest millimetre.

5. The volume of a prism with a square base of side $x$ is $1000 \text{ cm}^3$. Find its surface area in terms of $x$.

6. Prove that in any regular $n$-sided polygon the size of each angle is \(180 - \frac{360}{n}\)°.

7. A parallelogram $ABCD$ has $AB$ produced to $E$ and diagonal $AC$ produced to $F$ so that $EF \parallel BC$. Prove that $\Delta AEF$ is similar to $\Delta ADC$.

8. $ABCD$ is a rhombus with $A(0, 0), B(a, b), C(2a, 0)$ and $D(a, -b)$. Show
   (a) the diagonals bisect each other at right angles
   (b) all sides are equal
   (c) $AC$ bisects $\angle BCD$.

9. In the parallelogram $ABCD$, $AC$ is perpendicular to $BD$. Prove that $AB = AD$.

10. Triangle $ABC$ has $P$, $Q$ and $R$ as midpoints of the sides, as shown in the diagram below. Prove that $\Delta PQR \equiv \Delta CPR$.

11. A pair of earrings is made with a wire surround holding a circular stone, as shown. Find the total length of wire needed for the earrings.

12. The sides of a quadrilateral $ABCD$ have midpoints $P$, $Q$, $R$ and $S$, as shown below.
   (a) Show that $\Delta DPS$ is similar to $\Delta DAC$.
   (b) Show $PS \parallel QR$.
   (c) Show that $PQRS$ is a parallelogram.
13. A plastic frame for a pair of glasses is designed as below. Find the length of plastic needed for the frame, to the nearest centimetre.
Geometrical Applications of Calculus

TERMINOLOGY

Anti-differentiation: The process of finding a primitive (original) function from the derivative. It is the inverse operation to differentiation

Concavity: The shape of a curve as it bends around (it can be concave up or concave down)

Differentiation: The process of finding the gradient of a tangent to a curve or the derivative

Gradient of a secant: The gradient (slope) of a line between two points that lie close together on a curve

Gradient of a tangent: The gradient (slope) of a line that is a tangent to a curve at a point on a function. It is the derivative of the function

Horizontal point of inflexion: A stationary point (where the first derivative is zero) where the concavity of the curve changes

Instantaneous rate of change: The derivative of a function

Maximum turning point: A local stationary point (where the first derivative is zero) and where the curve is concave down. The gradient of the tangent is zero

Minimum turning point: A local stationary point (where the first derivative is zero) and where the curve is concave up. The gradient of the tangent is zero

Monotonic increasing or decreasing function: A function is always increasing or decreasing

Point of inflexion: A point at which the curve is neither concave upwards nor downwards, but where the concavity changes

Primitive function: The original function found by working backwards from the derivative. Found by anti-differentiation

Rate of change: The rate at which the dependent variable changes as the independent variable changes

Stationary (turning) point: A local point at which the gradient of the tangent is zero and the tangent is horizontal. The first derivative is zero
INTRODUCTION

YOU LEARNED ABOUT differentiation in the Preliminary Course. This is the process of finding the gradient of a tangent to a curve. This chapter looks at how the gradient of a tangent can be used to describe the shape of a curve. Knowing this will enable us to sketch various curves and find their maximum and minimum values. The theory also allows us to solve various problems involving maximum and minimum values.

DID YOU KNOW?

Although Newton and Leibniz are said to have discovered calculus, elements of calculus were around before then. It was Newton and Leibniz who perfected the processes and developed the notation associated with calculus. Pierre de Fermat (1601–65) used coordinate geometry to find maximum and minimum values of functions. His method is very close to calculus. He also worked out a way of finding the tangent to a curve.

The 17th-century mathematicians who developed calculus knew that it worked, but it was not fully understood. Limits were not introduced into calculus until the nineteenth century.

Gradient of a Curve

To learn about the shape of a curve, we first need to revise what we know about the gradient of a tangent. The gradient (slope) of a straight line measures the rate of change of $y$ with respect to the change in $x$.

- positive gradient
- negative gradient
- zero gradient

Since the gradient of a curve varies, we find the gradient of the tangent at each point along the curve.
In the examples on the previous page, where the gradient is positive, the curve is going up, or increasing (reading from left to right).

Where the gradient is negative, the curve is going downwards, or decreasing.

The gradient is zero at particular points of the curves. At these points the curve isn’t increasing or decreasing. We say the curve is stationary at these points.

If \( f'(x) > 0 \), the curve is increasing
If \( f'(x) < 0 \), the curve is decreasing
If \( f'(x) = 0 \), the curve is stationary

A curve is monotonic increasing or decreasing if it is always increasing or decreasing; that is,
if \( f'(x) > 0 \) for all \( x \) (monotonic increasing)
or \( f'(x) < 0 \) for all \( x \) (monotonic decreasing)

**EXAMPLES**

1. Find all \( x \) values for which the curve \( f(x) = x^2 - 4x + 1 \) is increasing.

**Solution**

\[
f'(x) = 2x - 4
\]
\[
f'(x) > 0 \text{ for increasing curve}
\]
i.e. \( 2x - 4 > 0 \)
\[
2x > 4
\]
\[
x > 2
\]

So the curve is increasing for \( x > 2 \).
2. Find the stationary point on the parabola \( y = x^2 - 6x + 3 \).

**Solution**

\[ \frac{dy}{dx} = 2x - 6 \]

For stationary points, \( \frac{dy}{dx} = 0 \)

i.e. \( 2x - 6 = 0 \)

\( 2x = 6 \)

\( x = 3 \)

When \( x = 3 \), \( y = 3^2 - 6(3) + 3 \)

\( = -6 \)

So the stationary point is \((3, -6)\).

3. Find any stationary points on the curve \( y = x^3 - 48x - 7 \).

**Solution**

\( y' = 3x^2 - 48 \)

For stationary points, \( y' = 0 \)

i.e. \( 3x^2 - 48 = 0 \)

\( 3x^2 = 48 \)

\( x^2 = 16 \)

\( \therefore \)

\( x = \pm 4 \)

When \( x = 4 \), \( y = 4^3 - 48(4) - 7 \)

\( = -135 \)

When \( x = -4 \), \( y = (-4)^3 - 48(-4) - 7 \)

\( = 121 \)

So the stationary points are \((4, -135)\) and \((-4, 121)\).
**PROBLEM**

*What is wrong with this working out?*

Find the stationary point on the curve $y = 2x^2 + x - 1$.

**Solution**

$y' = 4x + 1$

For stationary points, $y' = 0$

i.e.

\[ 4x + 1 = 0 \]

\[ 4x = -1 \]

\[ x = -0.25 \]

When $x = -0.25$, $y = 4(-0.25) + 1$

\[ = -1 + 1 \]

\[ = 0 \]

So the stationary point is $(-0.25, 0)$.

---

**2.1 Exercises**

1. Find the parts of each curve where the gradient of the tangent is positive, negative or zero. Label each curve with $+$, $-$ or $0$.

   (a) ![Graph (a)](image1)

   (b) ![Graph (b)](image2)

   (c) ![Graph (c)](image3)

   (d) ![Graph (d)](image4)

2. Find all values of $x$ for which the curve $y = 2x^2 - x$ is decreasing.

3. Find the domain over which the function $f(x) = 4 - x^2$ is increasing.

4. Find values of $x$ for which the curve $y = x^2 - 3x - 4$ is
   (a) decreasing
   (b) increasing
   (c) stationary.

5. Show that the function $f(x) = -2x - 7$ is always (monotonic) decreasing.
6. Prove that \( y = x^3 \) is monotonic increasing for all \( x \neq 0 \).

7. Find the stationary point on the curve \( f(x) = x^3 \).

8. Find all \( x \) values for which the curve \( y = 2x^3 + 3x^2 - 36x + 9 \) is stationary.

9. Find all stationary points on the curve
   (a) \( y = x^2 - 2x - 3 \)
   (b) \( f(x) = 9 - x^2 \)
   (c) \( y = 2x^3 - 9x^2 + 12x - 4 \)
   (d) \( y = x^4 - 2x^2 + 1 \).

10. Find any stationary points on the curve \( y = (x - 2)^4 \).

11. Find all values of \( x \) for which the curve \( f(x) = x^3 - 3x + 4 \) is decreasing.

12. Find the domain over which the curve \( y = x^4 + 12x^2 + 45x - 30 \) is increasing.

13. Find any values of \( x \) for which the curve \( y = 2x^3 - 21x^2 + 60x - 3 \) is stationary
   (a) stationary
   (b) decreasing
   (c) increasing.

14. The function \( f(x) = 2x^2 + px + 7 \) has a stationary point at \( x = 3 \). Evaluate \( p \).

15. Evaluate \( a \) and \( b \) if \( y = x^3 - ax^2 + bx - 3 \) has stationary points at \( x = -1 \) and \( x = 2 \).

16. (a) Find the derivative of \( y = x^3 - 3x^2 + 27x - 3 \).
   (b) Show that the curve is monotonic increasing for all values of \( x \).

17. Sketch a function with \( f''(x) > 0 \) for \( x < 2 \), \( f''(2) = 0 \) and \( f''(x) < 0 \) when \( x > 2 \).

18. Draw a sketch showing a curve with \( \frac{dy}{dx} < 0 \) for \( x < 4 \), \( \frac{dy}{dx} = 0 \) when \( x = 4 \) and \( \frac{dy}{dx} > 0 \) for \( x > 4 \).

19. Sketch a curve with \( \frac{dy}{dx} > 0 \) for all \( x \neq 1 \) and \( \frac{dy}{dx} = 0 \) when \( x = 1 \).

20. Draw a sketch of a function that has \( f''(x) > 0 \) for \( x < -2 \), \( x > 5 \), \( f'(x) = 0 \) for \( x = -2, 5 \) and \( f''(x) < 0 \) for \( -2 < x < 5 \).

21. A function has \( f(3) = 2 \) and \( f''(3) < 0 \). Show this information on a sketch.

22. The derivative is positive at the point \((-2, -1)\). Show this information on a graph.

23. Find the stationary points on the curve \( y = (3x - 1)(x - 2)^4 \).

24. Differentiate \( y = x\sqrt{x + 1} \). Hence find the stationary point on the curve, giving the exact value.

25. The curve \( f(x) = ax^4 - 2x^3 + 7x^2 - x + 5 \) has a stationary point at \( x = 1 \). Find the value of \( a \).

26. Show that \( f(x) = \sqrt{x} \) has no stationary points.

27. Show that \( f(x) = \frac{1}{x^3} \) has no stationary points.
Types of Stationary Points

There are three types of stationary points.

**Local minimum point**

There are three types of stationary points. The curve is decreasing on the left and increasing on the right of the minimum turning point.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$LHS$</th>
<th>$Minimum$</th>
<th>$RHS$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$&lt; 0$</td>
<td>$0$</td>
<td>$&gt; 0$</td>
</tr>
</tbody>
</table>

**Local maximum point**

The curve is increasing on the left and decreasing on the right of the maximum turning point.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$LHS$</th>
<th>$Maximum$</th>
<th>$RHS$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>$&gt; 0$</td>
<td>$0$</td>
<td>$&lt; 0$</td>
</tr>
</tbody>
</table>

Local maximum and minimum points are also called turning points, as the curve turns around at these points. They can also be called relative maxima or minima.

**Point of horizontal inflexion**

The curve is either increasing on both sides of the inflexion or it is decreasing on both sides. It is not called a turning point as the curve does not turn around at this point.
The stationary points are important to the shape of a curve. A reasonably accurate sketch of the curve can be made by finding these points, together with the intercepts on the axes if possible.

### EXAMPLES

1. Find the stationary point on the curve \( y = x^3 \) and determine which type it is.

**Solution**

\[
\frac{dy}{dx} = 3x^2
\]

For stationary points, \( \frac{dy}{dx} = 0 \)

i.e. \( 3x^2 = 0 \)

\( x = 0 \)

When \( x = 0, y = 0^3 = 0 \)

So the stationary point is \((0, 0)\).

To determine its type, check the curve on the LHS and the RHS.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dy}{dx} )</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Since the curve is increasing on both sides, \((0, 0)\) is a point of inflexion.
2. Find any stationary points on the curve \( f(x) = 2x^3 - 15x^2 + 24x - 7 \) and distinguish between them.

**Solution**

\[ f'(x) = 6x^2 - 30x + 24 \]

For stationary points, \( f'(x) = 0 \)

i.e.

\[ 6x^2 - 30x + 24 = 0 \]
\[ x^2 - 5x + 4 = 0 \]
\[ (x - 1)(x - 4) = 0 \]

\[ \therefore \quad x = 1 \text{ or } 4 \]

\[ f(1) = 2(1)^3 - 15(1)^2 + 24(1) - 7 = 4 \]

So \((1, 4)\) is a stationary point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(x) )</td>
<td>24</td>
<td>0</td>
<td>-12</td>
</tr>
</tbody>
</table>

\[ \therefore (1, 4) \text{ is a maximum stationary point} \]

\[ f(4) = 2(4)^3 - 15(4)^2 + 24(4) - 7 = -23 \]

So \((4, -23)\) is a stationary point.

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f''(x) )</td>
<td>-12</td>
<td>0</td>
<td>24</td>
</tr>
</tbody>
</table>

\[ \therefore (4, -23) \text{ is a minimum stationary point.} \]
1. Find the stationary point on the curve \( y = x^2 - 1 \) and show that it is a minimum point by checking the derivative on both sides of it.

2. Find the stationary point on the curve \( y = x^4 \) and determine its type.

3. Find the stationary point on the curve \( y = x^3 + 2 \) and determine its nature.

4. The function \( f(x) = 7 - 4x - x^2 \) has one stationary point. Find its coordinates and show that it is a maximum turning point.

5. Find the turning point on the curve \( y = 3x^2 + 6x + 1 \) and determine its nature.

6. For the curve \( y = (4 - x)^2 \) find the turning point and determine its nature.

7. The curve \( y = x^4 - 6x^2 + 5 \) has 2 turning points. Find them and use the derivative to determine their nature.

8. Show that the curve \( f(x) = x^5 + 1 \) has a point of inflexion at \((0, 1)\).

9. Find the turning points on the curve \( y = x^3 - 3x^2 + 5 \) and determine their nature.

10. Find any stationary points on the curve \( f(x) = x^4 - 2x^2 - 3 \). What type of stationary points are they?

11. The curve \( y = x^3 - 3x + 2 \) has 2 stationary points. Find their coordinates and determine their type.

12. The curve \( y = x^3 + mx^2 - 7x + 5 \) has a stationary point at \( x = -2 \). Find the value of \( m \).

13. For a certain function, \( f''(x) = 3 + x \). For what value of \( x \) does the function have a stationary point? What type of stationary point is it?

14. A curve has \( f''(x) = x(x + 1) \). For what \( x \) values does the curve have stationary points? What type are they?

15. For a certain curve, \( \frac{dy}{dx} = (x - 1)^2(x - 2) \). Find the \( x \) values of its stationary points and determine their nature.

16. (a) Differentiate \( P = 2x + \frac{50}{x} \) with respect to \( x \).

(b) Find any stationary points on the curve and determine their nature.

17. For the function \( A = \frac{h^2 - 2h + 5}{8} \), find any stationary points and determine their nature.

18. Find any stationary points for the function \( V = 40r - \pi r^4 \) and determine their nature (correct to 2 decimal places).

19. Find any stationary points on the curve \( S = 2\pi + \frac{120}{r} \) (correct to 2 decimal places) and determine their nature.

20. (a) Differentiate \( A = x\sqrt{3600 - x^2} \).

(b) Find any stationary points for \( A = x\sqrt{3600 - x^2} \) (to 1 decimal place) and determine their nature.
Higher Derivatives

A function can be differentiated several times:

- differentiating \( f(x) \) gives \( f'(x) \)
- differentiating \( f'(x) \) gives \( f''(x) \)
- differentiating \( f''(x) \) gives \( f'''(x) \), and so on
- the other notation is \( \frac{dy}{dx} \), \( \frac{d^2y}{dx^2} \), and so on

**EXAMPLES**

1. Find the first 4 derivatives of \( f(x) = x^3 - 4x^2 + 3x - 2 \).

   **Solution**
   
   \[
   \begin{align*}
   f'(x) &= 3x^2 - 8x + 3 \\
   f''(x) &= 6x - 8 \\
   f'''(x) &= 6 \\
   f''''(x) &= 0
   \end{align*}
   \]

2. Find the second derivative of \( y = (2x + 5)^7 \).

   **Solution**
   
   \[
   \begin{align*}
   \frac{dy}{dx} &= 7(2x + 5)^6 \cdot 2 \\
   &= 14(2x + 5)^6 \\
   \frac{d^2y}{dx^2} &= 14 \cdot 6(2x + 5)^5 \cdot 2 \\
   &= 168(2x + 5)^5
   \end{align*}
   \]

3. Find \( f''(-1) \) if \( f(x) = x^4 - 1 \)

   **Solution**
   
   \[
   \begin{align*}
   f'(x) &= 4x^3 \\
   f''(x) &= 12x^2 \\
   \therefore f''(-1) &= 12(-1)^2 \\
   &= 12
   \end{align*}
   \]

Only the first 2 derivatives are used in sketching graphs of curves.
2.3 Exercises

1. Find the first 4 derivatives of $x^7 - 2x^3 + x^4 - x - 3$.

2. If $f(x) = x^9 - 5$, find $f''''(x)$.

3. Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^3 - x^3 + 1$.

4. Find $f'(1)$ and $f''(-2)$, given $f(t) = 3t^4 - 2t^3 + 5t - 4$.

5. Find the first 3 derivatives of $x^7 - 2x^6 + 4x^4 - 7$.

6. Differentiate $y = 2x^2 - 3x + 3$ twice.

7. If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$.

8. Differentiate $x^4$ twice.

9. If $g(x) = \sqrt{x}$, find $g''(4)$.

10. Given $h = 5t^3 - 2t^2 + t + 5$, find $\frac{d^2h}{dt^2}$ when $t = 1$.

11. Find the value of $x$ for which $\frac{d^2y}{dx^2} = 3$ given $y = 3x^4 - 2x^2 + 5x$.

12. Find all values of $x$ for which $f''''(x) > 0$ given that $f(x) = x^3 - x^2 + x + 9$.

13. Differentiate $(4x - 3)^6$ twice.

14. Find $f'(x)$ and $f''''(x)$ if $f(x) = \sqrt{2 - x}$.

15. Find the first and second derivatives of $f(x) = \frac{x + 5}{3x - 1}$.

16. Find $\frac{dv}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

17. Find the value of $b$ in $y = bx^3 - 2x^2 + 5x + 4$ if $\frac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$.

18. Find the exact value of $f''(2)$ if $f(x) = x\sqrt{3x - 4}$.

19. Find $f''(1)$ if $f(t) = t(2t - 1)^7$.

20. Find the value of $b$ if $f(x) = 5bx^2 - 4x^3$ and $f''''(-1) = -3$.

Sign of the Second Derivative

The second derivative gives extra information about a curve that helps us to find its shape.

Since $f''(x)$ is the derivative of $f'(x)$, then $f''''(x)$ and $f''(x)$ have the same relationship as $f'(x)$ and $f(x)$.

That is if $f''(x) > 0$ then $f'(x)$ is increasing

if $f''(x) < 0$ then $f'(x)$ is decreasing

if $f''(x) = 0$ then $f'(x)$ is stationary
Concavity

If \( f''(x) > 0 \) then \( f'(x) \) is increasing. This means that the gradient of the tangent is increasing, that is, the curve is becoming steeper.

Notice the upward shape of these curves. The curve lies above the tangents. We say that the curve is **concave upwards**.

If \( f''(x) < 0 \) then \( f'(x) \) is decreasing. This means that the gradient of the tangent is decreasing. That is, the curve is becoming less steep.

Notice the downward shape of these curves. The curve lies below the tangents. We say that the curve is **concave downwards**.

If \( f''(x) = 0 \) then \( f'(x) \) is stationary. That is, it is neither increasing nor decreasing. This happens when the curve goes from being concave upwards to concave downwards, or when the curve changes from concave downwards to concave upwards. We say that the curve is **changing concavity**.

The curve has a point of inflexion as long as **concavity changes**.
Class Investigation

How would you check that concavity changes?

- If \( f''(x) > 0 \), the curve is **concave upwards**.
- If \( f''(x) < 0 \), the curve is **concave downwards**.
- If \( f''(x) = 0 \) and concavity changes, there is a **point of inflexion**.

**EXAMPLES**

1. Does the curve \( y = x^4 \) have a point of inflexion?

   **Solution**

   \[
   \frac{dy}{dx} = 4x^3 \\
   \frac{d^2y}{dx^2} = 12x^2
   \]

   For inflexions, \( \frac{d^2y}{dx^2} = 0 \)
   
   i.e. \( 12x^2 = 0 \)
   
   \( x = 0 \)

   When \( x = 0 \), \( y = 0^4 \)
   
   \( = 0 \)

   So \( (0, 0) \) is a possible point of inflexion.

   Check that concavity changes:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d^2y}{dx^2} )</td>
<td>12</td>
<td>0</td>
<td>12</td>
</tr>
</tbody>
</table>

   Since concavity doesn’t change, \( (0, 0) \) is not a point of inflexion.
2. Find all values of \( x \) for which the curve \( f(x) = 2x^3 - 7x^2 - 5x + 4 \) is concave downwards.

**Solution**

\[
\begin{align*}
    f'(x) &= 6x^2 - 14x - 5 \\
    f''(x) &= 12x - 14
\end{align*}
\]

For concave downwards, \( f''(x) < 0 \)

\[
\begin{align*}
    12x - 14 &< 0 \\
    12x &< 14 \\
    x &< 1\frac{1}{6}
\end{align*}
\]

\[\therefore \quad x < 1\frac{1}{6}\]

3. Find the point of inflexion on the curve \( y = x^3 - 6x^2 + 5x + 9 \).

**Solution**

\[
\begin{align*}
    y' &= 3x^2 - 12x + 5 \\
    y'' &= 6x - 12
\end{align*}
\]

For inflexions, \( y'' = 0 \)

\[
\begin{align*}
    6x - 12 &= 0 \\
    6x &= 12 \\
    x &= 2
\end{align*}
\]

\[\therefore \quad x = 2\]

When \( x = 2 \), \( y = 2^3 - 6(2)^2 + 5(2) + 9 = 3\)

\[\therefore \quad (2, 3) \text{ is a possible point of inflexion}\]

Check that concavity changes:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d^2y}{dx^2} )</td>
<td>-6</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

Since concavity changes, \( (2, 3) \) is a point of inflexion.

### 2.4 Exercises

1. For what values of \( x \) is the curve \( y = x^4 + x^2 - 2x - 1 \) concave upwards?

2. Find all values of \( x \) for which the curve \( y = (x - 3)^3 \) is concave downwards.

3. Prove that the curve \( y = 8 - 6x - 4x^2 \) is always concave downwards.

4. Show that the curve \( y = x^2 \) is always concave upwards.

5. Find the domain over which the curve \( f(x) = x^3 - 7x^2 + 1 \) is concave downwards.

6. Find any points of inflexion on the curve \( g(x) = x^3 - 3x^2 + 2x + 9 \).

7. Find the points of inflexion on the curve \( y = x^4 - 6x^2 + 12x - 24 \).
8. Find the stationary point on the curve \( y = x^3 - 2 \). Show that it is an inflexion.

9. Find all values of \( x \) for which the function \( f(x) = x^4 + 2x^3 - 12x^2 + 12x - 1 \) is concave downwards.

10. Determine whether there are any points of inflexion on the curve
(a) \( y = x^6 \)
(b) \( y = x^7 \)
(c) \( y = x^4 \)
(d) \( y = x^3 \)
(e) \( y = x^{12} \)

11. Sketch a curve that is always concave up.

12. Sketch a curve where \( f''(x) < 0 \) for \( x > 1 \) and \( f''(x) > 0 \) for \( x < 1 \).

13. Find any points of inflexion on the curve \( y = x^4 - 8x^3 + 24x^2 - 4x - 9 \).

14. Show that \( f(x) = \frac{2}{x^2} \) is concave upwards for all \( x \neq 0 \).

15. For the function \( f(x) = 3x^5 - 10x^3 + 7 \)
(a) Find any points of inflexion.

(b) Find which of these points are horizontal points of inflexion (stationary points).

16. (a) Show that the curve \( y = x^4 + 12x^2 - 20x + 3 \) has no points of inflexion.
(b) Describe the concavity of the curve.

17. If \( y = ax^3 - 12x^2 + 3x - 5 \) has a point of inflexion at \( x = 2 \), evaluate \( a \).

18. Evaluate \( p \) if \( f(x) = x^4 - 6px^2 - 20x + 11 \) has a point of inflexion at \( x = -2 \).

19. The curve \( y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3 \) has points of inflexion at \( x = 2 \) and \( x = -1 \). Find the values of \( a \) and \( b \).

20. The curve \( y = x^6 - 3x^4 + 21x - 8 \) has two points of inflexion.
(a) Find these points.
(b) Show that these points of inflexion are not stationary points.

If we combine the information from the first and second derivatives, this will tell us about the shape of the curve.

**EXAMPLES**

1. For a particular curve, \( f(2) = -1 \), \( f'(2) > 0 \) and \( f''(2) < 0 \). Sketch the curve at this point, showing its shape.

**Solution**

\( f(2) = -1 \) means that the point \((2, -1)\) lies on the curve.

If \( f'(2) > 0 \), the curve is increasing at this point.

If \( f''(2) < 0 \), the curve is concave downwards at this point.
2. The curve below shows the number of unemployed people $P$ over time $t$ months.

(a) Describe the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$.

(b) How is the number of unemployed people changing over time?

(c) Is the unemployment rate increasing or decreasing?

**Solution**

(a) The curve is decreasing, so $\frac{dP}{dt} < 0$ and the curve is concave upwards, so $\frac{d^2P}{dt^2} > 0$.

(b) As the curve is decreasing, the number of unemployed people is decreasing.

(c) Since the curve is concave upwards, the gradient is increasing. This means that the unemployment rate is increasing.
2.5 Exercises

1. For each curve, describe the sign of \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \).

   (a) 
   ![Curve A]

   (b) 
   ![Curve B]

   (c) 
   ![Curve C]

   (d) 
   ![Curve D]

   (e) 
   ![Curve E]

2. The curve below shows the population of a colony of seals.

   (a) Describe the sign of the first and second derivatives.
   (b) How is the population rate changing?

3. Inflation is increasing, but the rate of increase is slowing. Draw a graph to show this trend.

4. Draw a sketch to show the shape of each curve:
   (a) \( f'(x) < 0 \) and \( f''(x) < 0 \)
   (b) \( f'(x) > 0 \) and \( f''(x) < 0 \)
   (c) \( f'(x) < 0 \) and \( f''(x) > 0 \)
   (d) \( f'(x) > 0 \) and \( f''(x) > 0 \)

5. The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing. Draw a graph to show this.

6. As an iceblock melts, the rate at which it melts increases. Draw a graph to show this information.

7. The graph shows the decay of a radioactive substance.

Describe the sign of \( \frac{dM}{dt} \) and \( \frac{d^2M}{dt^2} \).
8. The population $P$ of fish in a certain lake was studied over time, and at the start the number of fish was 2500.

(a) During the study, $\frac{dP}{dt} < 0$. What does this say about the number of fish during the study?

(b) If at the same time, $\frac{d^2P}{dt^2} > 0$, what can you say about the population rate?

(c) Sketch the graph of the population $P$ against $t$.

9. The graph shows the level of education of youths in a certain rural area over the past 100 years.

Describe how the level of education has changed over this period of time. Include mention of the rate of change.

10. The graph shows the number of students in a high school over several years.

Describe how the school population is changing over time, including the rate of change.

Here is a summary of the shape of a curve given the first and second derivatives.

<table>
<thead>
<tr>
<th>$\frac{dy}{dx} &gt; 0$</th>
<th>$\frac{dy}{dx} &lt; 0$</th>
<th>$\frac{dy}{dx} = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{d^2y}{dx^2} &gt; 0$</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>$\frac{d^2y}{dx^2} &lt; 0$</td>
<td><img src="image3" alt="Graph" /></td>
<td><img src="image4" alt="Graph" /></td>
</tr>
<tr>
<td>$\frac{d^2y}{dx^2} = 0$</td>
<td><img src="image5" alt="Graph" /></td>
<td><img src="image6" alt="Graph" /></td>
</tr>
</tbody>
</table>
Determining Types of Stationary Points

We can determine the type of a stationary point by looking at the first and second derivatives together.

- If \( f'(x) = 0 \) and \( f''(x) > 0 \), there is a minimum turning point.

- If \( f'(x) = 0 \) and \( f''(x) < 0 \), there is a maximum turning point.

- If \( f'(x) = 0, f''(x) = 0 \) and concavity changes, then there is a horizontal point of inflexion.

Class Investigation

There are three mistakes in this argument. Can you find all of them?

\[
y = 2\sqrt{x} = 2x^{\frac{1}{2}}
\]

\[
dy \over dx = 2\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2\sqrt{x}}
\]

For stationary points, \( \frac{dy}{dx} = 0 \)

i.e.

\[
\frac{1}{2\sqrt{x}} = 0
\]

\[
1 = \sqrt{x}
\]

\[
\pm 1 = x
\]

So there are stationary points at \( x = 1 \) and \( x = -1 \).
EXAMPLES

1. Find the stationary points on the curve \( f(x) = 2x^3 - 3x^2 - 12x + 7 \) and distinguish between them.

**Solution**

\[
 f'(x) = 6x^2 - 6x - 12 \\
\text{For stationary points, } f''(x) = 0 \\
\text{i.e. } 6x^2 - 6x - 12 = 0 \\
\quad x^2 - x - 2 = 0 \\
\quad (x - 2)(x + 1) = 0 \\
\therefore \quad x = 2 \text{ or } -1 \\
\]

\[
 f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 7 = -13 \\
\text{So } (2, -13) \text{ is a stationary point.} \\
 f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 7 = 14 \\
\text{So } (-1, 14) \text{ is a stationary point.} \\
\]

Now \( f''(x) = 12x - 6 \)

\[
 f''(2) = 12(2) - 6 = 18 \\
> 0 \quad \text{(concave upwards)} \\
\]

So \((2, -13) \) is a minimum turning point.

\[
 f''(-1) = 12(-1) - 6 = -18 \\
< 0 \quad \text{(concave downwards)} \\
\]

So \((-1, 14) \) is a maximum turning point.

2. Find the stationary point on the curve \( y = 2x^3 - 3 \) and determine its nature.

**Solution**

\[
 y' = 10x^4 \\
\text{For stationary points, } y' = 0 \\
\text{i.e. } 10x^4 = 0 \\
\therefore \quad x = 0 
\]
When \( x = 0 \), \( y = 2(0)^3 - 3 \)
\[ = -3 \]
So \((0, -3)\) is a stationary point.

Now \( y'' = 40x^3 \)
When \( x = 0 \), \( y'' = 40(0)^3 \)
\[ = 0 \]
So \((0, -3)\) is a possible point of inflexion.

Check concavity on the LHS and RHS:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d^2y}{dx^2} )</td>
<td>-40</td>
<td>0</td>
<td>40</td>
</tr>
</tbody>
</table>

Since concavity changes, \((0, -3)\) is a horizontal point of inflexion.

### 2.6 Exercises

1. Find the stationary point on the curve \( y = x^2 - 2x + 1 \) and determine its nature.

2. Find the stationary point on the curve \( y = 3x^4 + 1 \) and determine what type of point it is.

3. Find the stationary point on the curve \( y = 3x^2 - 12x + 7 \) and show that it is a minimum point.

4. Determine the stationary point on the curve \( y = x - x^2 \) and show that it is a maximum turning point.

5. Show that the curve \( f(x) = 2x^3 - 5 \) has an inflexion and find its coordinates.

6. Does the function \( f(x) = 2x^3 + 3 \) have a stationary point? If it does, determine its nature.

7. Find any stationary points on the curve \( f(x) = 2x^3 + 15x^2 + 36x - 50 \) and determine their nature.

8. Find the stationary points on the curve \( y = 3x^4 - 4x^3 - 12x^2 + 1 \) and determine whether they are maximum or minimum turning points.

9. Find any stationary points on the curve \( y = (4x^2 - 1)^3 \) and determine their nature.

10. (a) Find any stationary points on the curve \( y = 2x^3 - 27x^2 + 120x \) and distinguish between them.
    (b) Find any points of inflexion on the curve.

11. Find any stationary points on the curve \( y = (x - 3)^4 - 4 - x \) and determine their nature.

12. Find any stationary points on the curve \( y = x^4 + 8x^3 + 16x^2 - 1 \) and determine their nature.
13. The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = -3$.
   (a) Find the value of $a$.
   (b) Hence, or otherwise, determine the nature of the stationary point.

14. The curve $y = x^3 - mx^2 + 8x - 7$ has a stationary point where $x = -1$. Find the value of $m$.

15. The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflexion at $(1, -2)$. Find the values of $a$ and $b$.

**Curve Sketching**

We can sketch curves by finding all of their important features, such as stationary points, points of inflexion and intercepts. Here is a summary of strategies for sketching a curve.

1. Find **stationary points** $(y' = 0)$.
2. Find **points of inflexion** $(y'' = 0)$.
3. Find intercepts on axes.  
   - For $x$-intercept, $y = 0$
   - For $y$-intercept, $x = 0$
4. Find **domain** and **range**.
5. Find any **asymptotes** or **limits**.
6. Use **symmetry**, odd or even functions.
7. Draw up a **table of values**.

**EXAMPLES**

1. Find any stationary points and points of inflexion on the curve $f(x) = x^3 - 3x^2 - 9x + 1$ and hence sketch the curve.

   **Solution**

   \[
   f'(x) = 3x^2 - 6x - 9 \\
   \text{and } f''(x) = 6x - 6
   \]

   First, find the stationary points.

   For stationary points, $f'(x) = 0$

   i.e. \[3x^2 - 6x - 9 = 0\]
   \[x^2 - 2x - 3 = 0\]
   \[(x - 3)(x + 1) = 0\]
   \[
   \therefore \quad x = -1 \text{ or } 3
   \]
\[ f(3) = (3)^3 - 3(3)^2 - 9(3) + 1 \]
\[ = -26 \]
So \((3, -26)\) is a stationary point.

\[ f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 1 \]
\[ = 6 \]
So \((-1, 6)\) is a stationary point.

We use the second derivative to determine their type.

\[ f''(3) = 6(3) - 6 \]
\[ = 12 \]
> 0 \quad \text{(concave upwards)}

\[ \therefore (3, -26) \text{ is a minimum turning point} \]

\[ f''(-1) = 6(-1) - 6 \]
\[ = -12 \]
< 0 \quad \text{(concave downwards)}

\[ \therefore (-1, 6) \text{ is a maximum turning point} \]

Next, find any points of inflexion.

For inflexions, \(f''(x) = 0\)

i.e. \[ 6x - 6 = 0 \]
\[ 6x = 6 \]
\[ x = 1 \]

\[ f(1) = 1^3 - 3(1)^2 - 9(1) + 1 \]
\[ = -10 \]

\[
\begin{array}{|c|c|c|c|}
\hline
x & 0 & 1 & 2 \\
\hline
f''(x) & -6 & 0 & 6 \\
\hline
\end{array}
\]

Since concavity changes, \((1, -10)\) is a point of inflexion.

Next, try to find intercepts on the axes.

For \(y\)-intercept, \(x = 0\):

\[ f(0) = 0^3 - 3(0)^2 - 9(0) + 1 \]
\[ = 1 \]

For \(x\)-intercept, \(y = 0\):

i.e. \(x^3 - 3x^2 - 9x + 1 = 0\)

This is too hard to solve.
Now sketch the graph using an appropriate scale so that all stationary points and points of inflexion fit on the graph.

2. Sketch the curve $y = 2x^3 + 1$, showing any important features.

**Solution**

\[
\frac{dy}{dx} = 6x^2
\]

For stationary points, \(\frac{dy}{dx} = 0\)

i.e. \[6x^2 = 0\]

When \(x = 0\), \(y = 2(0)^3 + 1 = 1\)

So \((0, 1)\) is a stationary point.

\[
\frac{d^2y}{dx^2} = 12x
\]

At \((0, 1)\) \(\frac{d^2y}{dx^2} = 12(0) = 0\)

So \((0, 1)\) is a possible point of inflexion.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{d^2y}{dx^2})</td>
<td>(-12)</td>
<td>0</td>
<td>12</td>
</tr>
</tbody>
</table>

Since concavity changes, \((0, 1)\) is a horizontal inflexion.

For \(x\)-intercept, \(y = 0\)

i.e. \[2x^3 + 1 = 0\]

\[2x^3 = -1\]

\[x^3 = -0.5\]

\[x = -0.8\]

For \(y\)-intercept, \(x = 0\)
We already know this point. It is \((0, 1)\), the inflexion. We can also find a point on either side of the inflexion.

\[
\begin{align*}
\text{When } x = -1, \quad y &= 2(-1)^3 + 1 \\
&= -1 \\
\text{When } x = 1, \quad y &= 2(1)^3 + 1 \\
&= 3
\end{align*}
\]

2.7 Exercises

1. Find the stationary point on the curve \(f(x) = x^2 - 3x - 4\) and determine its type. Find the intercepts on the axes and sketch the curve.

2. Sketch \(y = 6 - 2x - x^2\), showing the stationary point.

3. Find the stationary point on the curve \(y = (x - 1)^3\) and determine its nature. Hence sketch the curve.

4. Sketch \(y = x^4 + 3\), showing any stationary points.

5. Find the stationary point on the curve \(y = x^3\) and show that it is a point of inflexion. Hence sketch the curve.

6. Sketch \(f(x) = x^2\).

7. Find any stationary points on the curve \(y = 2x^3 - 9x^2 - 24x + 30\) and sketch its graph.

8. (a) Determine any stationary points on the curve \(y = x^3 + 6x^2 - 7\).
   (b) Find any points of inflexion on the curve.
   (c) Sketch the curve.

9. Find any stationary points and inflexions on the curve \(y = x^3 - 6x^2 + 3\) and hence sketch the curve.

10. Find any stationary points and inflexions on the curve \(y = 2 + 9x - 3x^2 - x^3\). Hence sketch the curve.

11. Sketch the function \(f(x) = 3x^4 + 4x^3 - 12x^2 - 1\), showing all stationary points.

12. Find the stationary points on the curve \(y = (x - 4)(x + 2)^2\) and hence sketch the curve.
13. Find all stationary points and inflexions on the curve 
\( y = (2x + 1)(x - 2)^4 \). Sketch the curve.

14. Show that the curve 
\( y = \frac{2}{1 + x} \)
has no stationary points. By considering the domain and range of the function, sketch the curve.

15. Find any stationary points on the curve 
\( y = \frac{x^2}{x - 2} \). By also considering the domain of the curve, sketch its graph.

**Maximum and Minimum Values**

A curve may have maximum and minimum turning points, but they are not necessarily the maximum or minimum values of the function.

**EXAMPLE**

This curve has a maximum turning point at \((-1, 6)\). We call it a relative maximum point, since it does not give the maximum value of the graph. Similarly, the curve has a relative minimum turning point at \((3, -26)\). The curve does not have any absolute maximum or minimum value since it increases to infinity and decreases to negative infinity.

If we restrict the domain of the curve to, say, \(-4 \leq x \leq 4\), then the curve will have an absolute maximum and minimum value. The absolute maximum is the greatest value of the curve in the domain. The absolute minimum is the least value of the curve in the domain.

In order to find the maximum or minimum value of a curve, find the values of the function at the endpoints of the domain as well as its turning points.
When \( x = -4 \), \( y = (-4)^3 - 3(-4)^2 - 9(-4) + 1 \\
= -75 \\
When \( x = 4 \), \( y = 4^3 - 3(4)^2 - 9(4) + 1 \\
= -19 \\

By restricting the curve to \(-4 \leq x \leq 4\), the maximum value is 6 and the minimum value is \(-75\).

**EXAMPLES**

1. Find the maximum and minimum values of \( y \) for the function \( f(x) = x^2 - 4x + 3 \) in the domain \(-4 \leq x \leq 3\).

**Solution**

\( f'(x) = 2x - 4 \)

For stationary points, \( f'(x) = 0 \)

i.e. \( 2x - 4 = 0 \)

\( 2x = 4 \)

\( x = 2 \)

\( f(2) = 2^2 - 4(2) + 3 \)

\( = -1 \)

So \((2, -1)\) is a stationary point.

\( f''(x) = 2 \)

\( > 0 \) \hspace{1cm} (concave upwards)

\( \therefore (2, -1) \) is a minimum turning point
Endpoints: \( f(-4) = (-4)^2 - 4(-4) + 3 \)
\[ = 35 \]
\( f(3) = 3^2 - 4(3) + 3 \)
\[ = 0 \]

The maximum value of \( y \) is 35 and the minimum value is -1 in the domain \(-4 \leq x \leq 3\).

2. Find the maximum and minimum values of the curve \( y = x^4 - 2x^2 + 1 \) for \(-2 \leq x \leq 3\).

**Solution**

\( y' = 4x^3 - 4x \)

For stationary points, \( y' = 0 \)

i.e. \( 4x^3 - 4x = 0 \)

\( 4x(x^2 - 1) = 0 \)

\( 4x(x + 1)(x - 1) = 0 \)

\( \therefore \ x = 0 \) or \( \pm 1 \)

When \( x = 1 \), \( y = 1^4 - 2(1)^2 + 1 \)
\[ = 0 \]

So \((1, 0)\) is a stationary point.

When \( x = -1 \), \( y = (-1)^4 - 2(-1)^2 + 1 \)
\[ = 0 \]

So \((-1, 0)\) is a stationary point.

When \( x = 0 \), \( y = (0)^4 - 2(0)^2 + 1 \)
\[ = 1 \]

So \((0, 1)\) is a stationary point.

Now \( y'' = 12x^2 - 4 \)

When \( x = 1 \), \( y'' = 12(1)^2 - 4 \)
\[ = 8 \]
\[ > 0 \] (concave upwards)
So \((1, 0)\) is a minimum turning point.
When \(x = -1\), \(y'' = 12(-1)^2 - 4 = 8\)
\(> 0\) (concave upwards)
So \((-1, 0)\) is a minimum turning point.
When \(x = 0\), \(y'' = 12(0)^2 - 4 = -4\)
\(< 0\) (concave downwards)
So \((0, 1)\) is a maximum turning point.
Endpoints: When \(x = -2\), \(y = (-2)^4 - 2(-2)^2 + 1 = 9\)
When \(x = 3\), \(y = 3^4 - 2(3)^2 + 1 = 64\)

The maximum value is 64 and the minimum value is 0 in the domain \(-2 \leq x \leq 3\).

### 2.8 Exercises

1. Sketch \(y = x^2 + x - 2\) for \(-2 \leq x \leq 2\) and find the maximum value of \(y\).

2. Sketch \(f(x) = 9 - x^2\) over the domain \(-4 \leq x \leq 2\). Hence find the maximum and minimum values of the curve over this domain.

3. Find the maximum value of the curve \(y = x^2 - 4x + 4\) for \(-3 \leq x \leq 3\).

4. Sketch \(f(x) = 2x^3 + 3x^2 - 36x + 5\) for \(-3 \leq x \leq 3\), showing any stationary points. Find the maximum and minimum values of the function for \(-3 \leq x \leq 3\).
5. Find the maximum value of \( y \) for the curve \( y = x^3 - 3 \) for \(-2 \leq x \leq 1\).

6. Sketch the curve 
\[ f(x) = 3x^2 - 16x + 5 \text{ for } 0 \leq x \leq 4 \]
and find the maximum and minimum values of the function over this domain.

7. Find the relative and absolute maximum and minimum values of the function 
\[ f(x) = 3x^4 + 4x^3 - 12x^2 - 3 \text{ for } -2 \leq x \leq 2. \]

8. Sketch \( y = x^3 + 2 \) over the domain \(-3 \leq x \leq 3\) and find its minimum and maximum values in that domain.

9. Sketch \( y = \sqrt{x + 5} \) for \(-4 \leq x \leq 4\) and find its maximum and minimum values.

10. Show that \( y = \frac{1}{x - 2} \) has no stationary points. Find the maximum and minimum values of the curve for \(-3 \leq x \leq 3\).

### Problems Involving Maxima and Minima

Often a problem involves finding maximum or minimum values. For example, a salesperson wants to maximise profit; a warehouse manager wants to maximise storage; a driver wants to minimise petrol consumption; a farmer wants to maximise paddock size.

**PROBLEM**

One disc 20 cm in diameter and one 10 cm in diameter are cut from a disc of cardboard 30 cm in diameter. Can you find the largest disc that can be cut from the remainder of the cardboard?

**EXAMPLES**

1. The equation for the expense per year (in units of ten thousand dollars) of running a certain business is given by \( E = x^2 - 6x + 12 \), where \( x \) is the number (in hundreds) of items manufactured.
   (a) Find the expense of running the business if no items are manufactured.
   (b) Find the number of items needed to minimise the expense of the business.
   (c) Find the minimum expense of the business.
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Solution

(a) When \( x = 0, \ E = 0^2 - 6(0) + 12 \)
\[ = 12 \]
So the expense of running the business when no items are manufactured is \( 12 \times 10000 \), or \$120 000 per year.

(b) \( \frac{dE}{dx} = 2x - 6 \)

For stationary points, \( \frac{dE}{dx} = 0 \)

i.e. \( 2x - 6 = 0 \)
\[ x = 3 \]
\( \frac{d^2E}{dx^2} = 2 > 0 \) (concave upwards)

\( \therefore x = 3 \) gives a minimum value

So 300 items manufactured each year will give the minimum expenses.

(c) When \( x = 3, \ E = 3^2 - 6(3) + 12 \)
\[ = 3 \]
So the minimum expense per year is \$30 000.

2. The formula for the volume of a pond is given by
\[ V = 2h^3 - 12h^2 + 18h + 50, \] where \( h \) is the depth of the pond in metres.
Find the maximum volume of the pond.

Solution

\( V' = 6h^2 - 24h + 18 \)

For stationary points, \( V' = 0 \)

i.e. \( 6h^2 - 24h + 18 = 0 \)
\[ h^2 - 4h + 3 = 0 \]
\( (h - 1)(h - 3) = 0 \)

So \( h = 1 \) or 3

\( V'' = 12h - 24 \)

When \( h = 1, \ V'' = 12(1) - 24 \)
\[ = -12 \]
\[ < 0 \] (concave downwards)

So \( h = 1 \) gives a maximum \( V \).

When \( h = 3, \ V'' = 12(3) - 24 \)
\[ = 12 \]
\[ > 0 \] (concave upwards)
So \( h = 3 \) gives a minimum \( V \).

When \( h = 1 \), 
\[
V = 2(1)^3 - 12(1)^2 + 18(1) + 50 \\
= 58
\]
So the maximum volume is 58 m\(^3\).

In the above examples, the equation is given as part of the question. However, we often need to work out an equation before we can start answering the question. Working out the equation is often the hardest part!

**EXAMPLES**

1. A rectangular prism has a base with length twice its breadth. The volume is to be 300 cm\(^3\). Show that the surface area is given by 
\[
S = 4x^2 + \frac{900}{x}.
\]

**Solution**

![Diagram of a rectangular prism](image)

Volume: \( V = lbh \)
\[
= 2x \times x \times h \\
300 = 2x^2h
\]
\[
\therefore \frac{300}{2x^2} = h \quad (1)
\]

Surface area:
\[
S = 2(lb + bh + lh) \\
= 2(2x^2 + xh + xh) \\
= 2(2x^2 + 3xh) \\
= 4x^2 + 6xh
\]
Now we substitute (1) into this equation.
\[
S = 4x^2 + 6x \cdot \frac{300}{2x^2} \\
= 4x^2 + \frac{900}{x}
\]

2. \(ABCD\) is a rectangle with \(AB = 10\) cm and \(BC = 8\) cm. Length \(AE = x\) cm and \(CF = y\) cm.
(a) Show that triangles $AEB$ and $CBF$ are similar.
(b) Show that $xy = 80$.
(c) Show that triangle $EDF$ has area given by $A = 80 + 5x + \frac{320}{x}$.

**Solution**

(a) $\angle EAB = \angle BCF = 90^\circ$ (given)
$\angle BFC = \angle EBA$ (corresponding $\angle s$, $AB \parallel DC$)
$\angle BEA = \angle FBC$ (similarly, $AD \parallel BC$)
$\therefore \triangle AEB$ and $\triangle CBF$ are similar (AAA)

(b) $\therefore \frac{10}{y} = \frac{x}{8}$ (similar triangles have sides in proportion)
$xy = 80$

(c) Side $FD = y + 10$ and side $ED = x + 8$

\[ A = \frac{1}{2}bh \]
\[ = \frac{1}{2}(y + 10)(x + 8) \]
\[ = \frac{1}{2}(xy + 8y + 10x + 80) \]

We need to eliminate $y$ from this equation.

If $xy = 80$,
then $y = \frac{80}{x}$

Substituting $xy = 80$ and $y = \frac{80}{x}$ into the area equation gives:

\[ A = \frac{1}{2}(xy + 8y + 10x + 80) \]
\[ = \frac{1}{2}(80 + 8 \cdot \frac{80}{x} + 10x + 80) \]
\[ = \frac{1}{2}(160 + \frac{640}{x} + 10x) \]
\[ = 80 + \frac{320}{x} + 5x \]
2.9 Exercises

1. The area of a rectangle is to be 50 m². Show that the equation of its perimeter is given by $P = 2x + \frac{100}{x}$.

2. A rectangular paddock on a farm is to have a fence with a 120 m perimeter. Show that the area of the paddock is given by $A = 60x - x^2$.

3. The product of two numbers is 20. Show that the sum of the numbers is $S = x + \frac{20}{x}$.

4. A closed cylinder is to have a volume of 400 cm³. Show that its surface area is $S = 2\pi r^2 + \frac{800}{r}$.

5. A 30 cm length of wire is cut into 2 pieces and each piece bent to form a square. (a) Show that $y = 30 - x$. (b) Show that the total area of the 2 squares is given by $A = \frac{x^2 - 30x + 450}{8}$.

6. A timber post with a rectangular cross-sectional area is to be cut out of a log with a diameter of 280 mm as shown. (a) Show that $y = \sqrt{78400} - x^2$. (b) Show that the cross-sectional area is given by $A = x\sqrt{78400} - x^2$.

7. A 10 cm × 7 cm rectangular piece of cardboard has equal square corners with side $x$ cut out and folded up to make an open box. Show that the volume of the box is $V = 70x - 34x^2 + 4x^3$.

8. A travel agency calculates the expense $E$ of organising a holiday per person in a group of $x$ people as $E = 200 + 400x$. The cost $C$ for each person taking a holiday is $C = 900 - 100x$. Show that the profit to the travel agency on a holiday with a group of $x$ people is given by $P = 700x - 500x^2$. 

(Images of calculations and diagrams are included for the problems.)
9. Joel is 700 km north of a town, travelling towards it at an average speed of 75 km/h. Nick is 680 km east of the town, travelling towards it at 80 km/h.

Show that after $t$ hours, the distance between Joel and Nick is given by

$$d = \sqrt{952400 - 213800t + 12025t^2}.$$

10. Sean wants to swim from point A to point B across a 500 m wide river, then walk along the river bank to point C. The distance along the river bank is 7 km.

If he swims at 5 km/h and walks at 4 km/h, show that the time taken to reach point C is given by

$$t = \frac{\sqrt{x^2 + 0.25} + \frac{7 - x}{4}}{5}.$$

When you have found the equation, you can use calculus to find the maximum or minimum value. The process is the same as for finding stationary points on curves.

Always check that an answer gives a maximum or minimum value. Use the second derivative to find its concavity, or if the second derivative is too hard to find, check the first derivative either side of this value.

**EXAMPLES**

1. The council wanted to make a rectangular swimming area at the beach using a straight cliff on one side and a length of 300 m of sharkproof netting for the other three sides. What are the dimensions of the rectangle that encloses the greatest area?

**Solution**

Let the length of the rectangle be $y$ and the width be $x$.
Chapter 2 Geometrical Applications of Calculus

Perimeter: \(2x + y = 300\) m
\[
\therefore \quad y = 300 - 2x \quad \text{(1)}
\]
Area \(A = xy\)
\[
= x (300 - 2x) \quad \text{[substituting (1)]}
= 300x - 2x^2
\]
\[
\frac{dA}{dx} = 300 - 4x
\]
For stationary points, \(\frac{dA}{dx} = 0\)
i.e. \(300 - 4x = 0\)
\(300 = 4x\)
\(75 = x\)
\[
\frac{d^2A}{dx^2} = -4 < 0 \quad \text{(concave downwards)}
\]
So \(x = 75\) gives maximum \(A\).
When \(x = 75\), \(y = 300 - 2(75)\)
\(= 150\)
So the dimensions that give the maximum area are \(150\) m \(\times\) \(75\) m.

2. Trinh and Soi set out from two towns. They travel on roads that meet at right angles, and they walk towards the intersection. Trinh is initially \(15\) km from the intersection and walks at \(3\) km/h. Soi is initially \(10\) km from the intersection and walks at \(4\) km/h.
   (a) Show that their distance apart after \(t\) hours is given by
   \[
   D^2 = 25t^2 - 170t + 325.
   \]
   (b) Hence find how long it takes them to reach their minimum distance apart.
   (c) Find their minimum distance apart.

Solution

(a) After \(t\) hours, Trinh has walked \(3t\) km. She is now \(15 - 3t\) km from the intersection.
After \(t\) hours, Soi has walked \(4t\) km. He is now \(10 - 4t\) km from the intersection.
By Pythagoras’ theorem:

\[ D^2 = (15 - 3t)^2 + (10 - 4t)^2 \]
\[ = 225 - 90t + 9t^2 + 100 - 80t + 16t^2 \]
\[ = 25t^2 - 170t + 325 \]

(b) \[ \frac{dD^2}{dt} = 50t - 170 \]

For stationary points, \[ \frac{dD^2}{dt} = 0 \]

i.e. \[ 50t - 170 = 0 \]
\[ 50t = 170 \]
\[ t = 3.4 \]

\[ \frac{d^2D^2}{dt^2} = 50 > 0 \] (concave upwards)

So \[ t = 3.4 \] gives minimum \( D^2 \).

Now \( 0.4 \times 60 \) minutes = 24 minutes
\[ \therefore 3.4 \text{ hours} = 3 \text{ hours and } 24 \text{ minutes} \]
So Trinh and Soi are a minimum distance apart after 3 hours and 24 minutes.

(c) When \( t = 3.4 \),
\[ D^2 = 25(3.4)^2 - 170(3.4) + 325 \]
\[ = 36 \]
\[ D = \sqrt{36} \]
\[ = 6 \]

So the minimum distance apart is 6 km.

2.10 Exercises

1. The height, in metres, of a ball is given by the equation \( h = 16t - 4t^2 \), where \( t \) is time in seconds. Find when the ball will reach its maximum height, and what the maximum height will be.

2. The cost per hour of a bike ride is given by the formula \( C = x^2 - 15x + 70 \), where \( x \) is the distance travelled in km. Find the distance that gives the minimum cost.

3. The perimeter of a rectangle is 60 m and its length is \( x \) m. Show that the area of the rectangle is given by the equation \( A = 30x - x^2 \). Hence find the maximum area of the rectangle.

4. A farmer wants to make a rectangular paddock with an area of 4000 m\(^2\). However, fencing costs are high and she wants the paddock to have a minimum perimeter.
(a) Show that the perimeter is given by the equation
\[ P = 2x + \frac{8000}{x}. \]
(b) Find the dimensions of the rectangle that will give the minimum perimeter, correct to 1 decimal place.
(c) Calculate the cost of fencing the paddock, at $48.75 per metre.

5. Bill wants to put a small rectangular vegetable garden in his backyard using two existing walls as part of its border. He has 8 m of garden edging for the border on the other two sides. Find the dimensions of the garden bed that will give the greatest area.

6. Find two numbers whose sum is 28 and whose product is a maximum.

7. The difference of two numbers is 5. Find these numbers if their product is to be minimum.

8. A piece of wire 10 m long is broken into two parts, which are bent into the shape of a rectangle and a square as shown. Find the dimensions \(x\) and \(y\) that make the total area a maximum.

9. A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of side \(x\) cm from each corner. The edges are turned up to make an open box.

(a) Show that the volume of the box is given by the equation
\[ V = 4x^3 - 220x^2 + 2400x. \]
(b) Find the value of \(x\) that gives the box its greatest volume.
(c) Find the maximum volume of the box.

10. The formula for the surface area of a cylinder is given by
\[ S = 2\pi r(r + h), \]
where \(r\) is the radius of its base and \(h\) is its height. Show that if the cylinder holds a volume of \(54\pi\) m\(^3\), the surface area is given by the equation
\[ S = 2\pi r^2 + \frac{108\pi}{r}. \]
Hence find the radius that gives the minimum surface area.

11. A silo in the shape of a cylinder is required to hold 8600 m\(^3\) of wheat.
12. A rectangle is cut from a circular disc of radius 6 cm. Find the area of the largest rectangle that can be produced.

13. A poster consists of a photograph bordered by a 5 cm margin. The area of the poster is to be 400 cm².

14. The sum of the dimensions of a box with a square base is 60 cm. Find the dimensions that will give the box a maximum volume.

15. A surfboard is in the shape of a rectangle and semicircle, as shown. The perimeter is to be 4 m. Find the maximum area of the surfboard, correct to 2 decimal places.

16. A half-pipe is to be made from a rectangular piece of metal of length $x$ m. The perimeter of the rectangle is 30 m.

17. Find the least surface area, to the nearest cm², of a closed cylinder that will hold a volume of 400 cm³.

18. The picture frame shown below has a border of 2 cm at the top and bottom and 3 cm at the sides. If the total area of the border is to be 100 cm², find the maximum area of the frame.

19. A 3 m piece of wire is cut into two pieces and bent around to form a square and a circle. Find the size of the two lengths, correct to 2 decimal places, that will make the total area of the square and circle a minimum.
20. Two cars are travelling along roads that intersect at right angles to one another. One starts 200 km away and travels towards the intersection at 80 kmh\(^{-1}\), while the other starts at 120 km away and travels towards the intersection at 60 kmh\(^{-1}\).

Show that their distance apart after \(t\) hours is given by
\[d = 10000t^2 - 46400t + 54400\]
and hence find their minimum distance apart.

21. \(X\) is a point on the curve \(y = x^2 - 2x + 5\). Point \(Y\) lies directly below \(X\) and is on the curve \(y = 4x - x^2\).

(a) Show that the distance \(d\) between \(X\) and \(Y\) is given by
\[d = 2x^2 - 6x + 5.\]
(b) Find the minimum distance between \(X\) and \(Y\).

22. A park is to have a dog-walking enclosure in the shape of a rectangle with a semi-circle at the end as shown, and a perimeter of 1200 m.

(a) Show that
\[x = \frac{2400 - 2y - \pi y}{4}.\]
(b) Show that the area of the enclosure is given by
\[A = \frac{4800y - 4y^2 - \pi y^2}{8}.\]
(c) Find the dimensions \(x\) and \(y\) (to the nearest metre) that maximises the area.

23. Points \(A(-a, 0), B(0, b)\) and \(O(0, 0)\) form a triangle as shown and \(AB\) always passes through the point \((-1, 2)\).

(a) Show that \(b = \frac{2a}{a - 1}\).
(b) Find values of \(a\) and \(b\) that give the minimum area of triangle \(OAB\).

24. Grant is at point \(A\) on one side of a 20 m wide river and needs to get to point \(B\) on the other side 80 m along the bank as shown. Grant swims to any point on the other bank and then runs along the side of the river to point \(B\). If he can swim at 7 km/h and run at 11 km/h, find the distance he
swims \( x \) to minimise the time taken to reach point \( B \). Answer to the nearest metre.

25. A truck travels 1500 km at an hourly cost given by \( s^2 + 9000 \) cents where \( s \) is the average speed of the truck.
   (a) Show that the cost for the trip is given by
   \[ C = 1500 \left( s + \frac{9000}{s} \right) \]
   (b) Find the speed that minimises the cost of the trip.
   (c) Find the cost of the trip to the nearest dollar.

---

Class Challenge

Can you solve either of these problems?

1. **Heron’s problem**
   One boundary of a farm is a straight river bank, and on the farm stands a house and some distance away, a shed; each is sited away from the river bank. Each morning the farmer takes a bucket from his house to the river, fills it with water, and carries the water to the shed.
   Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed. Further, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting.

2. **Lewis Carroll’s problem**
   After a battle at least 95% of the combatants had lost a tooth, at least 90% had lost an eye, at least 80% had lost an arm, and at least 75% had lost a leg. At least how many had lost all four?

---

**Primitive Functions**

This chapter uses differentiation to find the gradient of tangents and stationary points of functions.
Sometimes you may know $f'(x)$ and need to find the original function, $f(x)$. This process is called **anti-differentiation**, and the original function is called the **primitive function**.

**EXAMPLE**

Sketch the primitive function (the original function) given the derivative function below.

**Solution**

Reversing what you would do to sketch the derivative function, the parts at the $x$-axis have a zero gradient, so show stationary points on the original function. There are stationary points at $x_1$ and $x_2$.

The parts of the graph above the $x$-axis show a positive gradient, so the original function is increasing. This happens to the left of $x_1$ and to the right of $x_2$.

The parts of the graph below the $x$-axis show a negative gradient, so the original function is decreasing. This happens between $x_1$ and $x_2$.

Sketching this information gives:
The only problem is, we do not have enough information to know where the curve turns around. There are many choices of where we could put the graph:

The primitive function gives a family of curves.

The examples below use what we know about differentiation to find out how to reverse this to find the primitive function.

**EXAMPLES**

1. Differentiate $x^2$. Hence find a primitive function of $2x$.

   **Solution**

   The derivative of $x^2$ is $2x$
   \[ \therefore \] a primitive function of $2x$ is $x^2$

2. Differentiate $x^2 + 5$. Hence find a primitive function of $2x$.

   **Solution**

   The derivative of $x^2 + 5$ is $2x$
   \[ \therefore \] a primitive function of $2x$ is $x^2 + 5$

3. Differentiate $x^2 - 3$. Hence find a primitive function of $2x$.

   **Solution**

   The derivative of $x^2 - 3$ is $2x$
   \[ \therefore \] a primitive function of $2x$ is $x^2 - 3$
Thus $2x$ has many different primitive functions. In general, the primitive of $2x$ is $x^2 + C$, where $C$ is a constant (real number).

**EXAMPLES**

1. Find the primitive of $x$.

**Solution**

The derivative of $x^2$ is $2x$. 

The derivative of $\frac{x^2}{2}$ is $x$. So the primitive of $x$ is $\frac{x^2}{2} + C$.

2. Find the primitive of $x^2$.

**Solution**

The derivative of $x^3$ is $3x^2$. 

The derivative of $\frac{x^3}{3}$ is $x^2$. 

So the primitive of $x^2$ is $\frac{x^3}{3} + C$.

Continuing this pattern gives the general primitive function of $x^n$.

If $\frac{dy}{dx} = x^n$ then $y = \frac{x^{n+1}}{n+1} + C$

where $C$ is a constant

**Proof**

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{(n+1)x^{n+1-1}}{n+1} = x^n$$

**EXAMPLES**

1. The gradient of a curve is given by $\frac{dy}{dx} = 6x^2 + 8x$. If the curve passes through the point $(1, -3)$, find the equation of the curve.

**Solution**

$$\frac{dy}{dx} = 6x^2 + 8x$$

$$y = 6\left(\frac{x^3}{3}\right) + 8\left(\frac{x^2}{2}\right) + C$$

$$\therefore y = 2x^3 + 4x^2 + C$$
The curve passes through $(1, -3)$
\[
\therefore -3 = 2(1)^3 + 4(1)^2 + C
\]
\[
= 2 + 4 + C
\]
\[
-9 = C
\]
Equation is \( y = 2x^3 + 4x^2 - 9 \).

2. If \( f''(x) = 6x + 2 \) and \( f'(1) = f(-2) = 0 \), find \( f(3) \).

**Solution**

\[
f''(x) = 6x + 2
\]
\[
f'(x) = 6 \left( \frac{x^2}{2} \right) + 2x + C
\]
\[
= 3x^2 + 2x + C
\]

Now \( f'(1) = 0 \)
\[
\begin{align*}
0 &= 3(1)^2 + 2(1) + C \\
-5 &= C
\end{align*}
\]
\[
\therefore f'(x) = 3x^2 + 2x - 5
\]
\[
\begin{align*}
f(x) &= 3 \left( \frac{x^3}{3} \right) + 2 \left( \frac{x^2}{2} \right) - 5x + C \\
&= x^3 + x^2 - 5x + C
\end{align*}
\]

Now \( f(-2) = 0 \)
\[
\begin{align*}
0 &= (-2)^3 + (-2)^2 - 5(-2) + C \\
-6 &= C
\end{align*}
\]
\[
\therefore f(x) = x^3 + x^2 - 5x - 6
\]
\[
\begin{align*}
f(3) &= 3^3 + 3^2 - 5(3) - 6 \\
&= 27 + 9 - 15 - 6 \\
&= 15
\end{align*}
\]

### 2.11 Exercises

1. Find the primitive function of
   (a) \(2x - 3\)
   (b) \(x^2 + 8x + 1\)
   (c) \(x^3 - 4x^3\)
   (d) \((x - 1)^2\)
   (e) \(6\)

2. Find \( f(x) \) if
   (a) \( f'(x) = 6x^2 - x\)
   (b) \( f'(x) = 2x^3 + 7\)
   (c) \( f'(x) = x - 2\)
   (d) \( f'(x) = (x + 1)(x - 3)\)
   (e) \( f'(x) = x^2\)
3. Express $y$ in terms of $x$ if
(a) $\frac{dy}{dx} = 5x^4 - 9$
(b) $\frac{dy}{dx} = x^4 - 2x^2$
(c) $\frac{dy}{dx} = \frac{x^3}{5} - x^2$
(d) $\frac{dy}{dx} = \frac{2}{x^2}$
(e) $\frac{dy}{dx} = x^3 - \frac{2x}{3} + 1$

4. Find the primitive function of
(a) $\sqrt{x}$
(b) $x^{-3}$
(c) $\frac{1}{x^8}$
(d) $x^{\frac{1}{2}} + 2x^{-\frac{2}{3}}$
(e) $x^{-7} - 2x^{-2}$

5. If $\frac{dy}{dx} = x^3 - 3x^2 + 5$ and $y = 4$ when $x = 1$, find an equation for $y$ in terms of $x$.

6. If $f'(x) = 4x - 7$ and $f(2) = 5$, find $f(x)$.

7. Given $f''(x) = 3x^2 + 4x - 2$ and $f(-3) = 4$, find $f(1)$.

8. Given that the gradient of the tangent to a curve is given by $\frac{dy}{dx} = 2 - 6x$ and the curve passes through $(-2, 3)$, find the equation of the curve.

9. If $\frac{dx}{dt} = (t - 3)^2$ and $x = 7$ when $t = 0$, find $x$ when $t = 4$.

10. Given $\frac{d^2y}{dx^2} = 8$ and $\frac{dy}{dx} = 0$ and $y = 3$ when $x = 1$, find the equation of $y$ in terms of $x$.

11. If $\frac{d^2y}{dx^2} = 12x + 6$ and $\frac{dy}{dx} = 1$ at the point $(-1, -2)$, find the equation of the curve.

12. If $f''(x) = 6x - 2$ and $f''(2) = f(2) = 7$, find the function $f(x)$.

13. Given $f''(x) = 5x^4$, $f'(0) = 3$ and $f(-1) = 1$, find $f(2)$.

14. If $\frac{d^2y}{dx^2} = 2x + 1$ and there is a stationary point at $(3, 2)$, find the equation of the curve.

15. A curve has $\frac{d^2y}{dx^2} = 8x$ and the tangent at $(-2, 5)$ makes an angle of 45° with the x-axis. Find the equation of the curve.

16. The tangent to a curve with $\frac{d^2y}{dx^2} = 2x - 4$ makes an angle of 135° with the x-axis in the positive direction at the point $(2, -4)$. Find its equation.

17. A function has a tangent parallel to the line $4x - y - 2 = 0$ at the point $(0, -2)$ and $f''(x) = 12x^2 - 6x + 4$. Find the equation of the function.

18. A curve has $\frac{d^2y}{dx^2} = 6$ and the tangent at $(-1, 3)$ is perpendicular to the line $2x + 4y - 3 = 0$. Find the equation of the curve.

19. A function has $f'(1) = 3$ and $f(1) = 5$. Evaluate $f(-2)$ given $f''(x) = 6x + 18$.

20. A curve has $\frac{d^2y}{dx^2} = 12x - 24$ and a stationary point at $(1, 4)$. Evaluate $y$ when $x = 2$. 
1. Find the stationary points on the curve 
\[ y = x^3 + 6x^2 + 9x - 11 \] 
determine their nature.

2. Find all \( x \)-values for which the curve 
\[ y = 2x^3 - 7x^2 - 3x + 1 \] 
is concave upwards.

3. A curve has \( \frac{dy}{dx} = 6x^2 + 12x - 5 \). If the curve passes through the point \( (2, -3) \), find the equation of the curve.

4. If \( f(x) = 3x^5 - 2x^4 + x^3 - 2 \), find 
(a) \( f(-1) \) 
(b) \( f'(-1) \) 
(c) \( f''(-1) \)

5. The height in metres of an object thrown up into the air is given by 
\[ h = 20t - 2t^2 \] 
where \( t \) is time in seconds. Find the maximum height that the object reaches.

6. Find the stationary point on the curve 
\[ y = 2x^2 \] 
determine its nature.

7. Find the domain over which the curve 
\[ y = 5 - 6x - 3x^2 \] 
is decreasing.

8. For the curve 
\[ y = 3x^4 + 8x^3 - 48x^2 + 1 \] 
(a) find any stationary points and determine their nature 
(b) sketch the curve.

9. Find the point of inflexion on the curve 
\[ y = 2x^3 - 3x^2 + 3x - 2 \].

10. If \( f''(x) = 15x + 12 \), and \( f(2) = f'(2) = 5 \), find \( f(x) \).

11. A soft drink manufacturer wants to minimise the amount of aluminium in its cans while still holding 375 mL of soft drink. Given that 375 mL has a volume of 375 cm\(^3\), (a) show that the surface area of a can is given by 
\[ S = 2\pi r^2 + \frac{750}{r} \] 
(b) find the radius of the can that gives the minimum surface area.

12. For the curve 
\[ y = 3x^4 + 8x^3 + 6x^2 \] 
(a) find any stationary points 
(b) determine their nature 
(c) sketch the curve for \(-3 \leq x \leq 3\).

13. A rectangular prism with a square base is to have a surface area of 250 cm\(^2\).
(a) Show that the volume is given by 
\[ V = \frac{125x - x^3}{2} \] 
(b) Find the dimensions that will give the maximum volume.

14. Find all \( x \)-values for which the curve 
\[ y = 3x^2 - 18x + 4 \] 
is decreasing.

15. If \( \frac{dy}{dx} = 6x + 6 \), and there is a stationary point at \( (0, 3) \), find the equation of the curve.

16. The cost to a business of manufacturing \( x \) products a week is given by 
\[ C = x^2 - 300x + 9000 \]
Find the number of products that will give the minimum cost each week.

17. Show that \( y = -x^3 \) is monotonic decreasing for all \( x \neq 0 \).
18. Sketch a primitive function for each graph.

(a) Show that the area of the garden is 
\[ A = \frac{1}{2} x \sqrt{25 - x^2}. \]

(b) Find the greatest possible area of the garden bed.

20. For the curve \( y = x^3 + 3x^2 - 72x + 5 \),
   (a) find any stationary points on the curve and determine their nature
   (b) find any points of inflexion
   (c) sketch the curve.

21. Find the domain over which the curve \( y = x^3 - 3x^2 + 7x - 5 \) is concave downwards.

22. A function has \( f'(3) = 5 \) and \( f(3) = 2 \). If \( f''(x) = 12x - 6 \), find the equation of the function.

23. Find any points of inflexion on the curve \( y = x^4 - 6x^3 + 2x + 1 \).

24. Find the maximum value of the curve \( y = x^3 + 3x^2 - 24x - 1 \) in the domain \(-5 \leq x \leq 6\).

25. A function has \( f'(2) < 0 \) and \( f''(2) < 0 \). Sketch the shape of the function near \( x = 2 \).
1. Find the first and second derivatives of \( \frac{5 - x}{(4x^2 + 1)^3} \).

2. Sketch the curve \( y = x(x - 2)^3 \), showing any stationary points and inflexions.

3. Find all values of \( x \) for which the curve \( y = 4x^3 - 21x^2 - 24x + 5 \) is increasing.

4. Find the maximum possible area if a straight 8 m length of fencing is placed across a corner to enclose a triangular space.

5. Find the greatest and least values of the function \( f(x) = 4x^3 - 3x^2 - 18x \) in the domain \(-2 \leq x \leq 3\).

6. Show that the function \( f(x) = 2(5x - 3)^3 \) has a horizontal point of inflexion at \((0.6, 0)\).

7. Two circles have radii \( r \) and \( s \) such that \( r + s = 25 \). Show that the sum of areas of the circles is least when \( r = s \).

8. The rate of change of \( V \) with respect to \( t \) is given by \( \frac{dV}{dt} = (2t - 1)^2 \). If \( V = 5 \) when \( t = \frac{1}{2} \), find \( V \) when \( t = 3 \).

9. (a) Show that the curve \( y = \sqrt{x - 1} \) has no stationary points.
   (b) Find the domain and range of the curve.
   (c) Hence sketch the curve.

10. Find the radius and height, correct to 2 decimal places, of a cylinder that holds 200 cm\(^3\), if its surface area is to be a minimum.

11. A curve passes through the point \((0, -1)\) and the gradient at any point is given by \((x + 3)(x - 5)\). Find the equation of the curve.

12. The cost of running a car at an average speed of \( v \) km/h is given by \( c = 150 + \frac{v^2}{80} \) cents per hour. Find the average speed, to the nearest km/h, at which the cost of a 500 km trip is a minimum.

13. Find the equation of a curve that is always concave upwards with a stationary point at \((-1, 2)\) and \(y\)-intercept 3.

14. Given \( f'(x) = x(x - 3)(x + 1) \) and \( f(0) = 0 \)
   (a) find the equation for \( f(x) \)
   (b) find any stationary points on the function
   (c) sketch the function.

15. The volume of air in a cubic room on a submarine is given by the formula \( V = -4x^3 + 27x^2 - 24x + 2 \), where \( x \) is the side of the room in metres. Find the dimensions of the room that will give the maximum volume of air.

16. If \( f''(x) = x - 9 \) and \( f'(-1) = f(-2) = 7 \), find \( f(3) \).

17. Given \( \frac{dy}{dx} = \frac{1}{(x - 5)^2} \) and \( y = 1 \) when \( x = 4 \), find
   (a) \( \frac{d^2y}{dx^2} \) when \( x = 6 \)
   (b) \( y \) when \( x = 6 \).
18. Show that \( y = x^n \) has a stationary point at (0, 0) where \( n \) is a positive integer.
   (a) If \( n \) is even, show that (0, 0) is a minimum turning point.
   (b) If \( n \) is odd, show that (0, 0) is a point of inflexion.

19. Find the maximum possible volume if a rectangular prism with a length twice its breadth has a surface area of 48 cm\(^2\).

20. (a) Find the stationary point on the curve \( y = x^k + 1 \) where \( k \) is a positive integer.
   (b) What values of \( k \) give a minimum turning point?

21. Find the minimum and maximum values of \( y = \frac{x + 3}{x^2 - 9} \) in the domain \(-2 \leq x \leq 2\).

22. The cost of running a car at an average speed of \( V \) km/h is given by \( c = 100 + \frac{V^2}{75} \) cents per hour. Find the average speed (to the nearest km/h) at which the cost of a 1000 km trip is a minimum.
**TERMINOLOGY**

**Definite integral**: The integral or primitive function restricted to a lower and upper boundary. It has the notation $\int_a^b f(x) \, dx$ and geometrically represents the area between the curve $y = f(x)$, the $x$-axis and the ordinates $x = a$ and $x = b$.

**Even function**: A function where $f(-x) = f(x)$. It is symmetrical about the $y$-axis.

**Indefinite integral**: General primitive function represented by $\int f(x) \, dx$.

**Integration**: The process of finding a primitive function.

**Odd function**: A function where $f(-x) = -f(x)$. An odd function has rotational symmetry about the origin.
INTRODUCTION

INTEGRATION IS THE PROCESS of finding an area under a curve. It is an important process in many areas of knowledge, such as surveying, physics and the social sciences. In this chapter, you will look at approximation methods of integrating, as well as shorter methods that lead to finding areas and volumes. You will learn how integration and differentiation are related.

DID YOU KNOW?

Integration has been of interest to mathematicians since very early times. Archimedes (287–212 BC) found the area of enclosed curves by cutting them into very thin layers and finding their sum. He found the formula for the volume of a sphere this way. He also found an estimation of π, correct to 2 decimal places.

Approximation Methods

Mathematicians used rectangles in order to find the approximate area between a curve and the x-axis.

small rectangles  big rectangles
NOTATION
The area of each rectangle is \( f(x) \delta x \) where \( f(x) \) is the height and \( \delta x \) is the width of each rectangle. As \( \delta x \to 0 \), sum of rectangles \( \to \) exact area.

\[
\text{Area} = \lim_{\delta x \to 0} \sum f(x) \delta x = \int f(x) \, dx
\]

Now there are other, more accurate ways to find the area under a curve. However, the notation is still used.

Trapezoidal rule
The trapezoidal rule uses a trapezium for the approximate area under a curve. A trapezium generally gives a better approximation to the area than a rectangle.

\[
\int_a^b f(x) \, dx = \frac{1}{2} (b - a) [f(a) + f(b)]
\]

Proof
\[
A = \frac{1}{2} h(a + b)
= \frac{1}{2} (b - a) [f(a) + f(b)]
\]

EXAMPLES
1. Find an approximation for \( \int_1^4 \frac{dx}{x} \) using the trapezoidal rule.

Solution
\[
\int_1^4 \frac{dx}{x} = \int_1^4 \frac{1}{x} \, dx
\]
\[
\int_a^b f(x) \, dx = \frac{1}{2} (b - a) [f(a) + f(b)]
\]
\[
\int_1^4 \frac{dx}{x} = \frac{1}{2} (4 - 1) [f(1) + f(4)]
\]
\[
\frac{1}{2} (3) \left( \frac{1}{4} + \frac{1}{4} \right) \\
= \frac{3}{2} \times \frac{5}{4} \\
= 1.875
\]

2. Find an approximation for \( \int_0^1 x^3 \, dx \) using the trapezoidal rule with 2 subintervals.

**Solution**

\[
\int_0^1 x^3 \, dx = \frac{1}{2} (1 - 0) \left[ f(0) + f(1) \right]
\]

\[
\int_0^1 x^3 \, dx = \int_0^{0.5} x^3 \, dx + \int_{0.5}^1 x^3 \, dx
\]

\[
= \frac{1}{2} (0.5 - 0) \left[ f(0) + f(0.5) \right] + \frac{1}{2} (1 - 0.5) \left[ f(0.5) + f(1) \right]
\]

\[
= \frac{1}{2} (0.5) (0^3 + 0.5^3) + \frac{1}{2} (0.5) (0.5^3 + 1^3)
\]

\[
= 0.25 (0.125) + 0.25 (1.125)
\]

\[
= 0.3125
\]

There is a more general formula for \( n \) subintervals. Several trapezia give a more accurate area than one. However, you could use the first formula several times if you prefer using it.
Proof

\[
\int_a^b f(x) \, dx \approx \frac{h}{2} \left[ (y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1}) \right]
\]

where \( h = \frac{b-a}{n} \) (width of each trapezium)

**EXAMPLES**

1. Find an approximation for \( \int_0^{14} (t^2 + 3) \, dt \), using the trapezoidal rule with 7 subintervals.

**Solution**

Seven subintervals mean 7 trapezia.
Chapter 3 Integration

There are 4 trapezia.

\[ \int_a^b f(x) \, dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1})] \]

\[ \int_0^{14} (t^2 + 3) \, dt = \frac{h}{2} [(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)] \]

where \( h = \frac{14 - 0}{7} = 2 \)

\[ \int_0^{14} (t^2 + 3) \, dt = \frac{2}{2} \left( [(0^2 + 3) + (14^2 + 3)] + 2 [(2^2 + 3) + (4^2 + 3) + (6^2 + 3) + (8^2 + 3) + (10^2 + 3) + (12^2 + 3)] \right) \]

\[ = 966 \]

2. Find an approximation for \( \int_{\frac{2}{1}}^{3} \frac{x}{2} \, dx \), using the trapezoidal rule with 4 subintervals, correct to 3 decimal places.

**Solution**

\[ \int_a^b f(x) \, dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3)] \]

where \( h = \frac{b - a}{n} \)

\[ = \frac{3 - 2}{4} \]

\[ = 0.25 \]

\[ \int_{\frac{2}{1}}^{3} \frac{2}{x - 1} \, dx = \frac{0.25}{2} \left[ \left( \frac{2}{2 - 1} + \frac{2}{3 - 1} \right) + 2 \left( \frac{2}{2.25 - 1} + \frac{2}{2.5 - 1} + \frac{2}{2.75 - 1} \right) \right] \]

\[ \div 0.125[(2 + 1) + 2(1.6 + 1.3333 + 1.1429)] \]

\[ = 1.394 \]

**Application**

When surveyors need to find the area of an irregular piece of land, they measure regular strips and use an approximation method such as the trapezoidal rule.
EXAMPLE

The table below gives the measurements of a certain piece of land:

<table>
<thead>
<tr>
<th>x m</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y m</td>
<td>3.7</td>
<td>5.9</td>
<td>6.4</td>
<td>5.1</td>
<td>4.9</td>
</tr>
</tbody>
</table>

Using the trapezoidal rule to find its area, correct to 2 decimal places:

\[
\int_{a}^{b} f(x) \, dx \approx \frac{h}{2} \left[ y_0 + y_n + 2(y_1 + y_2 + \ldots + y_n) \right]
\]

Where \( h = \frac{b - a}{n} \)

\[
= \frac{4 - 0}{4} = 1
\]

\[
\int_{0}^{4} f(x) \, dx \approx \frac{1}{2} \left[ (3.7 + 4.9) + 2(5.9 + 6.4 + 5.1) \right]
\]

\[
= \frac{1}{2} \left[ 8.6 + 2 \times 18 \right] = 21.7
\]

So the area of the land is approximately 21.7 m\(^2\).

3.1 Exercises

Use the trapezoidal rule to find an approximation for

1. \( \int_{1}^{2} x^2 \, dx \) using 1 subinterval.

2. \( \int_{0}^{2} (x^3 + 1) \, dx \) using 1 subinterval.

3. \( \int_{1}^{3} \frac{dx}{x} \) using 1 subinterval.

4. \( \int_{1}^{2} \frac{dx}{x + 3} \) using 1 subinterval.

5. \( \int_{1}^{3} x^3 \, dx \) using
   (a) 1 subinterval
   (b) 2 subintervals.

6. \( \int_{2}^{3} \log x \, dx \) using 2 subintervals.

7. \( \int_{0}^{2} \frac{dx}{x + 4} \) using 2 strips.
8. \( \int_1^4 \log x \, dx \) using 3 subintervals.

9. \( \int_0^2 (x^2 - x) \, dx \) using 4 trapezia.

10. \( \int_0^1 \sqrt{x} \, dx \) using 5 subintervals.

11. \( \int_1^5 \frac{1}{x^2} \, dx \) using 4 subintervals.

12. \( \int_3^6 \frac{1}{x - 1} \, dx \) using 6 trapezia.

13. \( \int_1^9 f(x) \, dx \) where values of \( f(x) \) are given in the table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>3.2</td>
<td>5.9</td>
<td>8.4</td>
<td>11.6</td>
<td>20.1</td>
</tr>
</tbody>
</table>

**Simpson’s rule**

This is generally more accurate than the trapezoidal rule, since it makes use of parabolic arcs instead of straight lines.

A parabola is drawn through points \( A, B \) and \( C \) to give the formula

\[
\int_a^b f(x) \, dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
\]

**Proof**

This proof is difficult and is only included for completion of the topic. It uses results that will be studied later in the chapter.
Let $f(a) = y_o$, $f\left(\frac{a+b}{2}\right) = y_1$ and $f(b) = y_2$.

Let the parabola passing through points $A(-h, y_o), B(0, y_1)$ and $C(h, y_2)$ be given by $y = ax^2 + bx + c$.

Then

1. $y_o = a(-h)^2 + b(-h) + c$
2. $y_1 = a(0)^2 + b(0) + c = c$
3. $y_2 = a(h)^2 + b(h) + c$

$y_0 + 4y_1 + y_2 = ah^2 - bh + c + 4c + ah^2 + bh + c = 2ah^2 + 6c$ (4)

$$
\int_{-h}^{h} (ax^2 + bx + c) \, dx = \left[ \frac{ax^3}{3} + \frac{bx^2}{2} + cx \right]_{-h}^{h}
$$

$$
= \left( \frac{ah^3}{3} + \frac{bh^2}{2} + ch \right) - \left( \frac{-ah^3}{3} + \frac{bh^2}{2} - ch \right)
$$

$$
= \frac{2ah^3}{3} + 2ch
$$

$$
= \frac{h}{3} (2ah^2 + 6c)
$$

$$
= \frac{h}{3} (y_o + 4y_1 + y_2)
$$

Now $h = \frac{b-a}{2}$, $y_o = f(a)$, $y_1 = f\left(\frac{a+b}{2}\right)$ and $y_2 = f(b)$.

$$
\therefore \int_{a}^{b} f(x) \, dx = \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]
$$

- One application of Simpson’s rule uses 3 function values (ordinates).
EXAMPLES

1. Use Simpson’s rule with 3 function values to find an approximation for \( \int_{2}^{3} \sqrt{x} \, dx \).

**Solution**
2. Use Simpson’s rule with 5 function values to find an approximation for \( \int_0^2 \frac{dx}{x+1} \).

**Solution**

There is a general formula for \( n \) equal subintervals. There are different versions of this formula. This one uses the function values \( y_0 \) to \( y_n \).

The \( n + 1 \) function values give \( n \) subintervals.
\[ \int_{a}^{b} f(x) \, dx = \frac{h}{3} \left[ (y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots) + 2(y_2 + y_4 + \ldots) \right] \]

where \( h = \frac{b - a}{n} \) for \( n \) subintervals \((n + 1) \text{ function values}\)

**Proof**

\[
\int_{a}^{b} f(x) \, dx = \frac{2h}{6} (y_0 + 4y_1 + y_2) + \frac{2h}{6} (y_2 + 4y_3 + y_4) + \frac{2h}{6} (y_4 + 4y_5 + y_6) \\
+ \ldots + \frac{2h}{6} (y_{n-2} + 4y_{n-1} + y_n) \\
= \frac{h}{3} (y_0 + 4y_1 + y_2 + 4y_3 + y_4 + 4y_5 + y_6 + \ldots + y_{n-2} + 4y_{n-1} + y_n) \\
= \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots) + 2(y_2 + y_4 + \ldots)]
\]

You could also remember this rule as

\[ \int_{a}^{b} f(x) \, dx = \frac{h}{3} [(y_0 + y_n) + 4(\text{odds}) + 2(\text{evens})]. \]

**EXAMPLES**

1. Use Simpson’s rule with 7 ordinates to find an approximation for \( \int_{1}^{4} \log x \, dx \).

**Solution**

\[
\int_{1}^{4} \log x \, dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]
\]

where \( h = \frac{b - a}{n} \)

\[
= \frac{4 - 1}{6} = 0.5
\]

\[
\int_{1}^{4} \log x \, dx = \frac{0.5}{3} [\log 1 + \log 4 + 4(\log 1.5 + \log 2.5 + \log 3.5) \\
+ 2(\log 2 + \log 3)]
\]

\[
= 1.105
\]
2. Use Simpson’s rule with 9 function values to find an approximation for \( \int_{1}^{5} \frac{dx}{x^2} \).

**Solution**

\[
\int_{a}^{b} f(x) \, dx \approx \frac{h}{3} \left[ (y_0 + y_6) + 4(y_1 + y_5 + y_3) + 2(y_2 + y_4 + y_6) \right]
\]

where \( h = \frac{b - a}{n} = \frac{5 - 1}{8} = 0.5 \)

\[
\int_{1}^{5} \frac{1}{x^2} \, dx \approx \frac{0.5}{3} \left[ 1 + 1 + 4 \left( \frac{1}{1.5^2} + \frac{1}{2.5^2} + \frac{1}{3.5^2} + \frac{1}{4.5^2} \right) + 2 \left( \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \right) \right]
\]

\[
\approx 0.8
\]

3. Use Simpson’s rule to find an approximation for \( \int_{1}^{4} f(x) \, dx \) using the values in the table below:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>8.6</td>
<td>11.9</td>
<td>23.7</td>
<td>39.8</td>
<td>56.7</td>
<td>71.4</td>
<td>93.2</td>
</tr>
</tbody>
</table>

**Solution**

\[
\int_{1}^{4} f(x) \, dx \approx \frac{h}{3} \left[ (y_0 + y_6) + 4(y_1 + y_5 + y_3) + 2(y_2 + y_4) \right]
\]

where \( h = \frac{4 - 1}{6} = 0.5 \)

\[
\int_{1}^{4} f(x) \, dx \approx \frac{0.5}{3} \left[ (8.6 + 93.2) + 4(11.9 + 39.8 + 71.4) + 2(23.7 + 56.7) \right]
\]

\[
\approx 125.83
\]
### Exercises

Use Simpson’s rule to find an approximation for

1. \( \int_1^3 x^4 \, dx \) using 3 function values.
2. \( \int_3^5 (x^2 - 1) \, dx \) using 3 ordinates.
3. \( \int_1^3 \frac{dx}{x} \) using 2 subintervals.
4. \( \int_0^1 \frac{dx}{x + 2} \) using 3 function values.
5. \( \int_2^4 \sqrt{x} \, dx \)
   - (a) 3 function values
   - (b) 5 function values.
6. \( \int_3^7 \log x \, dx \) using 5 function values.
7. \( \int_2^6 \frac{dx}{x + 1} \) using 6 subintervals.
8. \( \int_3^7 \log x \, dx \) using 7 function values.
9. \( \int_0^4 (x^3 + x) \, dx \) using 9 ordinates.
10. \( \int_0^4 \sqrt{x} \, dx \) using 5 function values.
11. \( \int_1^7 \frac{1}{x^3} \, dx \) using 7 function values.
12. \( \int_2^6 \frac{1}{x^2 - 1} \, dx \) using 6 subintervals.
13. \( \int_0^4 f(x) \, dx \) where values of \( f(x) \) are given in the table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.7</td>
<td>1.3</td>
<td>5.4</td>
<td>–0.5</td>
<td>–3.8</td>
</tr>
</tbody>
</table>

14. \( \int_2^4 f(t) \, dt \) where values of \( f(t) \) are given in the table:

<table>
<thead>
<tr>
<th>( t )</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>3.7</td>
<td>1.2</td>
<td>9.8</td>
<td>4.1</td>
<td>2.7</td>
</tr>
</tbody>
</table>

15. \( \int_0^3 f(x) \, dx \) where values of \( f(x) \) are given in the table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>15.3</td>
<td>29.2</td>
<td>38.1</td>
<td>56.2</td>
<td>69.9</td>
<td>94.8</td>
<td>102.5</td>
</tr>
</tbody>
</table>

### Computer Application

There are computer application packages that can calculate the area under curves by using rectangles, the trapezoidal rule or Simpson’s rule.

A graphics calculator will also do this.

### Integration and the Primitive Function

Mathematicians found a link between finding areas under a curve and the primitive function. This made possible a simple method for finding exact areas.
The graph below shows the speed at which an object travels over time.

(a) Find the distance it travels in
   (i) 1 s
   (ii) 2 s
   (iii) 3 s

(b) Find the area under the line between
   (i) \( t = 0 \) and 1
   (ii) \( t = 0 \) and 2
   (iii) \( t = 0 \) and 3

Solution

(a) The speed is a constant 30 metres per second (ms\(^{-1}\))
   (i) The object travels 30 metres in 1 s
   (ii) The object travels 60 metres in 2 s
   (iii) The object travels 90 metres in 3 s

(b) (i) The area is \( 30 \times 1 = 30 \text{ units}^2 \)
   (ii) The area is \( 30 \times 2 = 60 \text{ units}^2 \)
   (iii) The area is \( 30 \times 3 = 90 \text{ units}^2 \)

In this example, the line graph gives a rate of change. The area under the curve gives the information about the original data for this rate of change.

In the same way, the area under a rate of change curve will give the original data. This original data is the primitive function of the curve.
DID YOU KNOW?

Many mathematicians in the 17th century were interested in the problem of finding areas under a curve. Isacc Barrow (1630–77) is said to be the first to discover that differentiation and integration are inverse operations. This discovery is called the fundamental theorem of calculus.

Barrow was an Englishman who was an outstanding Greek scholar as well as making contributions in the areas of mathematics, theology, astronomy and physics. However, when he was a schoolboy, he was so often in trouble that his father was overheard saying to God in his prayers that if He decided to take one of his children, he could best spare Isaac.

Sir Isaac Newton (1643–1727), another English mathematician, and scientist and astronomer, helped to discover calculus. He was not interested in his school work, but spent most of his time inventing things, such as a water clock and sundial.

He left school at 14 to manage the estate after his stepfather died. However, he spent so much time reading that he was sent back to school. He went on to university and developed his theories in mathematics and science which have made him famous today.

Fundamental theorem of calculus

The area enclosed by the curve \( y = f(x) \), the \( x \)-axis and the lines \( x = a \) and \( x = b \) is given by

\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]

where \( F(x) \) is the primitive of function \( f(x) \)

Proof

Consider a continuous curve \( y = f(x) \) for all values of \( x > a \)
Let area $ABCD$ be $A(x)$
Let area $ABGE$ be $A(x + h)$
Then area $DCGE$ is $A(x + h) - A(x)$

Area $DCFE < area \; DCGE < area \; DHGE$

\[
f(x) \cdot h < A(x+h) - A(x) < f(x+h) \cdot h
\]

\[
\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} < \lim_{h \to 0} \frac{f(x+h)}{h}
\]

\[
f(x) < A'(x) < f(x)
\]

\[
A'(x) = f(x)
\]

i.e. $A(x)$ is a primitive function of $f(x)$

$A(x) = F(x) + C$ where $F(x)$ is the primitive function of $f(x)$ (1)

Now $A(x)$ is the area between $a$ and $x$

\[
A(a) = 0
\]

Substitute in (1):

\[
A(a) = F(a) + C
\]

\[
0 = F(a) + C
\]

\[
- F(a) = C
\]

\[
A(x) = F(x) - F(a)
\]

If $x = b$ where $b > a,$

$A(b) = F(b) - F(a)$

---

**Definite Integrals**

By the fundamental theorem of calculus,

\[
\int_a^b f(x) \, dx = F(b) - F(a) \text{ where } F(x) \text{ is the primitive function of } f(x).
\]

The primitive function of $x^n$ is $\frac{x^{n+1}}{n+1} + C.$

Putting these pieces of information together, we can find areas under simple curves.
EXAMPLES

Evaluate

1. \( \int_3^4 (2x + 1) \, dx \)

Solution

\[
\int_3^4 (2x + 1) \, dx = \left[ x^2 + x \right]^4_3 \\
= (4^2 + 4) - (3^2 + 3) \\
= 20 - 12 \\
= 8
\]

Constant C will cancel out. That is, \((4^2 + 4 + C) - (3^2 + 3 + C) = 20 - 12 = 8\).

The graph shows the area that the definite integral calculates.

2. \( \int_0^5 3x^2 \, dx \)

Solution

\[
\int_0^5 3x^2 \, dx = \left[ x^3 \right]^5_0 \\
= (5^3) - (0^3) \\
= 125
\]

CONTINUED
3. \( \int_0^2 -3x^2 \, dx \)

**Solution**

\[
\int_0^2 -3x^2 \, dx = \left[-x^3\right]_0^2 \\
= (-2^3) - (-0^3) \\
= -8
\]

---

4. \( \int_{-1}^{1} x^3 \, dx \)

**Solution**

\[
\int_{-1}^{1} x^3 \, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} \\
= \frac{1^4}{4} - \frac{(-1)^4}{4} \\
= \frac{1}{4} - \frac{1}{4} \\
= 0
\]

---

Can you see why there is a negative solution?

The definite integral gives an area of zero. Can you see why?
Chapter 3: Integration

3.3 Exercises

Evaluate

1. \( \int_0^4 4x \, dx \)
2. \( \int_1^4 (2x + 1) \, dx \)
3. \( \int_0^5 3x^2 \, dx \)
4. \( \int_1^2 (4t - 7) \, dt \)
5. \( \int_1^3 (6y + 5) \, dy \)
6. \( \int_0^3 6x^2 \, dx \)
7. \( \int_1^2 (x^2 + 1) \, dx \)
8. \( \int_0^2 4x^3 \, dx \)
9. \( \int_{-1}^4 (3x^2 - 2x) \, dx \)
10. \( \int_1^3 (4x^2 + 6x - 3) \, dx \)
11. \( \int_{-1}^1 x^2 \, dx \)
12. \( \int_{-1}^3 (x^3 + 1) \, dx \)
13. \( \int_{-2}^2 x^5 \, dx \)
14. \( \int_1^4 \sqrt{x} \, dx \)
15. \( \int_0^1 (x^3 - 3x^2 + 4x) \, dx \)
16. \( \int_0^2 (2x - 1)^2 \, dx \)
17. \( \int_{-1}^1 (y^3 + y) \, dy \)
18. \( \int_{-3}^4 (2 - x)^2 \, dx \)
19. \( \int_2^2 4t^3 \, dt \)
20. \( \int_{-2}^4 \frac{x^2}{3} \, dx \)
21. \( \int_0^3 \frac{5x^4}{x} \, dx \)
22. \( \int_{-2}^2 \frac{x^4 - 3x}{x} \, dx \)
23. \( \int_0^3 \frac{4x^3 + x^2 + 5x}{x} \, dx \)
24. \( \int_0^4 \frac{x^3 - 2x^2 + 3x}{x} \, dx \)
25. \( \int_{-3}^3 \frac{x^2 + x + 3}{3x^5} \, dx \)

Class Investigation

Look at the results of definite integrals in the examples and exercises. Sketch the graphs where possible and shade in the areas found.

- Can you see why the definite integral sometimes gives a negative answer?
- Can you see why it will sometimes be zero?

Indefinite Integrals

Sometimes it is necessary to find a general or indefinite integral (primitive function).

\[ \int f(x) \, dx = F(x) + C \text{ where } F(x) \text{ is a primitive function of } f(x) \]

In Chapter 2 you learned the result for the primitive function of \( x^n \). This result is the same as the integral of \( x^n \).

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \]
EXAMPLES

Find each indefinite integral (primitive function)

1. \[ \int (x^4 - 3x^2 + 4x - 7) \, dx \]

Solution

\[ \int (x^4 - 3x^2 + 4x - 7) \, dx = \frac{x^5}{5} - 3 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} - 7x + C \]
\[ = \frac{x^5}{5} - x^3 + 2x^2 - 7x + C \]

2. \[ \int 5x^6 \, dx \]

Solution

\[ \int 5x^6 \, dx = 5 \left( \frac{x^{10}}{10} \right) + C \]
\[ = \frac{x^{10}}{2} + C \]

3. \[ \int \left( \frac{2x^5 + 7x^2 - 3x}{x} \right) \, dx \]

Solution

\[ \int \left( \frac{2x^5}{x} + \frac{7x^2}{x} - \frac{3x}{x} \right) \, dx = \int (2x^4 + 7x - 3) \, dx \]
\[ = \frac{2x^5}{5} + \frac{7x^2}{2} - 3x + C \]

4. \[ \int \left( \frac{1}{x^3} + \sqrt{x} \right) \, dx \]

Solution

\[ \int \left( \frac{1}{x^3} + \sqrt{x} \right) \, dx = \int (x^{-3} + x^{\frac{1}{2}}) \, dx \]
\[ = \frac{x^{-2}}{-2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C \]
\[ = -\frac{1}{2x^2} + \frac{2\sqrt{x^3}}{3} + C \]
DID YOU KNOW?

John Wallis (1616–1703) found that the area under the curve \( y = 1 + x + x^2 + x^3 + \ldots \) is given by
\[
x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots
\]
He found this result independently of the fundamental theorem of calculus.

3.4 Exercises

Find each indefinite integral (primitive function)

1. \( \int x^2 \, dx \)
2. \( \int 3x^3 \, dx \)
3. \( \int 2x^4 \, dx \)
4. \( \int (m + 1) \, dm \)
5. \( \int (t^2 - 7) \, dt \)
6. \( \int (h^2 + 5) \, dh \)
7. \( \int (y - 3) \, dy \)
8. \( \int (2x + 4) \, dx \)
9. \( \int (b^2 + b) \, db \)
10. \( \int (a^3 - a - 1) \, da \)
11. \( \int (x^2 + 2x + 5) \, dx \)
12. \( \int (4x^3 - 3x^2 + 8x - 1) \, dx \)
13. \( \int (6x^5 + x^4 + 2x^3) \, dx \)
14. \( \int (x^7 - 3x^6 - 9) \, dx \)
15. \( \int (2x^3 + x^2 - x - 2) \, dx \)
16. \( \int (x^5 + x^3 + 4) \, dx \)
17. \( \int (4x^2 - 5x - 8) \, dx \)
18. \( \int (3x^4 - 2x^3 + x) \, dx \)
19. \( \int (6x^3 + 5x^2 - 4) \, dx \)
20. \( \int (3x^4 + x^3 + 2x^2) \, dx \)
21. \( \int \frac{dx}{x^3} \)
22. \( \int x^3 \, dx \)
23. \( \int \frac{x^6 - 3x^4 + 2x^4}{x^3} \, dx \)
24. \( \int (1 - 2x)^2 \, dx \)
25. \( \int (x - 2)(x + 5) \, dx \)
26. \( \int \frac{3}{x^2} \, dx \)
27. \( \int \frac{dx}{x^3} \)
28. \( \int \frac{4x^3 - x^3 - 3x^2 + 7}{x^3} \, dx \)
29. \( \int (y^5 - y^2 + 5) \, dy \)
30. \( \int (t^2 - 4) \, (t - 1) \, dt \)
31. \( \int \sqrt{x} \, dx \)
32. \( \int \frac{2}{t^3} \, dt \)
33. \( \int \sqrt{x} \, dx \)
34. \( \int x \sqrt{x} \, dx \)
35. \( \int \sqrt{x} \left( 1 + \frac{1}{\sqrt{x}} \right) \, dx \)
Function of a function rule

**EXAMPLE**

Differentiate \((2x + 3)^3\).

Hence find \(\int 10(2x + 3)^4 \, dx\).

Find \(\int (2x + 3)^4 \, dx\).

**Solution**

\[
\frac{d}{dx} (2x + 3)^3 = 5(2x + 3)^4 \cdot 2 = 10(2x + 3)^4
\]

\[
\therefore \int 10(2x + 3)^4 \, dx = (2x + 3)^5 + C
\]

\[
\int (2x + 3)^4 \, dx = \frac{1}{10} \int 10(2x + 3)^4 \, dx
\]

\[
= \frac{1}{10} (2x + 3)^5 + C
\]

\[
\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C
\]

**Proof**

\[
\frac{d}{dx}(ax + b)^{n+1} = (n+1)(ax + b)^n \cdot a = a(n + 1)(ax + b)^n
\]

\[
\therefore \int a(n + 1)(ax + b)^n \, dx = (ax + b)^{n+1} + C
\]

\[
\int (ax + b)^n \, dx = \frac{1}{a(n + 1)} \int a(n + 1)(ax + b)^n \, dx
\]

\[
= \frac{1}{a(n + 1)} (ax + b)^{n+1} + C
\]

\[
= \frac{(ax + b)^{n+1}}{a(n + 1)} + C
\]

Remember: Integration is the reverse of differentiation.
EXAMPLES

Find

1. \( \int (5x - 9)^3 \, dx \)

Solution

\[
\int (5x - 9)^3 \, dx = \frac{(5x - 9)^4}{5 \times 4} + C
\]

\[
= \frac{(5x - 9)^4}{20} + C
\]

2. \( \int (3 - x)^8 \, dx \)

Solution

\[
\int (3 - x)^8 \, dx = \frac{(3 - x)^9}{-1 \times 9} + C
\]

\[
= - \frac{(3 - x)^9}{9} + C
\]

3. \( \int \sqrt{4x + 3} \, dx \)

Solution

\[
\int (4x + 3)^{\frac{1}{2}} \, dx = \frac{(4x + 3)^{\frac{3}{2}}}{4 \times \frac{3}{2}} + C
\]

\[
= \frac{\sqrt{(4x + 3)^3}}{6} + C
\]

3.5 Exercises

Find each indefinite integral (primitive function)

1. (a) \( \int (3x - 4)^2 \, dx \) by
   (i) expanding
   (ii) the function of a function rule

(b) \( \int (x + 1)^4 \, dx \)

(c) \( \int (5x - 1)^9 \, dx \)

(d) \( \int (3y - 2)^5 \, dy \)

(e) \( \int (4 + 3x)^3 \, dx \)

(f) \( \int (7x + 8)^{12} \, dx \)

(g) \( \int (1 - x)^6 \, dx \)

(h) \( \int \sqrt{2x - 5} \, dx \)

(i) \( \int 2(3x + 1)^{-4} \, dx \)

(j) \( \int 3(x + 7)^{-2} \, dx \)

(k) \( \int \frac{1}{2(4x - 5)^3} \, dx \)

(l) \( \int \sqrt[3]{4x + 3} \, dx \)
(m) \( \int (2 - x)^{\frac{1}{2}} \, dx \)
(n) \( \int \sqrt{(t + 3)^3} \, dt \)
(o) \( \int \sqrt{(5x + 2)^5} \, dx \)

2. Evaluate
(a) \( \int_1^2 (2x + 1)^4 \, dx \)
(b) \( \int_0^1 (3y - 2)^3 \, dy \)
(c) \( \int_1^2 (1 - x)^7 \, dx \)
(d) \( \int_0^2 (3 - 2x)^5 \, dx \)
(e) \( \int_1^3 \frac{(3x - 1)^2}{6} \, dx \)
(f) \( \int_1^3 (5 - x)^6 \, dx \)
(g) \( \int_3^6 \sqrt{x - 2} \, dx \)
(h) \( \int_0^1 \frac{dx}{(3x - 2)^2} \)
(i) \( \int_0^2 \frac{5}{(2n + 1)^3} \, dn \)
(j) \( \int_1^4 \frac{2}{\sqrt{(5x - 4)^3}} \, dx \)

Class Exercise

Differentiate \((x + 1)\sqrt{2x - 3}\). Hence find \( \int \frac{3x - 2}{\sqrt{2x - 3}} \, dx \).

Areas Enclosed by the x-axis

The definite integral gives the **signed area** under a curve.
Areas **above** the x-axis give a **positive definite** integral.
Areas **below** the x-axis give a **negative definite** integral.

We normally think of areas as positive. So to find areas below the x-axis, take the absolute value of the definite integral. That is,

\[
\text{Area} = \left| \int_a^b f(x) \, dx \right|
\]

**Always sketch the area to be found.**

### EXAMPLES

1. Find the area enclosed by the curve \( y = 2 + x - x^2 \) and the x-axis.

**Solution**

Sketch \( y = 2 + x - x^2 \).

\[
\text{Area} = \int_{-1}^{2} (2 + x - x^2) \, dx
\]

\[
= \left[ 2x + \frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^{2}
\]

\[
= \left( 2(2) + \frac{2^2}{2} - \frac{2^3}{3} \right) - \left( 2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right)
\]

\[
= \left( 4 + 2 - \frac{8}{3} \right) - \left( -2 + \frac{1}{2} + \frac{1}{3} \right)
\]

\[
= 4 \frac{1}{2}
\]

So the area is \( 4 \frac{1}{2} \) units\(^2\).
2. Find the area bounded by the curve \( y = x^2 - 4 \) and the \( x \)-axis.

**Solution**

Sketch \( y = x^2 - 4 \).

\[
\int_{-2}^{2} (x^2 - 4) \, dx = \left[ \frac{x^3}{3} - 4x \right]_{-2}^{2}
\]

\[
= \left( \frac{2^3}{3} - 4(2) \right) - \left( \frac{(-2)^3}{3} - 4(-2) \right)
\]

\[
= \left( \frac{8}{3} - 8 \right) - \left( \frac{-8}{3} + 8 \right)
\]

\[
= -10 \frac{2}{3}
\]

So the area is \( 10 \frac{2}{3} \) units\(^2\).

3. Find the area enclosed between the curve \( y = x^3 \), the \( x \)-axis and the lines \( x = -1 \) and \( x = 3 \).

**Solution**

Sketch \( y = x^3 \).

Some of the area is below the \( x \)-axis, so its integral will be negative.
We need to find the sum of the areas between \( x = -1 \) and 0, and \( x = 0 \) and 3.

\[
\int_{-1}^{0} x^3 \, dx = \left[ \frac{x^4}{4} \right]_{-1}^{0} = 0^4 - (-1)^4 = -\frac{1}{4}
\]

So \( A_1 = \frac{1}{4} \) units\(^2\).

\[
\int_{0}^{3} x^3 \, dx = \left[ \frac{x^4}{4} \right]_{0}^{3} = 3^4 - 0^4 = \frac{81}{4}
\]

So \( A_2 = \frac{81}{4} \) units\(^2\).

\( A_1 + A_2 = \frac{1}{4} + \frac{81}{4} = 20\frac{1}{2} \) units\(^2\)

### Even and odd functions

Some functions have special properties.

For odd functions, \( \int_{-a}^{a} f(x) \, dx = 0 \)

For even functions, \( \int_{-a}^{a} f(x) \, dx = 2\int_{0}^{a} f(x) \, dx \)
EXAMPLES

1. Find the area between the curve \( y = x^3 \), the \( x \)-axis and the lines \( x = -2 \) and \( x = 2 \).

Solution

\[
\int_{-2}^{2} x^3 \, dx = 0
\]

Area = \( 2 \int_{0}^{2} x^3 \, dx \)

\[
= 2 \left[ \frac{x^4}{4} \right]_{0}^{2} \\
= 2 \left( \frac{2^4}{4} - \frac{0^4}{4} \right) \\
= 8
\]

So the area is 8 units\(^2\).

2. Find the area between the curve \( y = x^2 \), the \( x \)-axis and the lines \( x = -4 \) and \( x = 4 \).

Solution

\[
\int_{-4}^{4} x^2 \, dx = 2 \int_{0}^{4} x^2 \, dx
\]

\[
= 2 \left[ \frac{x^3}{3} \right]_{0}^{4} \\
= 2 \left( \frac{4^3}{3} - \frac{0^3}{3} \right) \\
= \frac{42}{3}
\]

So the area is \( 42 \frac{2}{3} \) units\(^2\).

3.6 Exercises

1. Find the area enclosed between the curve \( y = 1 - x^2 \) and the \( x \)-axis.

2. Find the area bounded by the curve \( y = x^2 - 9 \) and the \( x \)-axis.

3. Find the area enclosed between the curve \( y = x^2 + 5x + 4 \) and the \( x \)-axis.

4. Find the area enclosed between the curve \( y = x^2 - 2x - 3 \) and the \( x \)-axis.

5. Find the area bounded by the curve \( y = -x^2 + 9x - 20 \) and the \( x \)-axis.

6. Find the area enclosed between the curve \( y = -2x^2 - 5x + 3 \) and the \( x \)-axis.

7. Find the area enclosed between the curve \( y = x^3 \), the \( x \)-axis and the lines \( x = 0 \) and \( x = 2 \).
8. Find the area enclosed between the curve \( y = x^4 \), the x-axis and the lines \( x = -1 \) and \( x = 1 \).

9. Find the area enclosed between the curve \( y = x^3 \), the x-axis and the lines \( x = -2 \) and \( x = 2 \).

10. Find the area enclosed between the curve \( y = x^3 \), the x-axis and the lines \( x = -3 \) and \( x = 2 \).

11. Find the area bounded between the curve \( y = 3x^2 \), the x-axis and the lines \( x = -1 \) and \( x = 1 \).

12. Find the area enclosed between the curve \( y = x^2 + 1 \), the x-axis and the lines \( x = -2 \) and \( x = 2 \).

13. Find the area enclosed between the curve \( y = x^2 \), the x-axis and the lines \( x = -3 \) and \( x = 2 \).

14. Find the area enclosed between the curve \( y = x^2 + x \), and the x-axis.

15. Find the area enclosed between the curve \( y = \frac{1}{x^2} \), the x-axis and the lines \( x = 1 \) and \( x = 3 \).

16. Find the area enclosed between the curve \( y = \frac{2}{(x - 3)^2} \), the x-axis and the lines \( x = 0 \) and \( x = 1 \).

17. Find the area bounded by the curve \( y = \sqrt{x} \), the x-axis and the line \( x = 4 \).

18. Find the area bounded by the curve \( y = \sqrt{x + 2} \), the x-axis and the line \( x = 7 \).

19. Find the area bounded by the curve \( y = \sqrt{4 - x^2} \), the x-axis and the y-axis in the first quadrant.

20. Find the area bounded by the curve \( y = x^3 \) and the x-axis, the curve \( y = x^3 \) and the lines \( x = -a \) and \( x = a \).

Areas Enclosed by the y-axis

To find the area between a curve and the y-axis, we change the subject of the equation of the curve to \( x \). That is,

\[ x = f(y). \]

The definite integral is given by

\[
\int_a^b f(y) \, dy \text{ or } \int_a^b x \, dy
\]

Since \( x \) is **positive** on the right-hand side of the y-axis, the definite integral is **positive**. Since \( x \) is **negative** on the left-hand side of the y-axis, the definite integral is **negative**.
EXAMPLES

1. Find the area enclosed by the curve \( x = y^2 \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 3 \).

Solution

Area \( = \int_1^3 y^2 \, dy \)
\[ = \left[ \frac{y^3}{3} \right]_1^3 \]
\[ = \frac{3^3}{3} - \frac{1^3}{3} \]
\[ = \frac{27}{3} - \frac{1}{3} \]
\[ = \frac{26}{3} \text{ units}^2 \]

The integral is positive since the area is to the right of the \( y \)-axis.

2. Find the area enclosed by the curve \( y = x^2 \), the \( y \)-axis and the lines \( y = 0 \) and \( y = 4 \) in the first quadrant.

Solution

Change the subject of the equation to \( x \).

\[ y = x^2 \]
\[ \therefore \pm \sqrt{y} = x \]

In the first quadrant,

\[ \sqrt{y} = x \]

i.e. \( y^{\frac{1}{2}} = x \)

Area \( = \int_0^4 \sqrt{y} \, dy \)
\[ = \left[ \frac{2\sqrt{y^3}}{3} \right]_0^4 \]
\[ = \frac{2\sqrt{4^3}}{3} - \frac{2\sqrt{0^3}}{3} \]
\[ = \frac{2 \cdot 4^3}{3} - \frac{2 \cdot 0^3}{3} \]
\[ = \frac{128}{3} \text{ units}^2 \]
3. Find the area enclosed between the curve \( y = \sqrt{x + 1} \), the \( y \)-axis and the lines \( y = 0 \) and \( y = 3 \).

Solution

\[
\begin{align*}
\text{Area} &= \int_0^1 (y^2 - 1) \, dy + \int_1^3 (y^2 - 1) \, dy \\
&= \left[ \frac{y^3}{3} - y \right]_0^1 + \left[ \frac{y^3}{3} - y \right]_1^3 \\
&= \left( \frac{1}{3} - 1 \right) - \left( 0 - 0 \right) + \left( \frac{3^3}{3} - 3 \right) - \left( \frac{1^3}{3} - 1 \right) \\
&= -\frac{2}{3} + 6 + \frac{2}{3} \\
&= \frac{2}{3} + \frac{6}{3} \\
&= 7 \frac{1}{3} \text{ units}^2
\end{align*}
\]

3.7 Exercises

1. Find the area bounded by the \( y \)-axis, the curve \( x = y^2 \) and the lines \( y = 0 \) and \( y = 4 \).

2. Find the area enclosed between the curve \( x = y^3 \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 3 \).

3. Find the area enclosed between the curve \( y = x^2 \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 4 \) in the first quadrant.

4. Find the area between the lines \( y = x - 1 \), \( y = 0 \) and \( y = 1 \) and the \( y \)-axis.

5. Find the area bounded by the line \( y = 2x + 1 \), the \( y \)-axis and the lines \( y = 3 \) and \( y = 4 \).

6. Find the area bounded by the curve \( y = \sqrt{x} \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 2 \).
7. Find the area bounded by the curve \( x = y^2 - 2y - 3 \) and the \( y \)-axis.

8. Find the area bounded by the curve \( x = -y^2 - 5y - 6 \) and the \( y \)-axis.

9. Find the area enclosed by the curve \( y = \sqrt{3x - 5} \), the \( y \)-axis and the lines \( y = 2 \) and \( y = 3 \).

10. Find the area enclosed between the curve \( y = \frac{1}{x^2} \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 4 \) in the first quadrant.

11. Find the area enclosed between the curve \( y = x^3 \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 8 \).

12. Find the area enclosed between the curve \( y = x^3 - 2 \) and the \( y \)-axis between \( y = -1 \) and \( y = 25 \).

13. Find the area enclosed between the lines \( y = 4 \) and \( y = 1 - x \) in the second quadrant.

14. Find the area enclosed between the \( y \)-axis and the curve \( x = y(y - 2) \).

15. Find the area bounded by the curve \( y = x^4 + 1 \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 3 \) in the first quadrant, correct to 2 significant figures.

---

**Sums and Differences of Areas**

**EXAMPLES**

1. Find the area enclosed between the curve \( y = x^2 \), the \( y \)-axis and the lines \( y = 0 \) and \( y = 4 \) in the first quadrant.

**Solution**

\[
A = \text{Area of rectangle} - \int_0^2 x^2 \, dx
\]

\[
= (4 \times 2) - \left[ \frac{x^3}{3} \right]_0^2
\]

\[
= 8 - \left( \frac{8}{3} - \frac{0}{3} \right)
\]

\[
= 5 \frac{1}{3} \text{ units}^2
\]

---

This example was done before by finding the area between the curve and the \( y \)-axis.
2. Find the area enclosed between the curves \( y = x^2, \ y = (x - 4)^2 \) and the \( x \)-axis.

**Solution**

Solve simultaneous equations to find the intersection of the curves.

\[
\begin{align*}
  y &= x^2 & & (1) \\
  y &= (x - 4)^2 & & (2)
\end{align*}
\]

\[
\therefore \quad x^2 = (x - 4)^2 \\
&= x^2 - 8x + 16 \\
&= 0 \Rightarrow -8x + 16 \\
&= 8x = 16 \\
&= x = 2
\]

Area = \( \int_0^2 x^2 \, dx + \int_2^4 (x - 4)^2 \, dx \)

\[
\begin{align*}
  &= \left[ \frac{x^3}{3} \right]_0^2 + \left[ \frac{(x - 4)^3}{3} \right]_2^4 \\
  &= \left( \frac{8}{3} - \frac{0}{3} \right) + \left( \frac{0}{3} - \frac{-8}{3} \right) \\
  &= \frac{8}{3} \text{ units}^2
\end{align*}
\]

3. Find the area enclosed between the curve \( y = x^2 \) and the line \( y = x + 2 \).

**Solution**

Solve simultaneous equations to find the intersection of the curve and line.

\[
\begin{align*}
  y &= x^2 & & (1) \\
  y &= x + 2 & & (2)
\end{align*}
\]

\[
\therefore \quad x^2 = x + 2 \\
&= x^2 - x - 2 = 0 \\
&= (x - 2)(x + 1) = 0 \\
&= x = 2 \text{ or } -1
\]

Area = \( \int_{-1}^2 (x + 2) \, dx - \int_{-1}^2 x^2 \, dx \)

\[
\begin{align*}
  &= \int_{-1}^2 (x + 2) \, dx - \int_{-1}^2 x^2 \, dx \\
  &= \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 \\
  &= \left( \frac{2^2}{2} + 2(2) - \frac{2^3}{3} \right) - \left( \frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3} \right) \\
  &= \left( \frac{2}{2} + 4 - \frac{8}{3} \right) - \left( \frac{1}{2} - 2 + \frac{1}{3} \right) \\
  &= 4 \frac{1}{2} \text{ units}^2
\end{align*}
\]
3.8 Exercises

1. Find the area bounded by the line $y = 1$ and the curve $y = x^2$.

2. Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

3. Find the area enclosed by the curve $y = x^2$ and the line $y = x$.

4. Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

5. Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

6. Find the area bounded by the curve $y = x^3$ and the line $y = 4x$.

7. Find the area enclosed between the curves $y = (x - 1)^2$ and $y = (x + 1)^2$.

8. Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

9. Find the area enclosed between the curve $y = x^3$, the $x$-axis and the line $y = -3x + 4$.

10. Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$.

11. Find the area enclosed between the curves $y = x^2$ and $y = x^4$.

12. Find the area enclosed by the curves $y = x^2$ and $x = y^2$.

13. Find the area bounded by the curve $y = x^2 + 2x - 8$ and the line $y = 2x + 1$.

14. Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

15. Find the exact area enclosed between the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$.

Volumes

The volume of a solid can be found by rotating an area under a curve about the $x$-axis or the $y$-axis.
Volumes about the $x$-axis

$$V = \pi \int_a^b y^2 \, dx$$

**Proof**

Take a disc of the solid with width $\delta x$.

The disc is approximately a cylinder with radius $y$ and height $\delta x$.

Volume $= \pi r^2 \, h$
$= \pi y^2 \, \delta x$

Taking the sum of an infinite number of these discs,

$$\text{Volume} = \lim_{\delta x \to 0} \pi y^2 \, \delta x$$
$= \int_a^b \pi y^2 \, dx$
$= \pi \int_a^b y^2 \, dx$
EXAMPLES

1. Find the volume of the solid of revolution formed when the curve \(x^2 + y^2 = 9\) is rotated about the \(x\)-axis between \(x = 1\) and \(x = 3\).

Solution

\[\begin{align*}
\text{Volume} & = \pi \int_a^b x^2 \, dx \\
& = \pi \int_1^3 (9 - x^2) \, dx \\
& = \pi \left[ 9x - \frac{x^3}{3} \right]_1^3 \\
& = \pi \left( 27 - 9 - \left(9 - \frac{1}{3}\right) \right) \\
& = \frac{28\pi}{3} \text{ units}^3
\end{align*}\]

2. Find the volume of the solid formed when the line \(y = 3x - 2\) is rotated about the \(x\)-axis from \(x = 1\) to \(x = 2\).

Solution

\[\begin{align*}
\text{Volume} & = \pi \int_a^b y^2 \, dx \\
& = \pi \int_1^2 (3x - 2)^2 \, dx \\
& = \pi \left[ \frac{(3x - 2)^3}{3} \right]_1^2 \\
& = \pi \left( \frac{4^3}{9} - \frac{1^3}{9} \right) \\
& = \frac{63\pi}{9} \\
& = 7\pi \text{ units}^3
\end{align*}\]

Use the function of a function rule
PROBLEM

What is wrong with the working out of this volume?
Find the volume of the solid of revolution formed when the curve
\( y = x^2 + 1 \) is rotated about the x-axis between \( x = 0 \) and \( x = 2 \).

Solution

\[
\begin{align*}
  y &= x^2 + 1 \\
  \therefore \quad y^2 &= (x^2 + 1)^2 \\
  V &= \pi \int_{a}^{b} y^2 \, dx \\
  &= \pi \int_{0}^{2} (x^2 + 1)^2 \, dx \\
  &= \pi \left[ \frac{(x^2 + 1)^3}{3} \right]_{0}^{b} \\
  &= \pi \left( \frac{(2^2 + 1)^3}{3} - \frac{(0^2 + 1)^3}{3} \right) \\
  &= \pi \left( \frac{5^3}{3} - \frac{1^3}{3} \right) \\
  &= \frac{124\pi}{3} \text{ units}^3
\end{align*}
\]

What is the correct volume?

Volumes about the y-axis

The formula for rotations about the y-axis is similar to the formula for the x-axis.

The proof of this result is similar to that for volumes about the x-axis.
EXAMPLE

Find the volume of the solid formed when the curve \( y = x^2 - 1 \) is rotated about the \( y \)-axis from \( y = -1 \) to \( y = 3 \).

**Solution**

\[
y = x^2 - 1
\]
\[
\therefore \quad y + 1 = x^2
\]

\[
V = \pi \int_a^b x^2 \, dy
\]
\[
= \pi \int_{-1}^{3} (y + 1) \, dy
\]
\[
= \pi \left[ \frac{y^2}{2} + y \right]_{-1}^{3}
\]
\[
= \pi \left[ \left( \frac{9}{2} + 3 \right) - \left( 1 - 1 \right) \right]
\]
\[
= 8\pi \text{ units}^3
\]

3.9 Exercises

1. Find the volume of the solid of revolution formed when the curve \( y = x^2 \) is rotated about the \( x \)-axis from \( x = 0 \) to \( x = 3 \).

2. Find the volume of the solid formed when the line \( y = x + 1 \) is rotated about the \( x \)-axis between \( x = 2 \) and \( x = 7 \).

3. Find the volume of the solid of revolution that is formed when the curve \( y = x^2 + 2 \) is rotated about the \( x \)-axis from \( x = 0 \) to \( x = 2 \).

4. Determine the volume of the solid formed when the curve \( y = x^3 \) is rotated about the \( x \)-axis from \( x = 0 \) to \( x = 1 \).
5. Find the volume of the solid formed when the curve \( x = y^2 - 5 \) is rotated about the \( x \)-axis between \( x = 0 \) and \( x = 3 \).

6. Find the volume of the solid of revolution formed by rotating the line \( y = 4x - 1 \) about the \( x \)-axis from \( x = 2 \) to \( x = 4 \).

7. Find the volume of the hemisphere formed when the curve \( x^2 + y^2 = 1 \) is rotated about the \( x \)-axis between \( x = 0 \) and \( x = 1 \).

8. The parabola \( y = (x + 2)^2 \) is rotated about the \( x \)-axis from \( x = 0 \) to \( x = 2 \). Find the volume of the solid formed.

9. Find the volume of the spherical cap formed when the curve \( x^2 + y^2 = 4 \) is rotated about the \( y \)-axis from \( y = 1 \) to \( y = 2 \).

10. Find the volume of the paraboloid formed when \( y = x^2 \) is rotated about the \( y \)-axis from \( y = 0 \) to \( y = 3 \).

11. Find the volume of the solid formed when \( y = x^2 - 2 \) is rotated about the \( y \)-axis from \( y = 1 \) to \( y = 4 \).

12. The line \( y = x + 2 \) is rotated about the \( y \)-axis from \( y = -2 \) to \( y = 2 \). Find the volume of the solid formed.

13. Determine the volume of the solid formed when the curve \( y = x^3 \) is rotated about the \( x \)-axis from \( x = -1 \) to \( x = 4 \).

14. Find the volume of the solid formed when the curve \( y = \sqrt{x} \) is rotated about the \( x \)-axis between \( x = 0 \) and \( x = 5 \).

15. Find the volume of the solid formed when the curve \( y = \sqrt{x} + 3 \) is rotated about the \( x \)-axis from \( x = 1 \) to \( x = 6 \).

16. Find the volume of the solid formed when the curve \( y = \sqrt{x} \) is rotated about the \( y \)-axis between \( y = 1 \) and \( y = 4 \).

17. Find the volume of the solid formed when the curve \( y = \sqrt{4 - x^2} \) is rotated about the \( y \)-axis from \( y = 1 \) to \( y = 2 \).

18. Find the volume of the solid formed when the line \( x + 3y - 1 = 0 \) is rotated about the \( y \)-axis from \( y = 1 \) to \( y = 2 \).

19. Find the volume of the solid formed when the line \( x + 3y - 1 = 0 \) is rotated about the \( x \)-axis from \( x = 0 \) to \( x = 8 \).

20. The curve \( y = x^3 \) is rotated about the \( y \)-axis from \( y = 0 \) to \( y = 1 \). Find the volume of the solid formed.

21. Find the volume of the solid of revolution formed if the area enclosed between the curves \( y = x^2 \) and \( y = (x - 2)^2 \) is rotated about the \( x \)-axis.

22. The area bounded by the curve \( y = x^2 \) and the line \( y = x + 2 \) is rotated about the \( x \)-axis. Find the exact volume of the solid formed.

23. Show that the volume of a sphere is given by the formula \( V = \frac{4}{3} \pi r^3 \) by rotating the semicircle \( y = \sqrt{r^2 - x^2} \) about the \( x \)-axis.
The volume of water in a small lake is to be measured by finding the depth of water at regular intervals across the lake.

What could the volume of this lake be?

The table below shows the results of the measurements, where $x$ stands for the regular intervals chosen, in metres, and $y$ is the depth in metres.

<table>
<thead>
<tr>
<th>$x$ (m)</th>
<th>0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$ (m)</td>
<td>10.2</td>
<td>39.1</td>
<td>56.9</td>
<td>43.2</td>
<td>28.5</td>
<td>15.7</td>
<td>9.8</td>
</tr>
</tbody>
</table>

Using Simpson’s rule to find an approximation for the volume of the lake, correct to 2 significant figures:

$$V = \pi \int_{a}^{b} y^2 \, dx$$
$$= \pi \int_{0}^{300} y^2 \, dx$$

$$\int_{a}^{b} f(x) \, dx \approx \frac{h}{3} \left[ (y_0 + y_6) + 4(y_1 + y_5) + 2(y_2 + y_4) \right]$$

where $h = \frac{b - a}{6}$
$$= \frac{300 - 0}{6}$$
$$= 50$$
\[ \int_{0}^{100} y^2 \, dx = \frac{50}{3} \left[ (10.2^2 + 9.8^2) + 4(39.1^2 + 43.2^2 + 15.7^2) + 2(56.9^2 + 28.5^2) \right] \]
\[ = 381099.3 \]

\[ \therefore V = \pi \int_{0}^{100} y^2 \, dx \]
\[ = \pi \times 381099.3 \]
\[ \div 1197258.866 \]
\[ = 1200000 \text{ correct to 2 significant figures} \]

So the volume of the lake is approximately 1 200 000 m$^3$ of water.
1. (a) Use the trapezoidal rule with 2 subintervals to find an approximation to \[ \int_1^2 \frac{dx}{x^2}. \]
(b) Use integration to find the exact value of \[ \int_1^2 \frac{dx}{x^2}. \]

2. Find the indefinite integral (primitive function) of
   (a) \(3x + 1\)
   (b) \(\frac{5x^2 - x}{x}\)
   (c) \(\sqrt{x}\)
   (d) \((2x + 5)^3\)

3. In an experiment, the velocity of a particle over time was found and the results placed in the table below.

<table>
<thead>
<tr>
<th>(t)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v)</td>
<td>1.3</td>
<td>1.7</td>
<td>2.1</td>
<td>3.5</td>
<td>2.8</td>
<td>2.6</td>
<td>2.2</td>
</tr>
</tbody>
</table>

Use Simpson’s rule to find the approximate value of \(\int_0^6 f(t) \, dt\)

4. Evaluate
   (a) \(\int_0^2 (x^3 - 1) \, dx\)
   (b) \(\int_1^3 x^3 \, dx\)
   (c) \(\int_0^1 (3x - 1)^9 \, dx\)

5. Draw a primitive function of the following curves.
   (a)

6. Find the area bounded by the curve \(y = x^3\), the \(x\)-axis and the lines \(x = -1\) and \(x = 2\).

7. Use Simpson’s rule with 5 function values to find the approximate area enclosed between the curve \(y = \frac{1}{x}\), the \(x\)-axis and the lines \(x = 1\) and \(x = 3\).

8. Find the area enclosed between the curves \(y = x^2\) and \(y = 2 - x^2\).

9. The line \(y = 2x - 3\) is rotated about the \(x\)-axis from \(x = 0\) to \(x = 3\). Find the volume of the solid of revolution.

10. Evaluate \(\int_1^2 \frac{3x^4 - 2x^3 + x^2 - 1}{x^2} \, dx\).

11. Find the area bounded by the curve \(y = x^3\), the \(y\)-axis and the lines \(y = 0\) and \(y = 1\).

12. Find the indefinite integral (primitive function) of \((7x + 3)^3\).

13. Find the area bounded by the curve \(y = x^2 - x - 2\), the \(x\)-axis and the lines \(x = 1\) and \(x = 3\).

14. Find the volume of the solid formed if the curve \(y = x^2 + 1\) is rotated about the (a) \(x\)-axis from \(x = 0\) to \(x = 2\) (b) \(y\)-axis from \(y = 1\) to \(y = 2\)
15. (a) Change the subject of \( y = (x + 3)^2 \) to \( x \).
(b) Find the area bounded by the curve \( y = (x + 3)^2 \), the \( y \)-axis and the lines \( y = 9 \) and \( y = 16 \) in the first quadrant.
(c) Find the volume of the solid formed if this area is rotated about the \( y \)-axis.

16. Evaluate \( \int_0^4 (3t^2 - 6t + 5) \, dt \).

17. Find the area bounded by the curve \( y = x^2 + 2x - 15 \) and the \( x \)-axis.

18. Find the volume of the solid formed if the curve \( y = x^3 \) is rotated about the \( y \)-axis from \( y = 0 \) to \( y = 1 \).

19. Find
(a) \( \int 5(2x - 1)^4 \, dx \)
(b) \( \int \frac{3x^5}{4} \, dx \)

20. Challenge Exercise 3

1. (a) Find the area enclosed between the curves \( y = x^3 \) and \( y = x^2 \).
(b) Find the volume of the solid formed if this area is rotated about the \( x \)-axis.

2. (a) Show that \( f(x) = x^3 + x \) is an odd function.
(b) Hence find the value of \( \int_{-2}^{2} f(x) \, dx \).
(c) Find the total area between \( f(x) \), the \( x \)-axis and the lines \( x = -2 \) and \( x = 2 \).

3. The curve \( y = 2^x \) is rotated about the \( x \)-axis between \( x = 1 \) and \( x = 2 \). Use Simpson’s rule with 3 function values to find an approximation of the volume of the solid formed, correct to 3 significant figures.

4. Find the area enclosed between the curves \( y = (x - 1)^2 \) and \( y = 5 - x^2 \).

5. (a) Differentiate \((x^4 - 1)^5\).
(b) Hence find \( \int x^5(x^4 - 1)^5 \, dx \).

6. (a) Differentiate \( \frac{2x^2 + 1}{3x^2 - 4} \).
(b) Hence evaluate \( \int_0^1 \frac{x}{(3x^2 - 4)^2} \, dx \).

7. Find the area enclosed between the curve \( y = \frac{3}{x - 2} \), the \( y \)-axis and the lines \( y = 1 \) and \( y = 3 \), using the trapezoidal rule with 4 subintervals.

8. Find the exact volume of the solid formed by rotating the curve \( y = \frac{1}{x} \) about the \( x \)-axis between \( x = 1 \) and \( x = 3 \).

9. Show that the area enclosed between the curve \( y = \frac{1}{x} \), the \( x \)-axis and the lines \( x = 0 \) and \( x = 1 \) is infinite by using Simpson’s rule with 3 function values.

10. (a) Sketch the curve \( y = x(x - 1)(x + 2) \).
(b) Find the total area enclosed between the curve and the \( x \)-axis.

11. Find the exact area bounded by the parabola \( y = x^2 \) and the line \( y = 4 - x \).

12. Find the volume of the solid formed when the curve \( y = (x + 5)^2 \) is rotated about the \( y \)-axis from \( y = 1 \) to \( y = 4 \).
13. (a) Find the derivative of $x\sqrt{x + 3}$.
(b) Hence find $\int \frac{x + 2}{\sqrt{x + 3}} \, dx$.

14. (a) Evaluate $\int_{1}^{3} (x^2 - x + 1) \, dx$.
(b) Use Simpson’s rule with 3 function values to evaluate $\int_{1}^{3} (x^2 - x + 1) \, dx$ and show that this gives the exact value.

15. Find the area enclosed between the curves $y = \sqrt{x}$ and $y = x^3$.

16. For the shaded region below, find
(a) the area
(b) the volume when this area is rotated about the $y$-axis.
Exponential equation: Equation where the pronumeral is the index or exponent such as $3^x = 9$

Exponential function: A function in the form $y = a^x$ where the variable $x$ is a power or exponent

Logarithm: A logarithm is an index. The logarithm is the power or exponent of a number to a certain base i.e. $2^x = 8$ is the same as $\log_2 8 = x$
INTRODUCTION

THIS CHAPTER INTRODUCES A new irrational number, ‘e’, that has special properties in calculus. You will learn how to differentiate and integrate the exponential function \( f(x) = e^x \).

The definition and laws of logarithms are also introduced in this chapter, as well as differentiation and integration involving logarithms.

DID YOU KNOW?

John Napier (1550–1617), a Scottish theologian and an amateur mathematician, was the first to invent logarithms. These ‘natural’, or ‘Naperian’, logarithms were based on ‘e’. Napier was also one of the first mathematicians to use decimals rather than fractions. He invented the notation of the decimal, using either a comma or a point. The point was used in England, but a few European countries still use a comma.

Henry Briggs (1561–1630), an Englishman who was a professor at Oxford, decided that logarithms would be more useful if they were based on 10 (our decimal system). These are called common logarithms. Briggs painstakingly produced a table of logarithms correct to 14 decimal places. He also produced sine tables—to 15 decimal places—and tangent tables—to 10 decimal places.

The work on logarithms was greatly appreciated by Kepler, Galileo and other astronomers at the time, since they allowed the computation of very large numbers.

Differentiation of Exponential Functions

When differentiating exponential functions \( f(x) = a^x \) from first principles, an interesting result can be seen. The derivative of any exponential function gives a constant which is multiplied by the original function.

EXAMPLE

Sketch the derivative (gradient) function of \( y = 10^x \).

Solution
The graph of \( y = 10^x \) always has a positive gradient that is becoming steeper. So the derivative function will always be positive, becoming steeper.

The derivative function of an exponential function will always have a shape similar to the original function.

We can use differentiation from first principles to find how close this derivative function is to the original function.

**EXAMPLE**

Differentiate \( f(x) = 10^x \) from first principles.

**Solution**

\[
 f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
 = \lim_{h \to 0} \frac{10^{x+h} - 10^x}{h} \\
 = \lim_{h \to 0} \frac{10^x(10^h - 1)}{h} \\
 = 10^x \lim_{h \to 0} \frac{10^h - 1}{h}
\]

Using the \( 10^x \) key on the calculator, and finding values of \( \frac{10^h - 1}{h} \) when \( h \) is small, gives the result:

\[
 f'(x) \doteq 2.3026 \times 10^x
\]

or

\[
 \frac{d}{dx}(10^x) \doteq 2.3026 \times 10^x
\]

Drawing the graphs of \( y = 2.3026 \times 10^x \) and \( y = 10^x \) together shows how close the derivative function is to the original graph.
Similar results occur for other exponential functions. In general,  
\[ \frac{d}{dx}(a^x) = ka^x \]  where \( k \) is a constant.

**Application**

If \( y = a^x \) then \[ \frac{dy}{dx} = ka^x \]

This means that the rate of change of \( y \) is proportional to \( y \) itself. That is, if \( y \) is small, its rate of change is small, but if \( y \) is large, then it is changing rapidly.

This is called **exponential growth** (or **decay**, if \( k \) is negative) and has many applications in areas such as population growth, radioactive decay, the cooling of objects, the spread of infectious diseases and the growth of technology.

Different exponential functions have different values of \( k \).

**EXAMPLES**

1. \[ \frac{d}{dx}(2^x) = 0.6931 \times 2^x. \]

   ![Graph of y = 2^x and y = 0.6931 \times 2^x](image)

2. \[ \frac{d}{dx}(3^x) = 1.0986 \times 3^x. \]

   ![Graph of y = 3^x and y = 1.0986 \times 3^x](image)
Notice that the derivative function of \( y = 3^x \) is very close to the original function.

We can find a number close to 3 that gives exactly the same graph for the derivative function. This number is approximately 2.71828, and is called \( e \).

\[
\frac{d}{dx}(e^x) = e^x
\]
2. Differentiate $5e^x$.

**Solution**

\[
\frac{d}{dx}(e^x) = e^x
\]

\[
\therefore \frac{d}{dx}(5e^x) = 5 \frac{d}{dx}(e^x) = 5e^x
\]

3. Find the equation of the tangent to the curve $y = 3e^x$ at the point $(0, 3)$.

**Solution**

\[
\frac{dy}{dx} = 3e^x
\]

At $(0, 3)$,

\[
\frac{dy}{dx} = 3e^0 = 3
\]

\[
\therefore m = 3
\]

Equation:

\[
y - y_1 = m(x - x_1)
\]

\[
y - 3 = 3(x - 0)
\]

\[
y = 3x + 3
\]
4. Differentiate \( \frac{2x + 3}{e^x} \).

Solution

\[
\frac{dy}{dx} = \frac{u'v - v'u}{v^2} = \frac{2e^x - e^x(2x + 3)}{(e^x)^2} = \frac{2e^x - 2xe^x - 3e^x}{e^{2x}} = \frac{-2xe^x}{e^{2x}} = \frac{-e^x(1 + 2x)}{e^{2x}} = \frac{-1 + 2x}{e^x}
\]

4.1 Exercises

1. Find, correct to 2 decimal places, the value of
   (a) \( e^{1.5} \)
   (b) \( e^{-2} \)
   (c) \( 2e^{0.3} \)
   (d) \( \frac{1}{e^3} \)
   (e) \(-3e^{-3.1}\)

2. Sketch the curve
   (a) \( y = 2e^x \)
   (b) \( y = e^{-x} \)
   (c) \( y = -e^x \)

3. Differentiate
   (a) \( 9e^x \)
   (b) \( -e^x \)
   (c) \( e^x + x^2 \)
   (d) \( 2x^3 - 3x^2 + 5x - e^x \)
   (e) \( (e^x + 1)^3 \)
   (f) \( (e^x + 5)^7 \)
   (g) \( (2e^x - 3)^2 \)
   (h) \( xe^x \)
   (i) \( \frac{e^x}{x} \)
   (j) \( x^2e^x \)
   (k) \( (2x + 1)e^x \)
   (l) \( \frac{e^x}{7x - 3} \)
   (m) \( \frac{5x}{e^x} \)

4. If \( f(x) = x^3 + 3x - e^x \), find \( f'(1) \) and \( f''(1) \) in terms of \( e \).

5. Find the exact gradient of the tangent to the curve \( y = e^x \) at the point \( (1, e) \).

6. Find the exact gradient of the normal to the curve \( y = e^x \) at the point where \( x = 5 \).

7. Find the gradient of the tangent to the curve \( y = 4e^x \) at the point where \( x = 1.6 \), correct to 2 decimal places.

8. Find the equation of the tangent to the curve \( y = -e^x \) at the point \( (1, -e) \).
9. Find the equation of the normal to the curve \( y = e^x \) at the point where \( x = 3 \), in exact form.

10. Find the stationary point on the curve \( y = xe^x \) and determine its nature. Hence sketch the curve.

### Function of a function rule

Remember that the function of a function rule uses the result

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.
\]

You studied this in Chapter 8 of the Preliminary Course book.

#### EXAMPLE

Differentiate \( e^{x^2 + 5x - 3} \).

**Solution**

Let \( u = x^2 + 5x - 3 \)

Then \( y = e^u \)

\[
\frac{du}{dx} = 2x + 5 \quad \text{and} \quad \frac{dy}{du} = e^u
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \cdot (2x + 5) = e^{x^2 + 5x - 3}(2x + 5) = (2x + 5)e^{x^2 + 5x - 3}
\]

**Proof**

Let \( u = f(x) \)

Then \( y = e^u \)

\[
\frac{dy}{du} = e^u \quad \text{and} \quad \frac{du}{dx} = f'(x)
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \cdot f'(x) = f'(x)e^{f(x)}
\]

11. Find the first and second derivatives of \( y = 7e^x \). Hence show that \( \frac{d^2y}{dx^2} = y \).

12. If \( y = 2e^x + 1 \), show that \( \frac{d^2y}{dx^2} = y - 1 \).
**EXAMPLES**

1. Differentiate $e^{5x-2}$

**Solution**

\[ y' = f'(x)e^{f(x)} = 5e^{5x-2} \]

2. Differentiate $x^2e^{3x}$.

**Solution**

\[ \frac{dy}{dx} = u'v + v'u = 2x \cdot e^{3x} + 3e^{3x} \cdot x^2 \]
\[ = xe^{3x}(2+3x) \]

3. Given $y = 2e^{3x} + 1$, show that $\frac{d^2y}{dx^2} = 9(y - 1)$.

**Solution**

\[ y = 2e^{3x} + 1 \]
\[ \frac{dy}{dx} = 6e^{3x} \]
\[ \frac{d^2y}{dx^2} = 18e^{3x} \]
\[ = 9(2e^{3x}) \]
\[ = 9(2e^{3x} + 1 - 1) \]
\[ = 9(y - 1) \]

4.2 Exercises

1. Differentiate
   (a) $e^{2x}$
   (b) $e^{-x}$
   (c) $e^{6x+2}$
   (d) $e^{x+1}$
   (e) $e^{x+5x+7}$
   (f) $e^{5x}$
   (g) $e^{-2x}$
   (h) $e^{10x}$
   (i) $e^{2x} + x$
   (j) $x^2 + 2x + e^{1-x}$
   (k) $(x + e^{4x})^5$
   (l) $xe^{2x}$
   (m) $\frac{e^{3x}}{x^2}$
   (n) $x^3e^{5x}$
   (o) $\frac{e^{2x+1}}{2x+5}$

2. Find the second derivative of $(e^{2x} + 1)^7$.

3. If $f(x) = e^{3x-2}$, find the exact value of $f'(1)$ and $f''(0)$.

4. Find the gradient of the tangent to the curve $y = e^{3x}$ at the point where $x = 0$. 
Integration of Exponential Functions

Since \( \frac{d}{dx} (e^x) = e^x \), then the reverse must be true.

\[ \int e^x \, dx = e^x + C \]

To find the indefinite integral (primitive function) when the function of a function rule is involved, look at the derivative first.

EXAMPLE

Differentiate \( e^{2x+1} \).

Hence find \( \int 2e^{2x+1} \, dx \).

Find \( \int e^{2x+1} \, dx \).

Solution

\[
\frac{d}{dx} (e^{2x+1}) = 2e^{2x+1} \\
\therefore \int 2e^{2x+1} \, dx = e^{2x+1} + C \\
\int e^{2x+1} \, dx = \frac{1}{2} \int 2e^{2x+1} \, dx \\
= \frac{1}{2} e^{2x+1} + C
\]
In general

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$$

**Proof**

$$\frac{d}{dx} (e^{ax+b}) = ae^{ax+b}$$

$$\therefore \int ae^{ax+b} \, dx = e^{ax+b} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a} \int ae^{ax+b} \, dx$$

$$= \frac{1}{a} e^{ax+b} + C$$

---

### EXAMPLES

1. Find \( \int (e^{2x} - e^{-x}) \, dx \).

**Solution**

\[
\int (e^{2x} - e^{-x}) \, dx = \frac{1}{2} e^{2x} - \frac{1}{(-1)} e^{-x} + C \\
= \frac{1}{2} e^{2x} + e^{-x} + C
\]

2. Find the exact area enclosed between the curve \( y = e^{3x} \), the x-axis and the lines \( x = 0 \) and \( x = 2 \).

**Solution**

Area = \[
\int_{0}^{2} e^{3x} \, dx \\
\left[ \frac{1}{3} e^{3x} \right]_{0}^{2} \\
\frac{1}{3} e^{6} - \frac{1}{3} e^{0} \\
\frac{1}{3} (e^{6} - e^{0}) \\
= \frac{1}{3} (e^{6} - 1) \text{ units}^2 \\
\]

3. Find the volume of the solid of revolution formed when the curve \( y = e^{x} \) is rotated about the x-axis from \( x = 0 \) to \( x = 2 \).

**Solution**

\[
y = e^{x} \\
\therefore y^2 = (e^{x})^2 \\
= e^{2x}
\]

Use index laws to simplify \((e^{x})^2\).
Chapter 4 Exponential and Logarithmic Functions

\[ V = \pi \int_a^b y^2 \, dx \]
\[ = \pi \int_0^2 e^{2x} \, dx \]
\[ = \pi \left[ \frac{1}{2} e^{2x} \right]_0^2 \]
\[ = \pi \left( \frac{1}{2} e^4 - \frac{1}{2} e^0 \right) \]
\[ = \pi \left( \frac{1}{2} e^4 - \frac{1}{2} \right) \]
\[ = \frac{\pi}{2} (e^4 - 1) \text{ units}^3 \]

4.3 Exercises

1. Find these indefinite integrals.
   (a) \( \int e^{2x} \, dx \)
   (b) \( \int e^{4x} \, dx \)
   (c) \( \int e^{-3x} \, dx \)
   (d) \( \int e^{5x} \, dx \)
   (e) \( \int e^{-2x} \, dx \)
   (f) \( \int e^{4x+1} \, dx \)
   (g) \( \int -3e^{5x} \, dx \)
   (h) \( \int e^{2x} \, dt \)
   (i) \( \int (e^{2x} - 2) \, dx \)
   (j) \( \int (e^{x-3} + x) \, dx \)

2. Evaluate in exact form.
   (a) \( \int_0^1 e^{5x} \, dx \)
   (b) \( \int_0^2 -e^{-x} \, dx \)
   (c) \( \int_1^4 2e^{3x+4} \, dx \)
   (d) \( \int_0^3 (3x^2 - e^{3x}) \, dx \)
   (e) \( \int_0^2 (e^{2x} + 1) \, dx \)
   (f) \( \int_1^2 (e^x - x) \, dx \)
   (g) \( \int_0^3 (e^{3x} - e^{-x}) \, dx \)

3. Evaluate correct to 2 decimal places.
   (a) \( \int_1^3 e^{-x} \, dx \)
   (b) \( \int_0^2 2e^{3y} \, dy \)
   (c) \( \int_5^6 (e^{x+5} + 2x - 3) \, dx \)
   (d) \( \int_0^1 (e^{3x+4} - t) \, dt \)
   (e) \( \int_1^2 (e^{4x} + e^{2x}) \, dx \)

4. Find the exact area enclosed by the curve \( y = 2e^{2x} \), the x-axis and the lines \( x = 1 \) and \( x = 2 \).

5. Find the exact area bounded by the curve \( y = e^{4x-3} \), the x-axis and the lines \( x = 0 \) and \( x = 1 \).

6. Find the area enclosed by the curve \( y = x + e^{-x} \), the x-axis and the lines \( x = 0 \) and \( x = 2 \), correct to 2 decimal places.

7. Find the area bounded by the curve \( y = e^{3x} \), the x-axis and the lines \( x = 0 \) and \( x = 1 \), correct to 3 significant figures.

8. Find the exact volume of the solid of revolution formed when the curve \( y = e^x \) is rotated about the x-axis from \( x = 0 \) to \( x = 3 \).

9. Find the volume of the solid formed when the curve \( y = e^{-x} + 1 \) is rotated about the x-axis from \( x = 1 \) to \( x = 2 \), correct to 1 decimal place.
10. Use Simpson’s rule with 3 function values to find an approximation to \( \int_1^2 xe^x \, dx \), correct to 1 decimal place.

11. (a) Differentiate \( x^2 e^x \).
(b) Hence find \( \int (2 + x)e^x \, dx \).

12. The curve \( y = \sqrt{e^x + 1} \) is rotated about the x-axis from \( x = 0 \) to \( x = 1 \). Find the exact volume of the solid formed.

13. Find the exact area enclosed between the curve \( y = e^{2x} \) and the lines \( y = 1 \) and \( x = 2 \).

**Application**

The exponential function occurs in many fields, such as science and economics.

- \( P = P_0 e^{kt} \) is a general formula that describes exponential growth.
- \( P = P_0 e^{-kt} \) is a general formula that describes exponential decay.

**Logarithms**

‘Logarithm’ is another name for the index or power of a number. Logarithms are related to exponential functions, and allow us to solve equations like \( 2^x = 5 \). They also allow us to change the subject of exponential equations such as \( y = e^x \) to \( x \).

**Definition**

If \( y = a^x \), then \( x \) is called the logarithm of \( y \) to the base \( a \).

\[
\text{If } y = a^x, \text{ then } x = \log_a y
\]

**Logarithm keys**

- \( \log \) is used for \( \log_{10} x \)
- \( \ln \) is used for \( \log_e x \)
EXAMPLES

1. Find \( \log_{10} 5.3 \) correct to 1 decimal place.

Solution

\[
\log_{10} 5.3 = 0.724275869
\]

= 0.7 correct to 1 decimal place

2. Evaluate \( \log_e 80 \) correct to 3 significant figures.

Solution

\[
\log_e 80 = 4.382026634
\]

= 4.38 correct to 3 significant figures

3. Evaluate \( \log_3 81 \).

Solution

Let \( \log_3 81 = x \)

Then \( 3^x = 81 \) (by definition)

i.e. \( 3^x = 3^4 \)

\( \therefore \) \( x = 4 \)

So \( \log_3 81 = 4. \)

4. Find the value of \( \log_2 \frac{1}{4} \).

Solution

Let \( \log_2 \frac{1}{4} = x \)

Then \( 2^x = \frac{1}{4} \)

\( = \frac{1}{2^2} \)

\( = 2^{-2} \)

\( \therefore \) \( x = -2 \)

So \( \log_2 \frac{1}{4} = -2. \)
Class Investigation

1. Sketch the graph of \( y = \log_2 x \).
   There is no calculator key for logarithms to the base 2. Use the definition of a logarithm to change the equation into index form, and the table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. On the same set of axes, sketch the curve \( y = 2^x \) and the line \( y = x \). What do you notice?

4.4 Exercises

1. Evaluate
   (a) \( \log_2 16 \)
   (b) \( \log_4 16 \)
   (c) \( \log_5 125 \)
   (d) \( \log_3 3 \)
   (e) \( \log_7 49 \)
   (f) \( \log_7 7 \)
   (g) \( \log_5 1 \)
   (h) \( \log_2 128 \)

2. Evaluate
   (a) \( 3 \log_3 8 \)
   (b) \( \log_3 25 + 1 \)
   (c) \( 3 \log_3 81 \)
   (d) \( 4 \log_2 27 \)
   (e) \( 2 \log_5 10000 \)
   (f) \( 1 + \log_6 64 \)
   (g) \( 3 \log_4 64 + 5 \)
   (h) \( 2 + 4 \log_6 216 \)
   (i) \( \frac{\log_5 9}{2} \)
   (j) \( \frac{\log_8 64 + 4}{\log_2 8} \)

3. Evaluate
   (a) \( \log_2 \frac{1}{2} \)
   (b) \( \log_3 \sqrt{3} \)
   (c) \( \log_4 2 \)
   (d) \( \log_5 \frac{1}{25} \)
   (e) \( \log_7 \sqrt[4]{7} \)
   (f) \( \log_3 \frac{1}{\sqrt{3}} \)
   (g) \( \log_4 \frac{1}{2} \)
   (h) \( \log_8 2 \)
   (i) \( \log_6 6\sqrt{6} \)
   (j) \( \log_2 \frac{\sqrt{2}}{4} \)

4. Evaluate correct to 2 decimal places.
   (a) \( \log_{10} 1200 \)
   (b) \( \log_{10} 875 \)
   (c) \( \log_{10} 25 \)
   (d) \( \ln 140 \)
   (e) \( 5 \ln 8 \)
   (f) \( \log_{10} 350 + 4.5 \)
   (g) \( \log_{10} 15 \)
   (h) \( \ln 9.8 + \log_{10} 17 \)
   (i) \( \frac{\log_{10} 30}{\log_{10} 30} \)
   (j) \( 4 \ln 10 - 7 \)
5. Write in logarithmic form.
   (a) \(3^x = y\)
   (b) \(5^x = z\)
   (c) \(x^2 = y\)
   (d) \(2^b = a\)
   (e) \(b^3 = d\)
   (f) \(y = 8^x\)
   (g) \(y = 6^x\)
   (h) \(y = e^x\)
   (i) \(y = a^x\)
   (j) \(Q = e^x\)

6. Write in index form.
   (a) \(\log x = 5\)
   (b) \(\log x = 7\)
   (c) \(\log a = b\)
   (d) \(\log y = 9\)
   (e) \(\log b = y\)
   (f) \(y = \log_2 6\)
   (g) \(y = \log_3 x\)
   (h) \(y = \log_{10} 9\)
   (i) \(y = \ln 4\)
   (j) \(y = \log_7 x\)

7. Solve for \(x\), correct to 1 decimal place where necessary.
   (a) \(\log_{10} x = 6\)
   (b) \(\log_x 5 = 3\)
   (c) \(\log_x 343 = 3\)
   (d) \(\log_6 64 = 6\)
   (e) \(\log_x 1 = x\)
   (f) \(\log_x \sqrt{3} = \frac{1}{2}\)
   (g) \(\ln x = 3.8\)
   (h) \(3 \log_{10} x - 2 = 10\)
   (i) \(\log_4 x = \frac{3}{2}\)
   (j) \(\log_x 4 = \frac{1}{3}\)

8. Evaluate \(y\) given that \(\log_y 125 = 3\).

9. If \(\log_{10} x = 1.65\), evaluate \(x\) correct to 1 decimal place.

10. Evaluate \(b\) to 3 significant figures if \(\log_e b = 0.894\).

11. Find the value of \(\log_2 1\). What is the value of \(\log_3 1\)?

12. Evaluate \(\log_5 5\). What is the value of \(\log_3 a\)?

13. (a) Evaluate \(\ln e\) without a calculator.
   (b) Using a calculator, evaluate
      (i) \(\log_e e^1\)
      (ii) \(\log_e e^2\)
      (iii) \(\log_e e^3\)
      (iv) \(\log_e \sqrt{e}\)
      (v) \(\log_e \frac{1}{e}\)
      (vi) \(e^{\ln 2}\)
      (vii) \(e^{\ln 3}\)
      (viii) \(e^{\ln 5}\)
      (ix) \(e^{\ln 7}\)
      (x) \(e^{\ln 1}\)
      (xi) \(e^{\ln e}\)

14. Sketch the graph of \(y = \log_x x\).
    What is its domain and range?

15. Sketch \(y = 10^x\), \(y = \log_{10} x\) and \(y = x\) on the same number plane.
    What do you notice about the relationship of the curves to the line?

16. Change the subject of \(y = \log_e x\) to \(x\).
Class Discussion

1. Investigate these questions on the calculator. Can you see some patterns?
   (a) \( \log_e e \)
   (b) \( \log_e e^2 \)
   (c) \( \log_e e^3 \)
   (d) \( \log_e e^4 \)
   (e) \( \log_e e^5 \)
   Can you write a rule for \( \log_e e^2 \)?

2. Evaluate using a calculator. Can you write a rule to show this pattern?
   (a) \( e^{\ln 1} \)
   (b) \( e^{\ln 2} \)
   (c) \( e^{\ln 3} \)
   (d) \( e^{\ln 4} \)
   (e) \( e^{\ln 5} \)
   Can you write a rule for \( e^{\ln x} \)?

3. Do these rules work if \( x \) is negative?

Logarithm laws

Because logarithms are closely related to indices there are logarithm laws that correspond to the index laws.

\[
\log_a (xy) = \log_a x + \log_a y
\]

Proof

Let \( x = a^m \) and \( y = a^n \)
Then \( m = \log_a x \) and \( n = \log_a y \)
\[
x y = a^m \times a^n = a^{m+n}
\]
\[
\therefore \log_a (xy) = m + n \quad \text{(by definition)}
\]
\[
= \log_a x + \log_a y
\]

\[
\log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y
\]
Chapter 4  Exponential and Logarithmic Functions

Proof

Let \( x = a^m \) and \( y = a^n \)
Then \( m = \log_a x \) and \( n = \log_a y \)

\[
\frac{x}{y} = a^m \cdot a^n = a^{m+n}
\]

\[
\therefore \log_a \left( \frac{x}{y} \right) = m - n \quad \text{(by definition)}
\]

\[
= \log_a x - \log_a y
\]

\[
\log_a x^n = n \log_a x
\]

Proof

Let \( x = a^m \)
Then \( m = \log_a x \)

\[
x^n = (a^m)^n = a^{mn}
\]

\[
\therefore \log_a x^n = mn \quad \text{(by definition)}
\]

\[
= n \log_a x
\]

EXAMPLES

1. Evaluate \( \log_6 3 + \log_6 12 \).

Solution

\[
\log_6 3 + \log_6 12 = \log_6 (3 \times 12)
\]

\[
= \log_6 36
\]

\[
= 2 
\]

2. Given \( \log_5 3 = 0.68 \) and \( \log_5 4 = 0.86 \), find
(a) \( \log_5 12 \)
(b) \( \log_5 0.75 \)
(c) \( \log_5 9 \)
(d) \( \log_5 20 \)

CONTINUED
Solution

(a) \( \log_5 12 = \log_5 (3 \times 4) \)
\[= \log_5 3 + \log_5 4 \]
\[= 0.68 + 0.86 \]
\[= 1.54 \]

(b) \( \log_5 0.75 = \log_5 \frac{3}{4} \)
\[= \log_5 3 - \log_5 4 \]
\[= 0.68 - 0.86 \]
\[= -0.18 \]

(c) \( \log_5 9 = \log_5 3^2 \)
\[= 2 \log_5 3 \]
\[= 2 \times 0.68 \]
\[= 1.36 \]

(d) \( \log_5 20 = \log_5 (5 \times 4) \)
\[= \log_5 5 + \log_5 4 \]
\[= 1 + 0.86 \]
\[= 1.86 \]

3. Solve \( \log_2 12 = \log_2 3 + \log_2 x \).

Solution

\[\log_2 12 = \log_2 3 + \log_2 x\]
\[= \log_2 3x\]
So \( 12 = 3x \)
\( 4 = x \)

4.5 Exercises

1. Use the logarithm laws to simplify
   (a) \( \log_a 4 + \log_a y \)
   (b) \( \log_a 4 + \log_a 5 \)
   (c) \( \log_a 12 - \log_a 3 \)
   (d) \( \log_a b - \log_a 5 \)
   (e) \( 3 \log_a y + \log_a z \)
   (f) \( 2 \log_a 3 + 3 \log_a y \)
   (g) \( 5 \log_a x - 2 \log_a y \)
   (h) \( \log_a x + \log_a y - \log_a z \)
   (i) \( \log_{10} a + 4 \log_{10} b + 3 \log_{10} c \)
   (j) \( 3 \log_3 p + \log_3 q - 2 \log_3 r \)

2. Given \( \log_2 2 = 0.36 \) and \( \log_7 5 = 0.83 \), find
   (a) \( \log_2 10 \)
   (b) \( \log_7 0.4 \)
   (c) \( \log_2 20 \)
   (d) \( \log_7 25 \)
   (e) \( \log_7 8 \)
   (f) \( \log_7 14 \)
   (g) \( \log_7 50 \)
   (h) \( \log_7 35 \)
   (i) \( \log_9 98 \)
   (j) \( \log_7 70 \)
3. Use the logarithm laws to evaluate
   (a) $\log_5 50 - \log_5 2$
   (b) $\log_2 16 + \log_2 4$
   (c) $\log_{10} 2 + \log_{10} 8$
   (d) $\log_{10} 500 - \log_{10} 4$
   (e) $\log_9 117 - \log_9 13$
   (f) $\log_3 32 + \log_3 16$
   (g) $3 \log_2 2 + 2 \log_2 4$
   (h) $2 \log_4 6 - (2 \log_4 3 + \log_4 2)$
   (i) $\log_6 4 - 2 \log_6 12$
   (j) $2 \log_3 6 + \log_3 18 - 3 \log_3 2$

4. If $\log_a 3 = x$ and $\log_a 5 = y$, find an expression in terms of $x$ and $y$ for
   (a) $\log_a 15$
   (b) $\log_a 0.6$
   (c) $\log_a 27$
   (d) $\log_a 25$
   (e) $\log_a 9$
   (f) $\log_a 75$
   (g) $\log_a 3a$
   (h) $\log_a \frac{a}{5}$
   (i) $\log_a 9a$
   (j) $\log_a \frac{125}{a}$

5. If $\log_a x = p$ and $\log_a y = q$, find, in terms of $p$ and $q$.
   (a) $\log_a xy$
   (b) $\log_a y^3$
   (c) $\log_a \frac{y}{x}$
   (d) $\log_a x^2$
   (e) $\log_a xy^5$
   (f) $\log_a \frac{x^2}{y}$
   (g) $\log_a ax$
   (h) $\log_a \frac{a}{y^2}$
   (i) $\log_a a^3y$
   (j) $\log_a \frac{x}{dy}$

6. If $\log_a b = 3.4$ and $\log_a c = 4.7$, evaluate
   (a) $\log_a \frac{c}{b}$
   (b) $\log_a bc^2$
   (c) $\log_a (bc)^2$
   (d) $\log_a abc$
   (e) $\log_a a^3c$
   (f) $\log_a b^7$
   (g) $\log_a \frac{a}{c}$
   (h) $\log_a a^3$
   (i) $\log_a bc^4$
   (j) $\log_a b^4c^2$

7. Solve
   (a) $\log_4 12 = \log_4 x + \log_4 3$
   (b) $\log_3 4 = \log_3 y - \log_3 7$
   (c) $\log_6 6 = \log_6 x - 3 \log_6 2$
   (d) $\log_2 81 = 4 \log_2 x$
   (e) $\log_3 54 = \log_3 k + 2 \log_3 3$

Change of base

Sometimes we need to evaluate logarithms such as $\log_{10} 7$. We use a change of base formula.

$$\log_a x = \frac{\log_b x}{\log_b a}$$
**Proof**

Let \( y = \log_a x \)

Then \( x = a^y \)

Take logarithms to the base \( b \) of both sides of the equation:

\[
\log_b x = \log_b a^y = y \log_b a
\]

\[
\therefore \frac{\log_b x}{\log_b a} = y = \log_a x
\]

You can use the change of base formula to find the logarithm of any number, such as \( \log_2 2 \). You change it to either \( \log_{10} x \) or \( \log_e x \), and use a calculator.

---

**EXAMPLE**

Find the value of \( \log_3 2 \), correct to 2 decimal places.

**Solution**

\[
\log_3 2 = \frac{\log 2}{\log 3}
\]

(by change of base)

\[
\div 0.430676558
\]

\[
= 0.43
\]

---

**Exponential equations**

You can also use the change of base formula to solve exponential equations such as \( 5^x = 7 \).

You studied exponential equations such as \( 2^x = 8 \) in the Preliminary Course. Exponential equations such as \( 2^x = 9 \) can be solved by taking logarithms of both sides, or by using the definition of a logarithm and the change of base formula.
EXAMPLES

1. Solve \(5^x = 7\) correct to 1 decimal place.

Solution

\(5^x = 7\)

Using the definition of a logarithm, this means:

\[
\log_5 7 = x
\]

\[
\log 7 = x
\]

(\text{using change of base formula})

\[
\frac{\log 7}{\log 5} = x
\]

\[
x = \frac{\log 7}{\log 5}
\]

\[
= 1.2 \text{ correct to 1 decimal place}
\]

If you do not like to solve the equation this way, you can use the logarithm laws instead.

Taking logs of both sides:

\[
\log 5^x = \log 7
\]

\[
x \log 5 = \log 7
\]

\[
\therefore x = \frac{\log 7}{\log 5}
\]

\[
= 1.2 \text{ correct to 1 decimal place}
\]

2. Solve \(4^{y-3} = 9\) correct to 2 decimal places.

Solution

\(4^{y-3} = 9\)

Using the logarithm definition and change of base:

\[
\log_4 9 = y - 3
\]

\[
\frac{\log 9}{\log 4} = y - 3
\]

\[
\log 9 = \log 4^{y-3}
\]

\[
\log 9 = (y-3) \log 4
\]

\[
\log 9 = y \log 4 - 3 \log 4
\]

\[
\log 9 = y \log 4 + 3
\]

\[
y = \frac{\log 9}{\log 4} + 3
\]

\[
y = 4.58 \text{ correct to 2 decimal place}
\]
4.6 Exercises

1. Use the change of base formula to evaluate to 2 decimal places.
   (a) \( \log_4 9 \)
   (b) \( \log_6 25 \)
   (c) \( \log_5 200 \)
   (d) \( \log_7 12 \)
   (e) \( \log_3 23 \)
   (f) \( \log_9 250 \)
   (g) \( \log_9 9.5 \)
   (h) \( 2 \log_9 23.4 \)
   (i) \( 7 – \log_9 108 \)
   (j) \( 3 \log_9 340 \)

2. By writing each equation as a logarithm and changing the base, solve the equation correct to 2 significant figures.
   (a) \( 4^x = 9 \)
   (b) \( 3^x = 5 \)
   (c) \( 7^x = 14 \)
   (d) \( 2^x = 15 \)
   (e) \( 5^x = 34 \)
   (f) \( 6^x = 60 \)
   (g) \( 2^x = 76 \)
   (h) \( 4^x = 50 \)
   (i) \( 3^x = 23 \)
   (j) \( 9^x = 210 \)

3. Solve, correct to 2 decimal places.
   (a) \( 2^x = 6 \)
   (b) \( 5^y = 15 \)
   (c) \( 3^r = 20 \)
   (d) \( 7^m = 32 \)
   (e) \( 4^t = 50 \)
   (f) \( 3^f = 4 \)
   (g) \( 8^h = 11 \)
   (h) \( 2^p = 57 \)
   (i) \( 4^s = 81.3 \)
   (j) \( 6^n = 102.6 \)

4. Solve, to 1 decimal place.
   (a) \( 3^{x+1} = 8 \)
   (b) \( 5^{3y} = 71 \)
   (c) \( 2^{x+3} = 12 \)
   (d) \( 4^{20-1} = 7 \)
   (e) \( 7^{5x+2} = 11 \)
   (f) \( 8^{3-n} = 5.7 \)
   (g) \( 2^{x/2} = 18.3 \)
   (h) \( 3^{x/3} = 32.9 \)
   (i) \( 9^2 = 50 \)
   (j) \( 6^{2x+1} = 61.3 \)

5. Solve each equation correct to 3 significant figures.
   (a) \( e^x = 200 \)
   (b) \( e^{3t} = 5 \)
   (c) \( 2e^t = 75 \)
   (d) \( 45 = e^x \)
   (e) \( 3000 = 100e^n \)
   (f) \( 100 = 20e^{3t} \)
   (g) \( 2000 = 50e^{0.15t} \)
   (h) \( 15000 = 2000e^{0.03t} \)
   (i) \( 3Q = Qe^{0.02t} \)
   (j) \( 0.5M = Me^{0.016t} \)
Derivative of the Logarithmic Function

Drawing the derivative (gradient) function of a logarithm function gives a hyperbola.

**EXAMPLE**

Sketch the derivative function of \( y = \log_2 x \).

![Graph of the derivative function of \( y = \log_2 x \).

**Solution**

The gradient is always positive but is decreasing.

![Graph showing the decreasing gradient of the derivative function.

If \( y = \log_e x \) then \( \frac{dy}{dx} = \frac{1}{x} \)

**Proof**

\[
\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}
\]

Given \( y = \log_e x \)

Then \( x = e^y \)

\[
\frac{dx}{dy} = e^y
\]

\[
\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}
\]
Function of a function rule

If \( y = \log_e f(x) \), then \( \frac{dy}{dx} = f'(x) \cdot \frac{1}{f(x)} \)

**Proof**

Let \( u = f(x) \)

Then \( y = \log_e u \)

\[ \therefore \frac{dy}{du} = \frac{1}{u} \]

Also \( \frac{du}{dx} = f'(x) \)

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

\[ = \frac{1}{u} \cdot f'(x) \]

\[ = \frac{1}{f(x)} \cdot f'(x) \]

**EXAMPLES**

1. Differentiate \( \log_e (x^2 - 3x + 1) \).

**Solution**

\[ \frac{d}{dx} [\log_e (x^2 - 3x + 1)] = \frac{2x - 3}{x^2 - 3x + 1} \]

2. Differentiate \( \log_e \frac{x + 1}{3x - 4} \).

**Solution**

Let \( y = \log_e \frac{x + 1}{3x - 4} \)

\[ = \log_e (x + 1) - \log_e (3x - 4) \]

\[ \frac{dy}{dx} = \frac{1}{x + 1} - \frac{3}{3x - 4} \]

\[ = \frac{1(3x - 4) - 3(x + 1)}{(x + 1)(3x - 4)} \]

\[ = \frac{3x - 4 - 3x - 3}{(x + 1)(3x - 4)} \]

\[ = \frac{-7}{(x + 1)(3x - 4)} \]
3. Find the gradient of the normal to the curve \( y = \log_e (x^3 - 5) \) at the point where \( x = 2 \).

**Solution**

\[
\frac{dy}{dx} = \frac{3x^2}{x^3 - 5}
\]

When \( x = 2 \),

\[
\frac{dy}{dx} = \frac{3(2)^2}{2^3 - 5} = \frac{12}{3} = 4
\]

The normal is perpendicular to the tangent

\[ m_1 m_2 = -1 \Rightarrow 4m_2 = -1 \]

\[ m_2 = -\frac{1}{4} \]

4. Differentiate \( y = \log_2 x \).

**Solution**

\[
y = \log_2 x = \frac{\log x}{\log 2} = \frac{1}{\log_2 x} \times \log x
\]

\[
\frac{dy}{dx} = \frac{1}{\log_2 x} \times \frac{1}{x} = \frac{1}{x \log_2 x}
\]

5. Find the derivative of \( 2^x \).

**Solution**

\[
2^x = e^{\ln 2}
\]

\[
\therefore 2^x = (e^{\ln 2})^x = e^{x \ln 2}
\]

\[
\frac{dy}{dx} = \ln 2 e^{x \ln 2} = 2^x \ln 2
\]
4.7 Exercises

1. Differentiate
   (a) $x + \log_e x$
   (b) $1 - \log_e 3x$
   (c) $\ln(3x + 1)$
   (d) $\log_e (x^2 - 4)$
   (e) $\ln(5x^4 + 3x - 9)$
   (f) $\log_e (5x + 1) + x^2$
   (g) $3x^2 + 5x - 5 + \ln 4x$
   (h) $\log_e (8x - 9) + 2$
   (i) $\log_e (2x + 4)(3x - 1)$
   (j) $\log_2 \frac{4x + 1}{2x - 7}$
   (k) $(1 + \log_e x)^5$
   (l) $(\ln x - x)^9$
   (m) $(\log_e x)^4$
   (n) $(x^2 + \log_e x)^6$
   (o) $x \log_e x$
   (p) $\frac{\log_e x}{x}$
   (q) $(2x + 1) \log_e x$
   (r) $x^3 \log_e (x + 1)$
   (s) $\log_e (\log_e x)$
   (t) $\frac{\ln x}{x - 2}$
   (u) $\frac{e^{2x}}{\log_e x}$
   (v) $e^x \ln x$
   (w) $5(\log_e x)^2$

2. If $f(x) = \log_e \sqrt{2 - x}$, find $f''(1)$.

3. Find the derivative of $\log_{10} x$.

4. Find the equation of the tangent to the curve $y = \log_e x$ at the point $(2, \log_e 2)$.

5. Find the equation of the tangent to the curve $y = \log_e (x - 1)$ at the point where $x = 2$.

6. Find the gradient of the normal to the curve $y = \log_e (x^4 + x)$ at the point $(1, \log_e 2)$.

7. Find the exact equation of the normal to the curve $y = \log_e x$ at the point where $x = 5$.

8. Find the equation of the tangent to the curve $y = \log_e (5x + 4)$ at the point where $x = 3$.

9. Find the point of inflexion on the curve $y = x \log_e x - x^2$.

10. Find the stationary point on the curve $y = \frac{\ln x}{x}$ and determine its nature.

11. Sketch, showing any stationary points and inflexions.
   (a) $y = x - \log_e x$
   (b) $y = (\log_e x - 1)^3$
   (c) $y = x \ln x$

12. Find the derivative of $\log_3 (2x + 5)$.

13. Differentiate
   (a) $3^x$
   (b) $10^x$
   (c) $2^{4x-4}$

14. Find the equation of the tangent to the curve $y = 4^{x+1}$ at the point $(0, 4)$.

15. Find the equation of the normal to the curve $y = \log_3 x$ at the point where $x = 3$.
Integration and the Logarithmic Function

\[ \int \frac{dx}{x} = \int \frac{1}{x} \, dx = \log_e x + C \]

\[ \int \frac{f'(x)}{f(x)} \, dx = \log_e f(x) + C \]

EXAMPLES

1. Find the area enclosed between the hyperbola \( y = \frac{1}{x} \), the x-axis and the lines \( x = 1 \) and \( x = 2 \), giving the exact value.

Solution

\[ A = \int_1^2 \frac{1}{x} \, dx = \left[ \log_e x \right]_1^2 = \log_e 2 - \log_e 1 = \log_e 2 \]

So area is \( \log_e 2 \) units².

2. Find \( \int \frac{x^2}{x^3 + 7} \, dx \).

Solution

\[ \int \frac{x^2}{x^3 + 7} \, dx = \frac{1}{3} \int \frac{3x^2}{x^3 + 7} \, dx = \frac{1}{3} \log_e (x^3 + 7) + C \]
3. Find \( \int \frac{x + 1}{x^2 + 2x + 4} \, dx \).

Solution

\[
\int \frac{x + 1}{x^2 + 2x + 4} \, dx = \frac{1}{2} \int \frac{2(x + 1)}{x^2 + 2x + 4} \, dx
= \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x + 4} \, dx
= \frac{1}{2} \log_e(x^2 + 2x + 4) + C
\]

4.8 Exercises

1. Find the indefinite integral (primitive function) of
   (a) \( \frac{2}{2x + 5} \)
   (b) \( \frac{4x}{2x^2 + 1} \)
   (c) \( \frac{5x^4}{x^3 - 2} \)
   (d) \( \frac{1}{2x} \)
   (e) \( \frac{2}{x} \)
   (f) \( \frac{5}{3x} \)
   (g) \( \frac{2x - 3}{x^2 - 3x} \)
   (h) \( \frac{x}{x^2 + 2} \)
   (i) \( \frac{3x}{x^2 + 7} \)
   (j) \( \frac{x + 1}{x^2 + 2x - 5} \)

2. Find
   (a) \( \int \frac{4}{4x - 1} \, dx \)
   (b) \( \int \frac{dx}{x + 3} \)
   (c) \( \int \frac{x^2}{2x^3 - 7} \, dx \)
   (d) \( \int \frac{x^5}{2x^6 + 5} \, dx \)
   (e) \( \int \frac{x + 3}{x^2 + 6x + 2} \, dx \)

3. Evaluate correct to 1 decimal place.
   (a) \( \int_1^4 \frac{2}{2x + 5} \, dx \)
   (b) \( \int_2^5 \frac{dx}{x + 1} \)
   (c) \( \int_1^7 \frac{x^2}{x^3 + 2} \, dx \)
   (d) \( \int_0^1 \frac{4x + 1}{2x^2 + x + 1} \, dx \)
   (e) \( \int_3^4 \frac{x - 1}{x^2 - 2x} \, dx \)

4. Find the exact area between the curve \( y = \frac{1}{x} \), the x-axis and the lines \( x = 2 \) and \( x = 3 \).

5. Find the exact area bounded by the curve \( y = \frac{1}{x - 1} \), the x-axis and the lines \( x = 4 \) and \( x = 7 \).

6. Find the exact area between the curve \( y = \frac{1}{x} \), the x-axis and the lines \( y = x \) and \( x = 2 \) in the first quadrant.

7. Find the area bounded by the curve \( y = \frac{x}{x^2 + 1} \), the x-axis and the lines \( x = 2 \) and \( x = 4 \), correct to 2 decimal places.
8. Find the exact volume of the solid formed when the curve \( y = \frac{1}{\sqrt{x}} \) is rotated about the \( x \)-axis from \( x = 1 \) to \( x = 3 \).

9. Find the volume of the solid formed when the curve \( y = \frac{2}{\sqrt{2x - 1}} \) is rotated about the \( x \)-axis from \( x = 1 \) to \( x = 5 \), giving an exact answer.

10. Find the area between the curve \( y = \ln x \), the \( y \)-axis and the lines \( y = 2 \) and \( y = 4 \), correct to 3 significant figures.

11. Find the exact volume of the solid formed when the curve \( y = \log x \) is rotated about the \( y \)-axis from \( y = 1 \) to \( y = 3 \).

12. (a) Show that \( \frac{3x + 3}{x^2 - 9} = \frac{1}{x + 3} + \frac{2}{x - 3} \).

(b) Hence find \( \int \frac{3x + 3}{x^2 - 9} \, dx \).

13. (a) Show that \( \frac{x - 6}{x - 1} = 1 - \frac{5}{x - 1} \).

(b) Hence find \( \int \frac{x - 6}{x - 1} \, dx \).

14. Find the indefinite integral (primitive function) of \( 3^{2x - 1} \).

15. Find, correct to 2 decimal places, the area enclosed by the curve \( y = \log_2 x \), the \( x \)-axis and the lines \( x = 1 \) and \( x = 3 \) by using Simpson's rule with 3 function values.
Test Yourself 4

1. Evaluate to 3 significant figures.
   (a) $e^5 - 1$
   (b) $\log_{10} 95$
   (c) $\log_e 26$
   (d) $\log_4 7$
   (e) $\log_4 3$
   (f) $\ln 50$
   (g) $e + 3$
   (h) $\frac{5e^3}{\ln 4}$
   (i) $e^{\ln 6}$
   (j) $e^{\ln 2}$

2. Differentiate
   (a) $e^{5x}$
   (b) $2e^{1-x}$
   (c) $\log_e 4x$
   (d) $\ln (4x + 5)$
   (e) $xe^x$
   (f) $\frac{\ln x}{x}$
   (g) $(e^x + 1)^{10}$

3. Find the indefinite integral (primitive function) of
   (a) $e^{4x}$
   (b) $\frac{x}{x^2 - 9}$
   (c) $e^{-x}$
   (d) $\frac{1}{x + 4}$

4. Find the equation of the tangent to the curve $y = 2 + e^{3x}$ at the point where $x = 0$.

5. Find the exact gradient of the normal to the curve $y = x - e^{-x}$ at the point where $x = 2$.

6. Find the exact area bounded by the curve $y = e^{2x}$, the x-axis and the lines $x = 2$ and $x = 5$.

7. Find the volume of the solid formed if the area bounded by $y = e^{3x}$, the x-axis and the lines $x = 1$ and $x = 2$ is rotated about the x-axis.

8. If $\log_e 2 = 0.36$ and $\log_e 3 = 0.56$, find the value of
   (a) $\log_e 6$
   (b) $\log_e 8$
   (c) $\log_e 1.5$
   (d) $\log_e 14$
   (e) $\log_e 3.5$

9. Find the area enclosed between the curve $y = \ln x$, the y-axis and the lines $y = 1$ and $y = 3$.

10. (a) Use Simpson’s rule with 3 function values to find the area bounded by the curve $y = \ln x$, the x-axis and the lines $x = 2$ and $x = 4$.
    (b) Change the subject of $y = \ln x$ to $x$.
    (c) Hence find the exact area in part (a).

11. Solve
    (a) $3^x = 8$
    (b) $2^{3x - 4} = 3$
    (c) $\log_8 181 = 4$
    (d) $\log_6 x = 2$
    (e) $12 = 10e^{0.01t}$

12. Evaluate
    (a) $\int_0^1 3e^{2t} \, dt$
    (b) $\int_1^4 \frac{dx}{3x - 2}$
    (c) $\int_1^2 \frac{2x^3 - x^2 + 5x + 3}{x} \, dx$

13. Find the equation of the tangent to the curve $y = e^x$ at the point $(4, e^4)$.

14. Evaluate $\log_e 8$ to 1 decimal place.
15. (a) Find the area bounded by the curve $y = e^x$, the $x$-axis and the lines $x = 1$ and $x = 2$.
(b) This area is rotated about the $x$-axis. Find the volume of the solid of revolution formed.

16. Simplify
(a) $5 \log_a x + 3 \log_a y$
(b) $2 \log_a k - \log_a 3 + \log_a p$

17. Find the equation of the normal to the curve $y = \ln x$ at the point $(2, \ln 2)$.

18. Find the stationary points on the curve $y = x^3 e^x$ and determine their nature.

19. Use the trapezoidal rule with 4 strips to find the area bounded by the curve $y = \ln (x^2 - 1)$, the $x$-axis and the lines $x = 3$ and $x = 5$.

20. Evaluate to 2 significant figures
(a) $\log_{10} 4.5$
(b) $\ln 3.7$

Challenge Exercise 4

1. Differentiate $\frac{\log x}{e^{2x} + x}$.

2. Find the exact gradient of the tangent to the curve $y = e^{x + \log x}$ at the point where $x = 1$.

3. If $\log_a 2 = 0.6$ and $\log_b 3 = 1.2$, find
   (a) $\log_a 6b$
   (b) $\log_b 8$
   (c) $\log_a 1.5b^2$

4. Differentiate $(e^{4x} + \log x)^9$.

5. Find the shaded area, correct to 2 decimal places.

6. Find the derivative of $\log x \frac{1}{2x - 3}$.

7. Use Simpson’s rule with 5 function values to find the volume of the solid formed when the curve $y = e^x$ is rotated about the $y$-axis from $y = 3$ to $y = 5$, correct to 2 significant figures.

8. Differentiate $5^x$.

9. Show that $\frac{d}{dx}(x^2 \log_e x) = x (1 + 2 \log_e x)$.
   Hence evaluate $\int_1^3 2x (1 + 2 \log_e x) \, dx$, giving an exact answer.

10. Find $\int 3^x \, dx$.

11. (a) Find the point of intersection of the curves $y = \log x$ and $y = \log_{10} x$.
   (b) Find the exact equations of the tangents to the two curves at this point of intersection.
   (c) Find the exact length of the interval $XY$ where $X$ and $Y$ are the $y$-intercepts of the tangents.
12. Use Simpson’s rule with 3 function values to find the area enclosed by the curve \( y = e^{2x} \), the \( y \)-axis and the line \( y = 3 \), correct to 3 significant figures.

13. Find the derivative of \( \frac{x \log_e x}{e^x} \).

14. If \( y = e^x + e^{-x} \), show \( \frac{d^2y}{dx^2} = y \).

15. Prove \( \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} - 5y - 10 = 0 \), given \( y = 3e^{4x} - 2 \).

16. Find the equation of the curve that has \( f''(x) = 12e^{2x} \) and a stationary point at \((0, 3)\).

17. Sketch \( y = \log_e(x - x^2) \).
Trigonometric Functions

**TERMINOLOGY**

**Amplitude:** Maximum displacement from the mean or centre position in vibrating or oscillating motion

**Circular measure:** The measurement of an angle as the length of an arc cut off by an angle at the centre of a unit circle. The units are called radians

**Period:** One complete cycle of a wave or other recurring function

**Radian:** A unit of angular measurement defined as the length of arc of one unit that an angle subtended at the centre of a unit circle cuts off

**Sector:** Part of a circle bounded by the centre and arc of a circle cut off by two radii

**Segment:** Part of a circle bounded by a chord and the circumference of a circle
INTRODUCTION

IN THIS CHAPTER YOU will learn about circular measure, which uses radians rather than degrees. Circular measure is very useful in solving problems involving properties of the circle, such as arc length and areas of sectors and segments, which you will study in this chapter.

You will also study trigonometric graphs in greater detail than in the Preliminary Course. These graphs have many applications in the movement of such things as sound, light and waves.

Circular measure is also useful in calculus, and you will learn about differentiation and integration of trigonometric functions in this chapter.

Circular Measure

Radians

We are used to degrees being used in geometry and trigonometry. However, they are limited. For example, when we try to draw the graph of a trigonometric function, it is difficult to mark degrees along the x- or y-axis as there is no concept of how large a degree is along a straight line. This is one of the reasons we use radians.

Other reasons for using radians are that they make solving circle problems simple and they are used in calculus.

A radian is based on the length of an arc in a unit circle (a circle with radius 1 unit), so we can visualise a radian as a number that can be measured along a line.
Proof

Circumference of the circle with radius 1 unit is given by:
\[ C = 2\pi r \]
\[ = 2\pi(1) \]
\[ = 2\pi \]

The arc length of the whole circle is \( 2\pi \).
\[ \therefore \text{there are } 2\pi \text{ radians in a whole circle.} \]
But there are 360° in a whole circle (angle of revolution).
So \( 2\pi = 360^\circ \)
\[ \pi = 180^\circ \]

EXAMPLES

1. Convert \( \frac{3\pi}{2} \) into degrees.

Solution

Since \( \pi = 180^\circ \),
\[ \frac{3\pi}{2} = \frac{3(180^\circ)}{2} \]
\[ = 270^\circ \]

2. Change 60° to radians, leaving your answer in terms in \( \pi \).

Solution

180° = \( \pi \) radians
So 1° = \( \frac{\pi}{180} \) radians
60° = \( \frac{\pi}{180} \times 60 \)
\[ = \frac{60\pi}{180} \]
\[ = \frac{\pi}{3} \]
3. Convert 50° into radians, correct to 2 decimal places.

Solution

\[ 180^\circ = \pi \text{ radians} \]

So \( 1^\circ = \frac{\pi}{180} \text{ radians} \)

\[ 50^\circ = \frac{50\pi}{180} \]

\[ = \frac{50\pi}{180} \approx 0.87 \]

4. Change 1.145 radians into degrees, to the nearest minute.

Solution

\[ \pi \text{ radians} = 180^\circ \]

\[ \therefore \quad 1 \text{ radian} = \frac{180^\circ}{\pi} \]

1.145 radians = \( \frac{180^\circ}{\pi} \times 1.145 \)

\[ = 65.6^\circ \]

\[ = 65^\circ 36' \]

Press 180 \( \div \) \( \pi \) \( \times \) 1.145 \( = \)

5. Convert 38° 41’ into radians, correct to 3 decimal places.

Solution

\[ 180^\circ = \pi \text{ radians} \]

\[ 1^\circ = \frac{\pi}{180^\circ} \]

\[ 38^\circ 41' = \frac{\pi}{180^\circ} \times 38^\circ 41' \]

\[ = 0.675 \]

6. Evaluate \( \cos 1.145 \) correct to 2 decimal places.

Solution

For \( \cos 1.145 \), use radian mode on your calculator.

\( \cos 1.145 = 0.413046135 \)

\[ = 0.41 \text{ correct to 2 decimal places} \]
Notice from the examples that \(1^\circ = \frac{\pi}{180}\) and 1 radian \(= \frac{180}{\pi}\). You can use these as conversions rather than starting each time from \(\pi = 180^\circ\).

To change from radians to degrees: Multiply by \(\frac{180}{\pi}\).

To change from degrees to radians: Multiply by \(\frac{\pi}{180}\).

Notice that \(1^\circ = \frac{\pi}{180}\)
\[= 0.017 \text{ radians}\]
Also \(1 \text{ radian} = \frac{180}{\pi}\)
\[= 57^\circ 18'\]

While we can convert between degrees and radians for any angle, there are some special angles that we use regularly in this course. It is easier if you know these without having to convert each time.

\[
\begin{align*}
\frac{\pi}{2} &= 90^\circ \\
\pi &= 180^\circ \\
\frac{3\pi}{2} &= 270^\circ \\
2\pi &= 360^\circ \\
\frac{\pi}{4} &= 45^\circ \\
\frac{\pi}{3} &= 60^\circ \\
\frac{\pi}{6} &= 30^\circ \\
\end{align*}
\]

5.1 Exercises

1. Change into degrees.
   (a) \(\frac{\pi}{5}\) (f) \(\frac{7\pi}{9}\)
   (b) \(\frac{2\pi}{3}\) (g) \(\frac{4\pi}{3}\)
   (c) \(\frac{5\pi}{4}\) (h) \(\frac{7\pi}{3}\)
   (d) \(\frac{7\pi}{6}\) (i) \(\frac{\pi}{9}\)
   (e) \(3\pi\) (j) \(\frac{5\pi}{18}\)
2. Convert into radians in terms of $\pi$.
   (a) $135^\circ$
   (b) $30^\circ$
   (c) $150^\circ$
   (d) $240^\circ$
   (e) $300^\circ$
   (f) $63^\circ$
   (g) $15^\circ$
   (h) $450^\circ$
   (i) $225^\circ$
   (j) $120^\circ$

3. Change into radians, correct to 2 decimal places.
   (a) $56^\circ$
   (b) $68^\circ$
   (c) $127^\circ$
   (d) $289^\circ$
   (e) $312^\circ$

4. Change into radians, to 2 decimal places.
   (a) $18^\circ34'$
   (b) $35^\circ12'$
   (c) $101^\circ56'$
   (d) $88^\circ29'$
   (e) $50^\circ39'$

5. Change these radians into degrees and minutes, to the nearest minute.
   (a) $1.09$
   (b) $0.768$
   (c) $1.16$
   (d) $0.99$
   (e) $0.32$
   (f) $3.2$
   (g) $2.7$
   (h) $4.31$
   (i) $5.6$
   (j) $0.11$

6. Find correct to 2 decimal places.
   (a) $\sin 0.342$
   (b) $\cos 1.5$
   (c) $\tan 0.056$
   (d) $\cos 0.589$
   (e) $\tan 2.29$
   (f) $\sin 2.8$
   (g) $\tan 5.3$
   (h) $\cos 4.77$
   (i) $\cos 3.9$
   (j) $\sin 2.98$

**Trigonometric Results**

All the trigonometry that you studied in the Preliminary Course can be done using radians instead of degrees.

**Special triangles**

The two triangles that give exact trigonometric ratios can be drawn using radians rather than degrees.
Using trigonometric ratios and these special triangles gives the results:

\[
\begin{align*}
\sin \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
\cos \frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
\tan \frac{\pi}{4} &= 1 \\
\sin \frac{\pi}{3} &= \frac{\sqrt{3}}{2} \\
\cos \frac{\pi}{3} &= \frac{1}{2} \\
\tan \frac{\pi}{3} &= \sqrt{3} \\
\sin \frac{\pi}{6} &= \frac{1}{2} \\
\cos \frac{\pi}{6} &= \frac{\sqrt{3}}{2} \\
\tan \frac{\pi}{6} &= \frac{1}{\sqrt{3}}
\end{align*}
\]

**EXAMPLE**

Find the exact value of \(3 \cos \frac{\pi}{6} - \csc \frac{\pi}{4}\).

**Solution**

\[
3 \cos \frac{\pi}{6} - \csc \frac{\pi}{4}
= 3 \cos \frac{\pi}{6} - \frac{1}{\sin \frac{\pi}{4}}
= 3 \left( \frac{\sqrt{3}}{2} \right) - \frac{1}{\frac{1}{\sqrt{2}}}
= \frac{3\sqrt{3}}{2} - \sqrt{2}
= \frac{3\sqrt{3} - 2\sqrt{2}}{2}
\]

**Angles of any magnitude**

The results for angles of any magnitude can also be looked at using radians. If we change degrees for radians, the ASTC rule looks like this:
We can summarise the trigonometric ratios for angles of any magnitude in radians as follows:

First quadrant:
Angle $\theta$:
- $\sin \theta$ is positive
- $\cos \theta$ is positive
- $\tan \theta$ is positive

Second quadrant:
Angle $\pi - \theta$:
- $\sin (\pi - \theta) = \sin \theta$
- $\cos (\pi - \theta) = -\cos \theta$
- $\tan (\pi - \theta) = -\tan \theta$

Third quadrant:
Angle $\pi + \theta$:
- $\sin (\pi + \theta) = -\sin \theta$
- $\cos (\pi + \theta) = -\cos \theta$
- $\tan (\pi + \theta) = \tan \theta$

Fourth quadrant:
Angle $2\pi - \theta$:
- $\sin (2\pi - \theta) = -\sin \theta$
- $\cos (2\pi - \theta) = \cos \theta$
- $\tan (2\pi - \theta) = -\tan \theta$
EXAMPLES

1. Find the exact value of \( \sin \frac{5\pi}{4} \).

**Solution**

\[
\frac{\pi}{4} = \frac{4\pi}{4} - \frac{\pi}{4}
\]

So

\[
\frac{5\pi}{4} = \frac{4\pi}{4} + \frac{\pi}{4}
\]

\[= \pi + \frac{\pi}{4}\]

In the 3rd quadrant, angles are in the form \( \pi + \theta \), so the angle is in the 3rd quadrant, and \( \sin \theta \) is negative in the 3rd quadrant.

\[
\sin \left( \frac{5\pi}{4} \right) = \sin \left( \pi + \frac{\pi}{4} \right)
\]

\[= -\sin \frac{\pi}{4}
\]

\[= -\frac{1}{\sqrt{2}}\]

2. Find the exact value of \( \cos \frac{11\pi}{6} \).

**Solution**

\[
2\pi = \frac{12\pi}{6}
\]

So

\[
\frac{11\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6}
\]

\[= 2\pi - \frac{\pi}{6}\]

In the 4th quadrant, angles are in the form \( 2\pi - \theta \), so the angle is in the 4th quadrant, and \( \cos \theta \) is positive.

\[
\cos \left( \frac{11\pi}{6} \right) = \cos \left( 2\pi - \frac{\pi}{6} \right)
\]

\[= \cos \frac{\pi}{6}
\]

\[= \frac{\sqrt{3}}{2}\]
3. Find the exact value of $\tan \left(-\frac{4\pi}{3}\right)$.

Solution

\[ \pi = \frac{3\pi}{3} \]
So \[ -\frac{4\pi}{3} = -\left(\frac{3\pi}{3} + \frac{\pi}{3}\right) \]
\[ = -\left(\pi + \frac{\pi}{3}\right) \]

In the 2nd quadrant, angles are in the form \(-\pi + \theta\), so the angle is in the 2nd quadrant, and $\tan \theta$ is negative in the 2nd quadrant.

\[ \tan \left(-\frac{4\pi}{3}\right) = -\tan \left[-\left(\pi + \frac{\pi}{3}\right)\right] \]
\[ = -\tan \frac{\pi}{3} \]
\[ = -\sqrt{3} \]

We can use the ASTC rule to solve trigonometric equations. You studied these in the Preliminary Course using degrees.

EXAMPLES

1. Solve $\cos x = 0.34$ for $0 \leq x \leq 2\pi$.

Solution

cos is positive in the 1st and 4th quadrants.
Using a calculator (with radian mode) gives $x = 1.224$.
i.e. $\cos 1.224 = 0.34$.
This is an angle in the 1st quadrant since $1.224 < \frac{\pi}{2}$.
In the 4th quadrant, angles are in the form of $2\pi - \theta$.
So the angle in the 4th quadrant will be $2\pi - 1.224$.
So $x = 1.224, 2\pi - 1.224$
\[ = 1.224, 5.06. \]

2. Solve $\sin \alpha = -\frac{1}{\sqrt{2}}$ in the domain $0 \leq \alpha \leq 2\pi$.

Solution

Here the sin of the angle is negative.
Since sin is positive in the 1st and 2nd quadrants, it is negative in the 3rd and 4th quadrants.
\[
\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}
\]

In the 3rd quadrant, angles are \(\pi + \theta\) and in the 4th quadrant, \(2\pi - \theta\).

So \(\alpha = \pi + \frac{\pi}{4}, 2\pi - \frac{\pi}{4}\)

\[= \frac{4\pi}{4} + \frac{\pi}{4} - \frac{\pi}{4}\]

\[= \frac{5\pi}{4}, \frac{7\pi}{4}\]

### 5.2 Exercises

1. **Copy and complete the table, giving exact values.**

<table>
<thead>
<tr>
<th></th>
<th>(\frac{\pi}{3})</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{6})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sin)</td>
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<tr>
<td>(\cos)</td>
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<td>(\sec)</td>
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<tr>
<td>(\cot)</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

2. **Find the exact value, with rational denominator where relevant.**

(a) \(\tan^2 \frac{\pi}{6}\)

(b) \(\left(\sin \frac{\pi}{4}\right)^2\)

(c) \(\left(\cos \frac{\pi}{6}\right)^3\)

(d) \(\tan \frac{\pi}{3} + \tan \frac{\pi}{6}\)

(e) \(\sin \frac{\pi}{4} - \cos \frac{\pi}{4}\)

(f) \(\tan \frac{\pi}{3} + \cos \frac{\pi}{3}\)

3. **Find the exact value, with rational denominator where relevant.**

(a) \(\cos^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{4}\)

(b) \(\sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}\)

(c) \(\cos \frac{\pi}{6} \cos \frac{\pi}{3} + \sin \frac{\pi}{6} \sin \frac{\pi}{3}\)

(d) \(\sin^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4}\)

(e) \(\sec^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{6}\)

4. **Find the exact value with rational denominator of**

(a) \(\sin \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos \frac{\pi}{3}\)

(b) \(\frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{3}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{3}}\)

5. **Show that**

\(\cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \)

\(\sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6} .\)

6. **(a) Show that** \(\frac{3\pi}{4} = \pi - \frac{\pi}{4}\).

**(b) Which quadrant is the angle** \(\frac{3\pi}{4}\) **in?**

**(c) Find the exact value of** \(\cos \frac{3\pi}{4}\).
7. (a) Show that $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$.
(b) Which quadrant is the angle $\frac{5\pi}{6}$ in?
(c) Find the exact value of $\sin \frac{5\pi}{6}$.

8. (a) Show that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$.
(b) Which quadrant is the angle $\frac{7\pi}{4}$ in?
(c) Find the exact value of $\tan \frac{7\pi}{4}$.

9. (a) Show that $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$.
(b) Which quadrant is the angle $\frac{4\pi}{3}$ in?
(c) Find the exact value of $\cos \frac{4\pi}{3}$.

10. (a) Show that $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$.
(b) Which quadrant is the angle $\frac{5\pi}{3}$ in?
(c) Find the exact value of $\sin \frac{5\pi}{3}$.

11. Find the exact value of each ratio.
(a) $\tan \frac{3\pi}{4}$
(b) $\cos \frac{11\pi}{6}$
(c) $\tan \frac{2\pi}{3}$
(d) $\sin \frac{5\pi}{4}$
(e) $\tan \frac{7\pi}{6}$

12. (a) (i) Show that $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$.
(ii) Which quadrant is the angle $\frac{13\pi}{6}$ in?
(iii) Find the exact value of $\cos \frac{13\pi}{6}$.

(b) Find the exact value of
(i) $\sin \frac{9\pi}{4}$
(ii) $\tan \frac{7\pi}{3}$
(iii) $\cos \frac{11\pi}{4}$
(iv) $\tan \frac{19\pi}{6}$
(v) $\sin \frac{10\pi}{3}$

13. Solve for $0 \leq x \leq 2\pi$
(a) $\cos x = \frac{1}{2}$
(b) $\sin x = -\frac{1}{\sqrt{2}}$
(c) $\tan x = 1$
(d) $\tan x = \sqrt{3}$
(e) $\cos x = -\frac{\sqrt{3}}{2}$

14. Simplify
(a) $\sin (\pi - \theta)$
(b) $\tan (2\pi - x)$
(c) $\cos (\pi + \alpha)$
(d) $\cos \left(\frac{\pi}{2} - x\right)$
(e) $\tan \left(\frac{\pi}{2} - \theta\right)$

15. Simplify $1 - \sin^2 \left(\frac{\pi}{2} - \theta\right)$

16. If $\tan x = \frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find the value of $\cos x$ and $\sin x$.

17. Solve $\tan^2 x - 1 = 0$ for $0 \leq x \leq 2\pi$
Circle Results

The area and circumference of a circle are useful in this section.

Area of a circle

\[ A = \pi r^2 \] where \( r \) is the radius of the circle

Circumference of a circle

\[ C = 2\pi r = \pi d \]

We use equal ratios to find the other circle results. The working out is simpler when using radians instead of degrees.

Length of arc

\[ l = r\theta \]

\( \theta \) is in radians

Proof

\[
\frac{\text{arc length } l}{\text{circumference}} = \frac{\text{angle } \theta}{\text{whole revolution}}
\]

\[
\frac{l}{2\pi r} = \frac{\theta}{2\pi}
\]

\[
\therefore \quad l = \frac{\theta}{2\pi} \times 2\pi = r\theta
\]

Using 360° instead of \( 2\pi \) gives a different formula. Can you find it?
EXAMPLES

1. Find the length of the arc formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

Solution

\[ l = r\theta \]
\[ = 5\left(\frac{\pi}{4}\right) \]
\[ = \frac{5\pi}{4} \text{ m} \]

2. The area of a circle is 450 cm$^2$. Find, in degrees and minutes, the angle subtended at the centre of the circle by a 2.7 cm arc.

Solution

\[ A = \pi r^2 \]
\[ 450 = \pi r^2 \]
\[ \frac{450}{\pi} = r^2 \]
\[ \sqrt{\frac{450}{\pi}} = r \]
\[ 11.97 = r \]

Now \( l = r\theta \)
\[ 2.7 = 11.97\theta \]
\[ \frac{2.7}{11.97} = \theta \]
\[ 0.226 = \theta \]

\( \pi \) radians = 180°

1 radian = $\frac{180^\circ}{\pi}$

0.226 radians = $\frac{180^\circ}{\pi} \times 0.226$
\[ = 12.93^\circ \]
\[ = 12^\circ 56' \]

So \[ \theta = 12^\circ 56'. \]
5.3 Exercises

1. Find the exact arc length of a circle if
   (a) radius is 4 cm and angle subtended is \( \pi \)
   (b) radius is 3 m and angle subtended is \( \frac{\pi}{3} \)
   (c) radius is 10 cm and angle subtended is \( \frac{5\pi}{6} \)
   (d) radius is 3 cm and angle subtended is 30°
   (e) radius is 7 mm and angle subtended is 45°.

2. Find the arc length, correct to 2 decimal places, given
   (a) radius is 1.5 m and angle subtended is 0.43
   (b) radius is 3.21 cm and angle subtended is 1.22
   (c) radius is 7.2 mm and angle subtended is 55°
   (d) radius is 5.9 cm and angle subtended is 23° 12′
   (e) radius is 2.1 m and angle subtended is 82° 35′.

3. The angle subtended at the centre of a circle of radius 3.4 m is 29° 51′. Find the length of the arc cut off by this angle, correct to 1 decimal place.

4. The arc length when a sector of a circle is subtended by an angle of \( \frac{\pi}{5} \) at the centre is \( \frac{3\pi}{2} \) m. Find the radius of the circle.

5. The radius of a circle is 3 cm and an arc is \( \frac{2\pi}{7} \) cm long. Find the angle subtended at the centre of the circle by the arc.

6. The circumference of a circle is 300 mm. Find the length of the arc that is formed by an angle of \( \frac{\pi}{6} \) subtended at the centre of the circle.

7. A circle with area 60 cm² has an arc 8 cm long. Find the angle that is subtended at the centre of the circle by the arc.

8. A circle with circumference 124 mm has a chord cut off it that subdends an angle of 40° at the centre. Find the length of the arc cut off by the chord.

9. A circle has a chord of 25 mm with an angle of \( \frac{\pi}{6} \) subtended at the centre. Find, to 1 decimal place, the length of the arc cut off by the chord.

10. A sector of a circle with radius 5 cm and an angle of \( \frac{\pi}{3} \) subtended at the centre is cut out of cardboard. It is then curved around to form a cone. Find its exact surface area and volume.
Area of sector

\[ A = \frac{1}{2} r^2 \theta \]
(\( \theta \) is in radians)

**Proof**

\[
\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{angle } \theta}{\text{whole revolution}}
\]

\[
\frac{A}{\pi r^2} = \frac{\theta}{2\pi}
\]

\[
\therefore \quad A = \frac{\theta \pi r^2}{2\pi}
\]

\[
= \frac{1}{2} r^2 \theta
\]

**EXAMPLES**

1. Find the area of the sector formed if an angle of \( \frac{\pi}{4} \) is subtended at the centre of a circle of radius 5 m.

**Solution**

\[
A = \frac{1}{2} r^2 \theta
\]

\[
= \frac{1}{2} (5)^2 \left( \frac{\pi}{4} \right)
\]

\[
= \frac{25\pi}{8} \text{ m}^2
\]
2. The area of the sector of a circle with radius 4 cm is $\frac{6\pi}{5}$ cm$^2$. Find the angle, in degrees, that is subtended at the centre of the circle.

**Solution**

\[
A = \frac{1}{2} r^2 \theta \\
6\pi = \frac{1}{2} (4)^2 \theta \\
= 8\theta \\
6\pi = \theta \\
\frac{6\pi}{40} = \theta \\
\frac{3\pi}{20} = \theta \\
\frac{3(180^\circ)}{20} = \theta \\
27^\circ = \theta
\]

**5.4 Exercises**

1. Find the exact area of the sector of a circle if
   (a) radius is 4 cm and angle subtended is $\pi$
   (b) radius is 3 m and angle subtended is $\frac{\pi}{3}$
   (c) radius is 10 cm and angle subtended is $\frac{5\pi}{6}$
   (d) radius is 3 cm and angle subtended is 30°
   (e) radius is 7 mm and angle subtended is 45°.

2. Find the area of the sector, correct to 2 decimal places, given
   (a) radius is 1.5 m and angle subtended is 0.43
   (b) radius is 3.21 cm and angle subtended is 1.22
   (c) radius is 7.2 mm and angle subtended is 55°
   (d) radius is 5.9 cm and angle subtended is 23° 12’
   (e) radius is 2.1 m and angle subtended is 82° 35’.

3. Find the area, correct to 3 significant figures, of the sector of a circle with radius 4.3 m and an angle of 1.8 subtended at the centre.

4. The area of a sector of a circle is 20 cm$^2$. If the radius of the circle is 3 cm, find the angle subtended at the centre of the circle by the sector.

5. The area of the sector of a circle that is subtended by an angle of $\frac{\pi}{3}$ at the centre is $6\pi$ m$^2$. Find the radius of the circle.

6. Find the
   (a) arc length
   (b) area of the sector of a circle with radius 7 cm if the sector is cut off by an angle of 30° subtended at the centre of the circle.
7. A circle has a circumference of 185 mm. Find the area of the sector cut off by an angle of $\frac{\pi}{5}$ subtended at the centre.

8. If the area of a circle is 200 cm$^2$ and a sector is cut off by an angle of $\frac{3\pi}{4}$ at the centre, find the area of the sector.

9. Find the area of the sector of a circle with radius 5.7 cm if the length of the arc formed by this sector is 4.2 cm.

10. The area of a sector is $\frac{3\pi}{10}$ cm$^2$ and the arc length cut off by the sector is $\frac{\pi}{5}$ cm. Find the angle subtended at the centre of the circle and find the radius of the circle.

**Area of minor segment**

$$A = \frac{1}{2}r^2(\theta - \sin \theta)$$

($\theta$ is in radians)

**Proof**

Area of minor segment = area of sector − area of triangle

$$= \frac{1}{2}r^2 \theta - \frac{1}{2}r^2 \sin \theta$$

$$= \frac{1}{2}r^2(\theta - \sin \theta)$$

The area of the triangle is given by $A = \frac{1}{2}ab \sin C$. 
EXAMPLES

1. Find the area of the minor segment formed if an angle of $\frac{\pi}{4}$ is subtended at the centre of a circle of radius 5 m.

Solution

\[
A = \frac{1}{2} r^2 (\theta - \sin \theta)
\]

\[
= \frac{1}{2} (5)^2 \left( \frac{\pi}{4} - \sin \frac{\pi}{4} \right)
\]

\[
= \frac{25}{2} \left( \frac{\pi}{4} - \frac{1}{\sqrt{2}} \right)
\]

\[
= \frac{25\pi}{8} - \frac{25}{2\sqrt{2}}
\]

\[
= \frac{25\pi}{8} - \frac{25\sqrt{2}}{4}
\]

\[
= \frac{25\pi - 50\sqrt{2}}{8} \text{ m}^2.
\]

2. An 80 cm piece of wire is bent into a circle shape, and a 10 cm piece of wire is joined to it to form a chord. Find the area of the minor segment cut off by the chord, correct to 2 decimal places.

Solution

\[
C = 2\pi r = 80
\]

\[
\therefore \quad r = \frac{80}{2\pi}
\]

\[
= \frac{40}{\pi}
\]

\[
= 12.7
\]
\[
\cos C = \frac{a^2 + b^2 - c^2}{2ab}
\]
\[
\cos \theta = \frac{12.7^2 + 12.7^2 - 10^2}{2(12.7)(12.7)} = 0.69
\]
\[
\therefore \quad \theta = 0.807
\]
\[
A = \frac{1}{2} r^2 (\theta - \sin \theta)
\]
\[
= \frac{1}{2} (12.7)^2 (0.807 - \sin 0.807)
\]
\[
= 6.88 \text{ cm}^2
\]

### 5.5 Exercises

1. Find the exact area of the minor segment of a circle if
   (a) radius is 4 cm and the angle subtended is \( \pi \)
   (b) radius is 3 m and the angle subtended is \( \frac{\pi}{3} \)
   (c) radius is 10 cm and the angle subtended is \( \frac{5\pi}{6} \)
   (d) radius is 3 cm and the angle subtended is 30°
   (e) radius is 7 mm and the angle subtended is 45°.

2. Find the area of the minor segment correct to 2 decimal places, given
   (a) radius is 1.5 m and the angle subtended is 0.43
   (b) radius is 3.21 cm and the angle subtended is 1.22
   (c) radius is 7.2 mm and the angle subtended is 55°
   (d) radius is 5.9 cm and the angle subtended is 23° 12’
   (e) radius is 2.1 m and the angle subtended is 82° 35’.

3. Find the area of the minor segment formed by an angle of 40° subtended at the centre of a circle with radius 2.82 cm, correct to 2 significant figures.

4. Find the
   (a) exact arc length
   (b) exact area of the sector
   (c) area of the minor segment, to 2 decimal places
   if an angle of \( \frac{\pi}{7} \) is subtended at the centre of a circle with radius 3 cm.

5. The area of the minor segment cut off by an angle of \( \frac{2\pi}{9} \) is 500 cm². Find the radius of the circle, correct to 1 decimal place.

6. Find the
   (a) length of the chord, to 1 decimal place
   (b) length of the arc
   (c) area of the minor segment, to 2 decimal places
   cut off by an angle of \( \frac{\pi}{6} \) subtended at the centre of a circle with radius 5 cm.

7. A chord 8 mm long is formed by an angle of 45° subtended at the centre of a circle. Find
   (a) the radius of the circle
   (b) the area of the minor segment cut off by the angle, correct to 1 decimal place.
8. An angle of $\frac{\pi}{8}$ is subtended at the centre of a circle. If this angle cuts off an arc of $\frac{5\pi}{4}$ cm, find
(a) the exact area of the sector
(b) the area of the minor segment formed, correct to 1 decimal place.

9. A triangle $OAB$ is formed where $O$ is the centre of a circle of radius 12 cm, and $A$ and $B$ are endpoints of a 15 cm chord.

(a) Find the angle ($\theta$) subtended at the centre of the circle, in degrees and minutes.
(b) Find the area of $\triangle OAB$, correct to 1 decimal place.
(c) Find the area of the minor segment cut off by the chord, correct to 2 decimal places.
(d) Find the area of the major segment cut off by the chord, correct to 2 decimal places.

10. The length of an arc is 8.9 cm and the area of the sector is 24.3 $\text{cm}^2$ when an angle of $\theta$ is subtended at the centre of a circle. Find the area of the minor segment cut off by $\theta$, correct to 1 decimal place.

11. An arc, centre $C$, cuts side $AC$ in $D$.

Find the exact
(a) length of arc $BD$
(b) area of $ABD$
(c) perimeter of $BDC$

12. The dartboard above has a radius of 23 cm. The shaded area has a height of 7 mm. If a player must hit the shaded area to win, find
(a) the area of the shaded part, to the nearest square centimetre
(b) the percentage of the shaded part in relation to the whole area of the dartboard, to 1 decimal place
(c) the perimeter of the shaded area.
13. The hour hand of a clock is 12 cm long. Find the exact length of the arc through which the hand would turn in 5 hours.
(a) area through which the hand would pass in 2 hours.
(b) area through which the hand would pass in 2 hours.

14. Arc BC subtends an angle of 100° at the centre A of a circle with radius 4 cm. Find
(a) the exact perimeter of sector ABC
(b) the approximate ratio of the area of the minor segment to the area of the sector.

15. A wedge is cut so that its cross-sectional area is a sector of a circle, radius 15 cm and subtending an angle of \( \frac{\pi}{6} \) at the centre. Find
(a) the volume of the wedge
(b) the surface area of the wedge.

Small Angles

When we use radians of small angles, there are some interesting results.

**EXAMPLES**

1. Find sin 0.0023.

   **Solution**

   Make sure your calculator is in radian mode
   \[ \sin 0.0023 = 0.0022999972 \]

2. Find tan 0.0023.

   **Solution**

   \[ \tan 0.0023 = 0.002300004056 \]

3. Find cos 0.0023.

   **Solution**

   \[ \cos 0.0023 = 0.999997355 \]
Maths In Focus Mathematics HSC Course

Class Investigation

Use the calculator to explore other small angles and their trigonometric ratios. What do you notice?

Can you see why?

In $\triangle OAC$, $OC \approx 1$ when $x$ is small.

Arc $AB = x$ (definition of a radian)

$\therefore AC \approx x$ when $x$ is small

\[
\sin x = \frac{AC}{OC} \quad ; \quad \frac{x}{1} = x
\]

\[
\cos x = \frac{OA}{OC} \quad ; \quad \frac{1}{1} = 1
\]

\[
\tan x = \frac{AC}{OA} \quad ; \quad \frac{x}{1} = x
\]

If $x$ (radians) is a small angle then

$\sin x \approx x$

$\tan x \approx x$

$\cos x \approx 1$

and $\sin x < x < \tan x$
**Proof**

Area \( \Delta OAB = \frac{1}{2} ab \sin C \)

\[ = \frac{1}{2} \times 1 \times 1 \times \sin x \]

\[ = \frac{1}{2} \sin x \]

Area sector \( OAB = \frac{1}{2} r^2 \)

\[ = \frac{1}{2} \times 1^2 \times x \]

\[ = \frac{1}{2} x \]

Area \( \Delta OAC = \frac{1}{2} bh \)

\[ = \frac{1}{2} \times 1 \times AC \]

\[ = \frac{1}{2} \times 1 \times \tan x \]

\[ = \frac{1}{2} \tan x \]

Area \( \Delta OAB < \) area sector \( OAB < \) area \( \Delta OAC \)

\[ \therefore \frac{1}{2} \sin x < \frac{1}{2} x < \frac{1}{2} \tan x \]

\[ \therefore \sin x < x < \tan x \]

**Class Investigation**

Check that \( \sin x < x < \tan x \) for \( 0 < x < \frac{\pi}{2} \) by using your calculator.

Remember that \( x \) is in radians. Does this work for \( x > \frac{\pi}{2} \)?

These results give the following result:

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \]

Also \( \lim_{x \to 0} \frac{\tan x}{x} = 1 \). Can you see why?

**Proof**

As \( x \to 0 \), \( \sin x \to x \)

\[ \therefore \frac{\sin x}{x} \to 1 \]
EXAMPLES

1. Evaluate \( \lim_{x \to 0} \frac{\sin 7x}{x} \).

Solution

\[
\lim_{x \to 0} \frac{\sin 7x}{x} = 7 \lim_{x \to 0} \frac{\sin 7x}{7x} \\
= 7 \lim_{x \to 0} \frac{\sin 7x}{7x} \\
= 7(1) \\
= 7
\]

2. The planet of Alpha Zeta in a faraway galaxy has a moon with a diameter of 145 000 km. If the distance between the centre of Alpha Zeta and the centre of its moon is \( 12.8 \times 10^6 \) km, find the angle subtended by the moon at the centre of the planet, in seconds.

Solution

The radius of the moon is \( 72,500 \) km.

\[
r = \frac{1}{2} \times 145,000 \\
= 72,500
\]

\[
\alpha = \frac{1}{2} \theta
\]

\[
\tan \alpha = \frac{72,500}{12.8 \times 10^6} \\
= 0.00566
\]

\[
\therefore \alpha = 0.00566 \\
\pi = 180^\circ
\]

\[
\therefore 1 = \frac{180^\circ}{\pi}
\]

\[
0.00566 = \frac{180^\circ}{\pi} \times 0.00566 \\
= 0.3245 ^\circ
\]

\[
\therefore \theta = 0.649^\circ \\
= 39^\prime
\]
5.6 Exercises

1. Evaluate, correct to 3 decimal places.
   (a) \( \sin 0.045 \)
   (b) \( \tan 0.003 \)
   (c) \( \cos 0.042 \)
   (d) \( \sin 0.065 \)
   (e) \( \tan 0.005 \)

2. Evaluate \( \lim_{x \to 0} \frac{\sin x}{4x} \).

3. Find \( \lim_{\theta \to 0} \frac{\tan \theta}{3} \).

4. Find the diameter of the sun to the nearest kilometre if its distance from the Earth is 149000000 km and it subtends an angle of 31' at the Earth.

5. Given that the wingspan of an aeroplane is 30 m, find the plane’s altitude to the nearest metre if the wingspan subtends an angle of 14' when it is directly overhead.

Trigonometric Graphs

You drew the graphs of trigonometric functions in the Preliminary Course, using degrees.

You can also draw these graphs using radians.

\( y = \sin x \)
\( y = \cos x \)

\[
\begin{array}{c|c|c|c|c|c}
\hline
x & 0 & \frac{\pi}{2} & \pi & \frac{3\pi}{2} & 2\pi \\
\hline
\text{cosec } x & & & & & \\
\text{sec } x & & & & & \\
\text{cot } x & & & & & \\
\hline
\end{array}
\]

\( y = \tan x \)

\( y = \tan x \) has asymptotes at \( x = \frac{\pi}{2} \) and \( \frac{3\pi}{2} \) since \( \tan x \) is undefined at those points. By finding values for \( x \) on each side of the asymptotes, we can see where the curve goes.

We can also sketch the graphs of the reciprocal trigonometric functions, by finding the reciprocals of \( \sin x \), \( \cos x \) and \( \tan x \) at important points on the graphs.

**Investigation**

1. Use cosec \( x = \frac{1}{\sin x} \), sec \( x = \frac{1}{\cos x} \) and cot \( x = \frac{1}{\tan x} \) to complete the table below.

\[
\begin{align*}
\text{cosec } x & = \frac{1}{\sin \frac{\pi}{2}} \\
& = \frac{1}{1} \\
& = 1 \\
\text{sec } x & = \frac{1}{\cos \frac{\pi}{2}} \\
& = \frac{1}{0} \\
& = \text{undefined} \\
\text{cot } x & = \frac{1}{\tan \frac{\pi}{2}} \\
& = \frac{1}{\text{undefined}} \\
& = \text{undefined}
\end{align*}
\]
2. Find any asymptotes

e.g. \[ \sec \frac{\pi}{2} = \frac{1}{\cos \frac{\pi}{2}} = \frac{1}{0} \quad \text{(undefined)} \]

Discover what the values are on either side of the asymptotes.

3. Sketch each graph of the reciprocal trigonometric functions.

Here are the graphs of the reciprocal ratios.
We have sketched these functions in the domain $0 \leq x \leq 2\pi$ which gives all the angles in one revolution of a circle. However, we could turn around the circle again and have angles greater than $2\pi$, as well as negative angles.

These graphs all repeat at regular intervals. They are called periodic functions.

**Investigation**

Here is a general sine function. Notice that the shape that occurs between 0 and $2\pi$ repeats as shown.

1. Draw a general cosine curve. How are the sine and cosine curves related? Do these curves have symmetry?
2. Draw a general tangent curve. Does it repeat at the same intervals as the sine and cosine curves?
3. Look at the reciprocal trigonometric curves and see if they repeat in a similar way.
4. Use a graphics calculator or a graphing computer program to draw the graphs of trigonometric functions. Choose different values of $k$ to sketch these families of trig functions. (Don’t forget to look at values where $k$ is a fraction or negative.)

(a) $f(x) = k \sin x$
(b) $f(x) = k \cos x$
(c) $f(x) = k \tan x$
(d) $f(x) = \sin kx$
(e) $f(x) = \cos kx$
(f) $f(x) = \tan kx$
(g) $f(x) = \sin x + k$
(h) $f(x) = \cos x + k$
(i) $f(x) = \tan x + k$
(j) $f(x) = \sin (x + k)$
(k) $f(x) = \cos (x + k)$
(l) $f(x) = \tan (x + k)$

Can you see patterns in these families?
Could you predict what the graph of $f(x) = a \sin bx$ looks like?

The trig functions have period and amplitude.
The period is the distance over which the curve moves along the $x$-axis before it repeats.
The amplitude is the maximum distance that the graph stretches out from the centre of the graph on the $y$-axis.

\[
\begin{align*}
y = \sin x & \text{ has amplitude 1 and period } 2\pi \\
y = \cos x & \text{ has amplitude 1 and period } 2\pi \\
y = \tan x & \text{ has no amplitude and period } \pi \\
\end{align*}
\]

\[
\begin{align*}
y = a \sin bx & \text{ has amplitude } a \text{ and period } \frac{2\pi}{b} \\
y = a \cos bx & \text{ has amplitude } a \text{ and period } \frac{2\pi}{b} \\
y = a \tan bx & \text{ has no amplitude and period } \frac{\pi}{b} \\
\end{align*}
\]
EXAMPLES

1. Sketch in the domain $0 \leq x \leq 2\pi$.
   (a) $y = 5 \sin x$ for
   (b) $y = \sin 4x$ for
   (c) $y = 5 \sin 4x$ for

Solution

(a) The graph of $y = 5 \sin x$ has amplitude 5 and period $2\pi$.

(b) The graph $y = \sin 4x$ has amplitude 1 and period $\frac{2\pi}{4}$ or $\frac{\pi}{2}$. This means that the curve repeats every $\frac{\pi}{2}$, so in the domain $0 \leq x \leq 2\pi$ there will be 4 repetitions.
(c) The graph \( y = 5 \sin 4x \) has amplitude 5 and period \( \frac{\pi}{2} \).

\[
\begin{align*}
\text{y} & \quad 5 & \quad -5 \\
\text{x} & \quad \frac{\pi}{4} & \quad \frac{\pi}{2} & \quad \frac{3\pi}{4} & \quad \pi & \quad \frac{5\pi}{4} & \quad \frac{3\pi}{2} & \quad \frac{7\pi}{4} & \quad 2\pi \\
\end{align*}
\]

2. Sketch \( f(x) = \sin \left(x + \frac{\pi}{2}\right) \) for \( 0 \leq x \leq 2\pi \).

**Solution**

Amplitude = 1
Period = \( \frac{2\pi}{1} = 2\pi \)

\( f(x) = \sin (x + k) \) translates the curve \( k \) units to the left (see Investigation on page 194–5).

So \( f(x) = \sin \left(x + \frac{\pi}{2}\right) \) will be moved \( \frac{\pi}{2} \) units to the left. The graph is the same as \( f(x) = \sin x \) but starts in a different position.

If you are not sure where the curve goes, you can draw a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( \frac{\pi}{2} )</th>
<th>( \pi )</th>
<th>( \frac{3\pi}{2} )</th>
<th>2( \pi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

This is the same as the graph \( y = \cos x \).
3. Sketch $y = 2 \tan \frac{x}{2}$ for $0 \leq x \leq 2\pi$.

Solution

There is no amplitude.

Period $= \frac{\pi}{\frac{2}{2}} = 2\pi$

4. (a) Sketch $y = 2 \cos x$ and $y = \cos 2x$ on the same set of axes for $0 \leq x \leq 2\pi$.

(b) Hence or otherwise, sketch $y = \cos 2x + 2 \cos x$ for $0 \leq x \leq 2\pi$.

Solution

(a) $y = 2 \cos x$ has amplitude 2 and period $2\pi$.

$y = \cos 2x$ has amplitude 1 and period $\frac{2\pi}{2}$ or $\pi$. 
Chapter 5  Trigonometric Functions

(b) Method 1: Use table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{2})</th>
<th>(\frac{3\pi}{4})</th>
<th>(\pi)</th>
<th>(\frac{5\pi}{4})</th>
<th>(\frac{3\pi}{2})</th>
<th>(\frac{7\pi}{4})</th>
<th>(2\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\cos 2x)</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(2 \cos x)</td>
<td>2</td>
<td>(\sqrt{2})</td>
<td>0</td>
<td>(-\sqrt{2})</td>
<td>-2</td>
<td>(-\sqrt{2})</td>
<td>0</td>
<td>(\sqrt{2})</td>
<td>2</td>
</tr>
<tr>
<td>(\cos 2x + 2 \cos x)</td>
<td>3</td>
<td>(\sqrt{2})</td>
<td>-1</td>
<td>(-\sqrt{2})</td>
<td>-1</td>
<td>(-\sqrt{2})</td>
<td>-1</td>
<td>(\sqrt{2})</td>
<td>3</td>
</tr>
</tbody>
</table>

Notice that the period of this graph is \(2\pi\).

Method 2: Add \(y\) values together on the graph itself

e.g. when \(x = 0\):
\[
y = 1 + 2 = 3
\]

These graphs can be sketched more accurately on a graphics calculator or in a computer program such as Autograph.

5.7 Exercises

1. Sketch for \(0 \leq x \leq 2\pi\)
   (a) \(y = \cos x\)
   (b) \(f(x) = 2 \sin x\)
   (c) \(y = 1 + \sin x\)
   (d) \(y = 2 - \sin x\)
   (e) \(f(x) = -3 \cos x\)
   (f) \(y = 4 \sin x\)
   (g) \(f(x) = \cos x + 3\)
   (h) \(y = 5 \tan x\)
   (i) \(f(x) = \tan x + 3\)
   (j) \(y = 1 - 2 \tan x\)

2. Sketch for \(0 \leq x \leq 2\pi\)
   (a) \(y = \cos 2x\)
   (b) \(y = \tan 2x\)
   (c) \(y = \sin 3x\)
   (d) \(f(x) = 3 \cos 4x\)
   (e) \(y = 6 \cos 3x\)
   (f) \(y = \tan \frac{x}{2}\)
   (g) \(f(x) = 2 \tan 3x\)
   (h) \(y = 3 \cos \frac{x}{2}\)
   (i) \(y = 2 \sin \frac{x}{2}\)
   (j) \(f(x) = 4 \cos \frac{x}{4}\)
3. Sketch for $-\pi \leq x \leq \pi$
   (a) $y = \sin 2x$
   (b) $y = 7 \cos 4x$
   (c) $f(x) = \tan 4x$
   (d) $y = 5 \sin 4x$
   (e) $f(x) = 2 \cos 2x$

4. Sketch $y = 8 \sin \frac{x}{2}$ in the domain $0 \leq x \leq 4\pi$.

5. Sketch for $0 \leq x \leq 2\pi$
   (a) $y = \sin (x + \pi)$
   (b) $y = \tan \left(x + \frac{\pi}{2}\right)$
   (c) $f(x) = \cos (x - \pi)$
   (d) $y = 3 \sin \left(x - \frac{\pi}{2}\right)$
   (e) $f(x) = 2 \cos \left(x + \frac{\pi}{2}\right)$
   (f) $y = 4 \sin \left(2x + \frac{\pi}{2}\right)$
   (g) $y = \cos \left(x - \frac{\pi}{4}\right)$
   (h) $y = \tan \left(x + \frac{\pi}{4}\right)$
   (i) $f(x) = 2 \cos \left(x + \frac{\pi}{2}\right) + 1$
   (j) $y = 2 - 3 \sin \left(x - \frac{\pi}{2}\right)$

6. Sketch for $-2 \leq x \leq 2$
   (a) $y = \sin \pi x$
   (b) $y = 3 \cos 2\pi x$

7. Sketch in the domain $0 \leq x \leq 2\pi$
   (a) $y = \sin x$ and $y = \sin 2x$ on the same set of axes
   (b) $y = \sin x + \sin 2x$

8. Sketch for $0 \leq x \leq 2\pi$
   (a) $y = 2 \cos x$ and $y = 3 \sin x$ on the same set of axes
   (b) $y = 2 \cos x + 3 \sin x$

9. By sketching $y = \cos x$ and $y = \cos 2x$ on the same set of axes for $0 \leq x \leq 2\pi$, sketch the graph of $y = \cos 2x - \cos x$.

10. Sketch the graph of
    (a) $y = \cos x + \sin x$
    (b) $y = \sin 2x - \sin x$
    (c) $y = \sin x + 2 \cos 2x$
    (d) $y = 3 \cos x - \cos 2x$
    (e) $y = \sin x - \sin \frac{x}{2}$

Applications

The sine and cosine curves are used in many applications including the study of waves. There are many different types of waves, including water, light and sound waves. Oscilloscopes display patterns of electrical waves on the screen of a cathode-ray tube.

Search waves and the oscilloscope on the Internet for further information.

Simple harmonic motion (such as the movement of a pendulum) is a wave-like or oscillatory motion when graphed against time. In 1581, when he was 17 years old, Gallileo noticed a lamp swinging backwards and forwards in Pisa cathedral. He found that the lamp took the same time to swing to and fro, no matter how much weight it had on it. This led him to discover the pendulum.

Gallileo also invented the telescope. Find out more information about Gallileo’s life and his other discoveries.
Some trigonometric equations are difficult to solve algebraically, but can be solved graphically by finding the points of intersection between two graphs.

**EXAMPLES**

1. Find approximate solutions to the equation $\sin x = x - 1$ by sketching the graphs $y = \sin x$ and $y = x - 1$ on a Cartesian plane.

   **Solution**

   When sketching the two graphs together, we use $\pi = 3.14, 2\pi = 6.28$ and so on to label the $x$-axis as shown.

   To sketch $y = x - 1$, find the gradient and $y$-intercept or find the $x$- and $y$-intercepts.

   $x$-intercept (where $y = 0$) is 1 and $y$-intercept (where $x = 0$) is −1.

   ![Graph of $y = \sin x$ and $y = x - 1$](graph.png)

   The solution to $\sin x = x - 1$ is at the point of intersection of the two graphs. ∴ $x \approx 2$.

2. How many solutions are there for $\cos 2x = \frac{x}{4}$ in the domain $0 \leq x \leq 2\pi$?

   **Solution**

   Sketch $y = \cos 2x$ and $y = \frac{x}{4}$ on the same set of axes.

   $y = \cos 2x$ has amplitude 1 and period $\pi$

   $y = \frac{x}{4}$ has $x$-intercept 0 and $y$-intercept 0. We can find another point on the line e.g. When $x = 4$

   $y = \frac{4}{4} = 1$
There are 3 points of intersection of the graphs, so the equation \( \cos 2x = \frac{x}{4} \) has 3 solutions.

We can also look at applications of trigonometric graphs in real life situations.

**EXAMPLE**

The table shows the highest average monthly temperatures in Sydney.

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>°C</td>
<td>26.1</td>
<td>26.1</td>
<td>25.1</td>
<td>22.8</td>
<td>19.8</td>
<td>17.4</td>
<td>16.8</td>
<td>18.0</td>
<td>20.1</td>
<td>22.2</td>
<td>23.9</td>
<td>25.6</td>
</tr>
</tbody>
</table>

(a) Draw a graph of this data, by hand or on a graphics calculator or computer.
(b) Is it periodic? Why would you expect it to be periodic?
(c) What is the period and amplitude?

**Solution**

(a) 

![Graph of monthly temperatures]
The graph looks like it is periodic, and we would expect it to be, since the temperature varies with the seasons. It goes up and down, and reaches a maximum in summer and a minimum in winter.

(c) This curve is approximately a cosine curve with one full period, so the period is 12 months.

The maximum temperature is around 26° and the minimum is around 18°, so the centre of the graph is 22° with 4° either side. So the amplitude is 4.

5.8 Exercises

1. Show graphically that \( \sin x = \frac{x}{2} \) has
   (a) 2 solutions for \( 0 \leq x \leq 2\pi \)
   (b) 3 solutions for \( -\pi \leq x \leq \pi \).

2. Solve \( \sin x = x \) for \( 0 \leq x \leq 2\pi \) graphically by sketching \( y = \sin x \) and \( y = x \) on the same number plane.

3. Solve \( \cos x = 2x - 3 \) for \( 0 \leq x \leq 2\pi \) by finding the points of intersection of the graphs \( y = \cos x \) and \( y = 2x - 3 \).

4. Solve \( \tan x = x \) graphically in the domain \( 0 \leq x \leq 2\pi \).

5. Solve by graphical means \( \sin 2x = x \) for \( 0 \leq x \leq 2\pi \).

6. Draw the graphs of \( y = \sin x \) and \( y = \cos x \) for \( 0 \leq x \leq 2\pi \) on the same set of axes. Use your graphs to solve the equation \( \sin x = \cos x \) for \( 0 \leq x \leq 2\pi \).

7. The graph below show the times of sunsets in a city over a period of 2 years.

(a) Find the period and amplitude of the graph.

(b) At approximately what time would you expect the sun to set in July?
8. The graph shows the incidence of crimes committed over 24 years in Gotham City.

(a) Approximately how many crimes were committed in the 10th year?
(b) What was the
   (i) highest and
   (ii) lowest number of crimes?
(c) Find the amplitude and the period of the graph.

9. Below is a table showing the average daylight hours over several months.

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>June</th>
<th>July</th>
<th>Aug</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daylight hours</td>
<td>15.3</td>
<td>14.7</td>
<td>13.2</td>
<td>13.1</td>
<td>12.7</td>
<td>12.2</td>
<td>12.5</td>
<td>13.8</td>
</tr>
</tbody>
</table>

(a) Draw a graph to show this data.
(b) Is it periodic? If so, what is the period?
(c) Find the amplitude.

10. The table below shows the high and low tides over a three-day period.

<table>
<thead>
<tr>
<th>Day</th>
<th>Friday</th>
<th>Saturday</th>
<th>Sunday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>6.20 am</td>
<td>11.55 am</td>
<td>6.20 am</td>
</tr>
<tr>
<td></td>
<td>6.15 pm</td>
<td>11.55 pm</td>
<td>11.48 pm</td>
</tr>
<tr>
<td>Tide (m)</td>
<td>3.2</td>
<td>1.1</td>
<td>3.2</td>
</tr>
<tr>
<td></td>
<td>3.4</td>
<td>1.3</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>1.3</td>
<td>3.2</td>
<td>1.1</td>
</tr>
<tr>
<td></td>
<td>3.4</td>
<td>1.2</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>3.5</td>
<td>1.3</td>
</tr>
</tbody>
</table>

(a) Draw a graph showing the tides.
(b) Find the period and amplitude.
(c) Estimate the height of the tide at around 8 am on Friday.
Differentiation of Trigonometric Functions

Derivative of $\sin x$

We can sketch the gradient (tangent) function of $y = \sin x$.

![Graph of sine function]

The gradient function is $y = \cos x$.

\[
\frac{d}{dx}(\sin x) = \cos x
\]

**Proof**

Let $f(x) = \sin x$

Then $f(x + h) = \sin(x + h)$

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}
\]

\[
= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}
\]

\[
= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}
\]

\[
= \lim_{h \to 0} \sin x \left( \frac{\cos h - 1}{h} \right) + \cos x \left( \frac{\sin h}{h} \right)
\]

\[
= \lim_{h \to 0} \sin x(0) + \cos x(1) \left( \text{since } \cos h \to 1 \text{ as } h \to 0 \text{ and } \lim_{h \to 0} \frac{\sin h}{h} = 1 \right)
\]

\[
= \cos x.
\]

Notice that the result $\lim_{h \to 0} \frac{\sin h}{h} = 1$ only works for radians. We always use radians when using calculus.
FUNCTION OF A FUNCTION RULE

\[
\frac{d}{dx}[\sin f(x)] = f'(x) \cos f(x)
\]

**Proof**

Let \( u = f(x) \)

Then \( y = \sin u \)

\[
\frac{du}{dx} = f'(x) \quad \text{and} \quad \frac{dy}{du} = \cos u
\]

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

\[
= \cos u \cdot f'(x)
\]

\[
= f'(x) \cos f(x)
\]

**Derivative of \( \cos x \)**

We can sketch the gradient function of \( y = \cos x \).

![Graph of \( y = \cos x \)](image)

The gradient function is \( y = -\sin x \).

![Graph of \( y = -\sin x \)](image)

\[
\frac{d}{dx} (\cos x) = -\sin x
\]

**Proof**

Since \( \cos x = \sin \left( \frac{\pi}{2} - x \right) \)

\[
\frac{d}{dx} (\cos x) = \frac{d}{dx} \left[ \sin \left( \frac{\pi}{2} - x \right) \right]
\]

\[
= -1 \times \cos \left( \frac{\pi}{2} - x \right)
\]

\[
= -\sin x
\]
FUNCTION OF A FUNCTION RULE

\[ \frac{d}{dx} [\cos(f(x))] = -f'(x) \sin f(x) \]

**Derivative of tan x**

The gradient function is harder to sketch for \( y = \tan x \).

\[ y = \sec^2 x \]

It is easier to see the rule for differentiating \( y = \tan x \) by using \( \tan x = \frac{\sin x}{\cos x} \) and the quotient rule.

\[ \frac{d}{dx} (\tan x) = \sec^2 x \]

**Proof**

\[
\begin{align*}
\tan x &= \frac{\sin x}{\cos x} \\
\frac{d}{dx} (\tan x) &= \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) \\
&= \frac{u'v - v'u}{v^2} \\
&= \frac{\cos x (\cos x) - (-\sin x) \sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{align*}
\]
EXAMPLES

1. Differentiate \( \sin(5x) \).

**Solution**

\[
\frac{d}{dx} \sin(f(x)) = f'(x) \cos(f(x))
\]

\[
\therefore \frac{d}{dx} \sin(5x) = 5 \cos(5x)
\]

2. Differentiate \( \sin x^\circ \).

**Solution**

\[
\frac{d}{dx} (\sin x^\circ) = \frac{d}{dx} \left(\sin \frac{\pi x}{180}\right)
\]

\[
= \pi \cos \frac{\pi x}{180}
\]

\[
= \pi \cos x^\circ
\]

3. Find the exact value of the gradient of the tangent to the curve \( y = x^2 \sin x \) at the point where \( x = \frac{\pi}{4} \).

**Solution**

\[
\frac{dy}{dx} = u'v + v'u
\]

\[
= \cos x (x^2) + 2x (\sin x)
\]

\[
= x^2 \cos x + 2x \sin x
\]

When \( x = \frac{\pi}{4} \),

\[
\frac{dy}{dx} = \left(\frac{\pi}{4}\right)^2 \cos \frac{\pi}{4} + 2\left(\frac{\pi}{4}\right) \sin \frac{\pi}{4}
\]

\[
= \frac{\pi^2}{16} \times \frac{1}{\sqrt{2}} + \frac{\pi}{2} \times \frac{1}{\sqrt{2}}
\]

\[
= \frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}}
\]

\[
= \frac{\pi^2 + 8\pi}{16\sqrt{2}}
\]

\[
= \frac{\pi \sqrt{2} (\pi + 8)}{32}
\]
5.9 Exercises

1. Differentiate
   (a) sin 4x
   (b) cos 3x
   (c) tan 5x
   (d) tan (3x + 1)
   (e) cos (−x)
   (f) 3 sin x
   (g) 4 cos (5x − 3)
   (h) 2 cos (x^2)
   (i) 7 tan (x^2 + 5)
   (j) sin 3x + cos 8x
   (k) tan (π + x) + x^2
   (l) x tan x
   (m) sin 2x tan 3x
   (n) sin x
   (o) 3x + 4
   (p) (2x + tan 7x)^9
   (q) sin^2 x
   (r) 3 cos^2 5x
   (s) e^x − cos 2x
   (t) sin (1 − log x)
   (u) sin (e^x + x)
   (v) log (sin x)
   (w) e^{3x} cos 2x
   (x) \frac{e^{2x}}{\tan 7x}

2. Find the derivative of cos x \sin^4 x.

3. Find the gradient of the tangent to the curve y = tan 3x at the point where x = \frac{\pi}{9}.

4. Find the equation of the tangent to the curve y = sin (\pi − x) at the point \((\frac{\pi}{6}, \frac{1}{2})\), in exact form.

5. Differentiate \log_x (cos x).

6. Find the exact gradient of the normal to the curve y = sin 3x at the point where x = \frac{\pi}{18}.

7. Differentiate e^{\tan x}.

8. Find the equation of the normal to the curve y = 3 sin 2x at the point where x = \frac{\pi}{8}, in exact form.

9. Show that \frac{d^2 y}{dx^2} = −25y if y = 2 cos 5x.

10. Given f(x) = −2 sin x, show that f''(x) = −f(x).

11. Show that \frac{d}{dx} [\log_t (\tan x)] = \tan x + cot x.

12. Find the coordinates of the stationary points on the curve y = 2 sin x − x for 0 ≤ x ≤ 2\pi.

13. Differentiate
   (a) \tan x^a
   (b) 3 cos x^a
   (c) \frac{\sin x^a}{5}

14. If y = 2 sin 3x − 5 cos 3x, show that \frac{d^2 y}{dx^2} = −9y.

15. Find values of a and b if
   \frac{d^2 y}{dx^2} = ae^{3x} \cos 4x + be^{3x} \sin 4x,
   given y = e^{3x} \cos 4x.

Integration of Trigonometric Functions

The integrals of the trigonometric functions are their primitive functions.

\[ \int \cos x \, dx = \sin x + C \]
\[ \int -\sin x \, dx = \cos x + C \]
\[ \therefore \int \sin x \, dx = -\cos x + C \]

\[ \int \sin x \, dx = -\cos x + C \]
\[ \int \sec^2 x \, dx = \tan x + C \]

**Function of a function rule**

\[ \int \cos (ax + b) \, dx = \frac{1}{a} \sin (ax + b) + C \]

**Proof**

\[ \frac{d}{dx} [\sin (ax + b)] = a \cos (ax + b) \]
\[ \therefore \int a \cos (ax + b) \, dx = \sin (ax + b) + C \]
\[ \int \cos (ax + b) \, dx = \frac{1}{a} \int a \cos (ax + b) \, dx \]
\[ = \frac{1}{a} \sin (ax + b) + C \]

Similarly,

\[ \int \sin (ax + b) \, dx = -\frac{1}{a} \cos (ax + b) + C \]
\[ \int \sec^2(ax + b) \, dx = \frac{1}{a} \tan(ax + b) + C \]

**EXAMPLES**

1. **Find** \( \int \sin (3x) \, dx \).

**Solution**

\[ \int \sin (3x) \, dx = -\frac{1}{3} \cos (3x) + C \]
2. Evaluate $\int_{0}^{\pi/2} \sin x \, dx$.

**Solution**

\[
\int_{0}^{\pi/2} \sin x \, dx = \left[-\cos x\right]_{0}^{\pi/2} = -\cos \frac{\pi}{2} - (-\cos 0) = 0 - (-1) = 1
\]

3. Find $\int \cos x^o \, dx$.

**Solution**

\[
\int \cos x^o \, dx = \int \cos \frac{\pi x}{180} \, dx = \frac{1}{\pi} \sin \frac{\pi x}{180} + C
\]

4. Find the area enclosed between the curve $y = \cos x$, the $x$-axis and the lines $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

**Solution**

\[
\int_{\pi/2}^{3\pi/2} \cos x \, dx = \left[\sin x\right]_{\pi/2}^{3\pi/2} = \sin \frac{3\pi}{2} - \sin \frac{\pi}{2} = -1 - 1 = -2
\]

\[
\therefore \text{area is } 2 \text{ units}^2.
\]
5. Find the volume of the solid formed if the curve \( y = \sec x \) is rotated about the \( x \)-axis from \( x = 0 \) to \( x = \frac{\pi}{4} \).

**Solution**

\[ y = \sec x \]
\[ \therefore y^2 = \sec^2 x \]

\[ V = \pi \int_{a}^{b} y^2 \, dx \]
\[ = \pi \int_{0}^{\frac{\pi}{4}} \sec^2 x \, dx \]
\[ = \pi \left[ \tan x \right]_{0}^{\frac{\pi}{4}} \]
\[ = \pi \left( \tan \frac{\pi}{4} - \tan 0 \right) \]
\[ = \pi \left( 1 - 0 \right) \]
\[ = \pi \]

So the volume is \( \pi \) units³.

---

5.10 **Exercises**

1. Find the indefinite integral (primitive function) of
   (a) \( \cos x \)
   (b) \( \sin x \)
   (c) \( \sec^2 x \)
   (d) \( \frac{\sin x^2}{4} \)
   (e) \( \sin 3x \)
   (f) \( -\sin 7x \)
   (g) \( \sec^2 5x \)
   (h) \( \cos (x + 1) \)
   (i) \( \sin (2x - 3) \)
   (j) \( \cos (2x - 1) \)
   (k) \( \sin (\pi - x) \)
   (l) \( \cos (x + \pi) \)
   (m) \( 2 \sec^2 7x \)
   (n) \( 4 \sin \frac{x}{2} \)
   (o) \( 3 \sec^2 \frac{x}{3} \)
   (p) \( -\sin (3 - x) \)

2. Evaluate, giving exact answers where appropriate.
   (a) \( \int_{0}^{\frac{\pi}{2}} \cos x \, dx \)
   (b) \( \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x \, dx \)
   (c) \( \int_{\frac{\pi}{2}}^{\pi} \sin \frac{x}{2} \, dx \)
   (d) \( \int_{\pi}^{0} \cos 3x \, dx \)
3. Find the area enclosed between the curve \( y = \sin x \) and the \( x \)-axis in the domain \( 0 \leq x \leq 2\pi \).

4. Find the exact area bounded by the curve \( y = \cos 3x \), the \( x \)-axis and the lines \( x = 0 \) and \( x = \frac{\pi}{12} \).

5. Find the area enclosed between the curve \( y = \sec^2 \frac{x}{4} \), the \( x \)-axis and the lines \( x = \frac{\pi}{4} \) and \( x = \frac{\pi}{2} \), correct to 2 decimal places.

6. Find the volume, correct to 2 decimal places, of the solid formed when the curve \( y = \sec \pi x \) is rotated about the \( x \)-axis from \( x = 0 \) to \( x = 0.15 \).

7. Find, in exact form, the volume of the solid of revolution formed by rotating the curve \( y = \sqrt{\sin 2x} \) about the \( x \)-axis from \( x = 0 \) to \( x = \frac{\pi}{6} \).

8. Find the exact area enclosed by the curve \( y = \sin x \) and the line \( y = \frac{1}{2} \) for \( 0 \leq x \leq 2\pi \).

9. Find the exact area bounded by the curves \( y = \sin x \) and \( y = \cos x \) in the domain \( 0 \leq x \leq 2\pi \).

10. (a) Show that the volume of the solid formed by rotating the curve \( y = \sqrt{\cos x} \) about the \( x \) axis between \( x = 0 \) and \( x = \frac{\pi}{2} \) is \( \pi \) units\(^3 \).

   (b) Use the trapezoidal rule with 4 subintervals to find an approximation to the volume of the solid formed by rotating the curve \( y = \sqrt{\cos x} \) about the \( x \)-axis from \( x = 0 \) to \( x = \frac{\pi}{2} \).

11. A curve has \( \frac{d^2y}{dx^2} = 18 \sin 3x \) and a stationary point at \( \left( \frac{\pi}{6}, -2 \right) \). Find the equation of the curve.
Test Yourself 5

1. A circle with radius 5 cm has an angle of $\frac{\pi}{6}$ subtended at the centre. Find
   (a) the exact arc length
   (b) the exact area of the sector
   (c) the area of the minor segment to 3 significant figures.

2. Find the exact value of
   (a) $\tan \frac{\pi}{3}$
   (b) $\cos \frac{\pi}{6}$
   (c) $\sin \frac{2\pi}{3}$
   (d) $\cos \frac{3\pi}{4}$

3. Solve for $0 \leq x \leq 2\pi$
   (a) $\tan x = -1$
   (b) $2 \sin x = 1$

4. Sketch for $0 \leq x \leq 2\pi$
   (a) $y = 3 \cos 2x$
   (b) $y = 7 \sin \frac{x}{2}$

5. Differentiate
   (a) $\cos x$
   (b) $2 \sin x$
   (c) $\tan x + 1$
   (d) $x \sin x$
   (e) $\frac{\tan x}{x}$
   (f) $\cos 3x$
   (g) $\tan 5x$

6. Find the indefinite integral (primitive function) of
   (a) $\sin 2x$
   (b) $3 \cos x$
   (c) $\sec^2 5x$
   (d) $1 + \sin x$

7. Evaluate
   (a) $\int_0^\frac{\pi}{2} \cos x \, dx$
   (b) $\int_0^\frac{\pi}{6} \sec^2 x \, dx$

8. Find the equation of the tangent to the curve $y = \sin 3x$ at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

9. If $x = \cos 2t$, show that $\frac{d^2x}{dt^2} = -4x$.

10. Find the exact area bounded by the curve $y = \sin x$, the $x$-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

11. Find the volume of the solid formed if the curve $y = \sec x$ is rotated about the $x$-axis from $x = 0$ to $x = \frac{\pi}{6}$.

12. Simplify
   (a) $\lim_{x \to 0} \frac{\sin 5x}{x}$
   (b) $\lim_{\theta \to 0} \frac{2 \tan \theta}{\theta}$

13. Find the gradient of the tangent to the curve $y = 3 \cos 2x$ at the point where $x = \frac{\pi}{6}$.

14. A circle has a circumference of $8\pi$ cm. If an angle of $\frac{\pi}{7}$ is subtended at the centre of the circle, find
   (a) the exact area of the sector
   (b) the area of the minor segment, to 2 decimal places.

15. (a) Sketch $y = \cos 2x$ and $y = \frac{2x}{3}$ on the same set of axes for $0 \leq x \leq 2\pi$.
   (b) Solve $\cos 2x = \frac{2x}{3}$ for $0 \leq x \leq 2\pi$. 

---

Magda arrived back in time for dinner, and the family was excited to see her. She explained how she managed to get there and the various obstacles she faced along the way. Her enthusiasm was contagious, and everyone was eager to hear more about her adventures.
16. Find the exact area with rational denominator bounded by the curve
\( y = \sin x \), the x-axis and the lines \( x = \frac{\pi}{6} \) and \( x = \frac{\pi}{4} \).

17. Find the area bounded by the curve \( y = \cos 2x \), the x-axis and the lines \( x = 0 \) to \( x = \pi \).

18. Find the equation of the normal to the curve \( y = \tan x \) at the point \( \left( \frac{\pi}{4}, 1 \right) \).

19. A curve has \( \frac{dy}{dx} = 6 \sin 2x \), and passes through the point \( \left( \frac{\pi}{2}, 3 \right) \). Find the equation of the curve.

20. (a) Sketch \( y = 7 \sin 3x \) and \( y = 2x - 1 \) on the same number plane for \( 0 \leq x \leq 2\pi \).
(b) Solve \( 7 \sin 3x = 2x - 1 \) for \( 0 \leq x \leq 2\pi \).

---

**Challenge Exercise 5**

1. Use Simpson’s rule with 5 function values to find an approximation to \( \int_{\frac{\pi}{2}}^{\pi} \tan x \, dx \), correct to 2 decimal places.

2. Evaluate \( \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \sec^2 2x \, dx \), in exact form.

3. The area of the sector of a circle is \( 4\pi \) units\(^2 \) and the length of the arc bounded by this sector is \( \frac{\pi}{8} \) units. Find the radius of the circle and the angle that is subtended at the centre.

4. If \( f(x) = 3 \cos \pi x \)
   (a) find the period and amplitude of the function
   (b) sketch \( f(x) \) for \( 0 \leq x \leq 4 \).

5. Given \( \frac{d^2y}{dx^2} = 9 \sin 3x \)
   (a) find \( y \) if there is a stationary point at \( \left( \frac{\pi}{2}, 1 \right) \)
   (b) Show that \( \frac{d^2y}{dx^2} + 9y = 0 \).

6. Sketch \( y = 5 \sin (x + \pi) \) for \( 0 \leq x \leq 2\pi \).

7. Find the derivative of \( \tan x^2 \).

8. (a) Show that \( \sec x \cosec x = \frac{\sec^2 x}{\tan x} \).
(b) Hence, or otherwise, find the exact value of \( \int_{\frac{\pi}{2}}^{\pi} \cosec x \sec x \, dx \).

9. Differentiate \( e^{\sin 2x} \).

10. (a) Find the stationary points on the curve \( y = \sin 2x + 3 \) over the domain \( 0 \leq x \leq \pi \).
   (b) What is the maximum value of the curve?
   (c) What is the amplitude?

11. The area of a sector in a circle of radius 4 cm is \( \frac{8\pi}{3} \) cm\(^2 \). Find the area of the minor segment, in exact form.

12. Find \( \int \sin x^2 \, dx \).

13. Find the gradient of the normal to the curve \( y = x^2 + \cos \pi x \) at the point where \( x = 1 \).
14. Use the trapezoidal rule with 4 subintervals to find, correct to 3 decimal places, an approximation to the volume of the solid formed by rotating the curve \( y = \sin x \) about the \( x \)-axis from \( x = 0.2 \) to \( x = 0.6 \).

15. Differentiate \( \log_e (\sin x + \cos x) \).

16. Find the exact area of the minor segment cut off by a quarter of a circle with radius 3 cm.

17. Sketch \( y = \tan \left( x - \frac{\pi}{4} \right) \) for \( 0 \leq x \leq 2\pi \).

18. Find all the points of inflexion on the curve \( y = 3 \cos \left( 2x + \frac{\pi}{4} \right) \) for \( 0 \leq x \leq 2\pi \).

19. Find the exact area bounded by the curve \( y = \cos x \), the \( x \)-axis and the lines \( x = \frac{\pi}{6} \) and \( x = \frac{\pi}{4} \).

20. If \( f(x) = 2 \cos 3x \), show that \( f''(x) = -9f(x) \).
Applications of Calculus to the Physical World

**TERMINOLOGY**

- **Acceleration**: The rate of change of velocity with respect to time
- **At rest**: Stationary (zero velocity)
- **Displacement**: The movement of an object in relation to its original position
- **Exponential growth**: Growth, or increase in a quantity, where the rate of change of a quantity is in direct proportion to the quantity itself. The growth becomes more rapid over time
- **Exponential decay**: Decay, or decrease in a quantity, where the rate of change of a quantity is in direct proportion to the quantity itself. The decay becomes less rapid over time
- **Rate of change**: The change of one variable with respect to another variable over time
- **Velocity**: The rate of change of displacement of an object with respect to time involving speed and direction
INTRODUCTION

CALCULUS IS USED IN many situations involving rates of change, such as physics or economics. This chapter looks at rates of change of a particle in motion, and exponential growth and decay. Both these types of rates of change involve calculus.

DID YOU KNOW?

Galileo (1564–1642) was very interested in the behaviour of bodies in motion. He dropped stones from the Leaning Tower of Pisa to try to prove that they would fall with equal speed. He rolled balls down slopes to prove that they move with uniform speed until friction slows them down. He showed that a body moving through the air follows a curved path at a fairly constant speed.

John Wallis (1616–1703) continued this study with his publication *Mechanica, sive Tractatus de Motu Geometricus*. He applied mathematical principles to the laws of motion and stimulated interest in the subject of mechanics.

Soon after Wallis’ publication, Christiaan Huygens (1629–1695) wrote *Horologium Oscillatorium sive de Motu Pendulorum*, in which he described various mechanical principles. He invented the pendulum clock, improved the telescope and investigated circular motion and the descent of heavy bodies.

These three mathematicians provided the foundations of mechanics.

Sir Isaac Newton (1642–1727) used calculus to increase the understanding of the laws of motion. He also used these concepts as a basis for his theories on gravity and inertia.

Rates of Change

Simple rates of change

The gradient of a straight line measures the rate of change of \( y \) with respect to the change in \( x \).

![Gradient of a straight line](image)

With a curve, the rate of change of \( y \) with respect to \( x \) is measured by the gradient of the tangent to the curve. That is, the derivative measures the rate of change.
EXAMPLES

1. The number of bacteria in a culture increases according to the formula $B = 2t^4 - t^2 + 2000$, where $t$ is time in hours. Find
   (a) the number of bacteria initially
   (b) the number of bacteria after 5 hours
   (c) the rate at which the number of bacteria will be increasing after 5 hours.

Solution

(a) $B = 2t^4 - t^2 + 2000$
   Initially, $t = 0$
   $\therefore B = 2(0)^4 - 0^2 + 2000$
   $= 2000$
   So there are 2000 bacteria initially.

(b) When $t = 5$,
   $B = 2(5)^4 - 5^2 + 2000$
   $= 3225$
   So there will be 3225 bacteria after 5 hours.

(c) The rate of change is given by the derivative $\frac{dB}{dt} = 8t^3 - 2t$
   When $t = 5$,
   $\frac{dB}{dt} = 8(5)^3 - 2(5)$
   $= 990$
   So the rate of increase after 5 hours will be 990 bacteria per hour.

2. The volume flow rate of water into a pond is given by $R = 4 + 3t^2$ litres per hour. If there is initially no water in the pond, find the volume of water after 12 hours.
Chapter 6
Applications of Calculus to the Physical World

Volume flow rate is the rate of change of volume over time.

Solution

\[ R = 4 + 3t^2 \]
i.e. \[ \frac{dV}{dt} = 4 + 3t^2 \]
\[ \therefore V = \int (4 + 3t^2)\,dt \]
\[ = 4t + t^3 + C \]
When \( t = 0, V = 0 \)
\[ \therefore 0 = 0 + 0 + C \]
\[ = C \]
\[ \therefore V = 4t + t^3 \]
When \( t = 12 \)
\[ V = 4(12) + (12)^3 \]
\[ = 1776 \text{ L} \]
So there will be 1776 L of water in the pond after 12 hours.

6.1 Exercises

1. Find a formula for the rate of change in each question.
   (a) \( h = 20t - 4t^2 \)
   (b) \( D = 5t^4 + 2t^2 + 1 \)
   (c) \( A = 16x - 2x^2 \)
   (d) \( x = 3t^3 - x^4 + 2x - 3 \)
   (e) \( V = e^t + 4 \)
   (f) \( S = 3 \cos 5\theta \)
   (g) \( S = 2\pi r + \frac{50}{r^2} \)
   (h) \( D = \sqrt{x^2 - 4} \)
   (i) \( S = 800r + \frac{400}{T} \)
   (j) \( V = \frac{4}{3}\pi r^3 \)

2. Find an original formula for each question given its rate of change.
   (a) \( R = 4t - 12t^2 \) is the rate at which the height \( h \) of an object changes over time \( t \)
   (b) \( R = 8x^3 + 1 \) is the rate at which the area \( A \) of a figure changes with side \( x \)
   (c) \( R = 4\pi r^2 \) is the rate at which the volume \( V \) changes with radius \( r \)
   (d) \( R = 7 \sin t \) is the rate at which the distance \( d \) of an object changes over time \( t \)
   (e) \( R = 8e^{2t} - 3 \) is the rate at which the speed \( s \) changes over time \( t \).

3. If \( h = t^4 - 7t + 5 \), find the rate of change of \( h \) when \( t = 3 \).

4. Given \( f(t) = \sin 2t \), find the rate of change when \( t = \frac{\pi}{6} \).

5. A particle moves so that its distance \( x \) is given by \( x = 2e^{3t} \). Find the exact rate of change when \( t = 4 \).

6. The volume of water flowing through a pipe is given by \( V = t^2 + 3t \). Find the rate at which the water is flowing when \( t = 5 \).
7. The rate of change of the angle sum $S$ of a polygon with $n$ sides is a constant 180. If $S$ is 360 when $n = 4$, find $S$ when $n = 7$.

8. A particle moves so that the rate of change of distance $D$ over time $t$ is given by $R = 2e^t - 1$. If $D = 10$ when $t = 0$, find $D$ when $t = 3$.

9. For a certain graph, the rate of change of $y$ values with respect to its $x$ values is given by $R = 3x^2 - 2x + 1$. If the graph passes through the point $(-1, 3)$, find its equation.

10. The mass in grams of a melting iceblock is given by the formula $M = t - 2t^2 + 100$, where $t$ is time in minutes. Show that the rate at which the iceblock is melting is given by $R = 4t^2 - t$ grams per minute and find the rate at which it will be melting after 5 minutes.

11. The rate of change in velocity over time is given by $\frac{dy}{dt} = 4t + t^2 - t^3$. If the initial velocity is 2 cm/s, find the velocity after 15 s.

12. The rate of flow of water into a dam is given by $R = 500 + 20t$ L/h. If there is 15 000 L of water initially in the dam, how much water will there be in the dam after 10 hours?

13. The surface area in cm$^2$ of a balloon being inflated is given by $S = t^3 - 2t^2 + 5t + 2$, where $t$ is time in seconds. Find the rate of increase in the balloon’s surface area after 8 s.

14. According to Boyle’s Law, the pressure of a gas is given by the formula $P = \frac{k}{V}$, where $k$ is a constant and $V$ is the volume of the gas. If $k = 100$ for a certain gas, find the rate of change in the pressure when $V = 20$.

15. A circular disc expands as it is heated. The area, in cm$^2$, of the disc increases according to the formula $A = 4t^2 + t$, where $t$ is time in minutes. Find the rate of increase in the area after 5 minutes.

16. A balloon is inflated so that its increase in volume is at a constant rate of 10 cm$^3$/s. If its volume is initially 1 cm$^3$, find its volume after 3 s.

17. The number of people in a certain city is given by $P = 200 000e^{0.2t}$, where $t$ is time in years. Find the rate at which the population of the city will be growing after 5 years.

18. A radioactive substance has a mass that decreases over time $t$ according to the formula $M = 200e^{-0.1t}$. Find (a) the mass of the substance after 20 years, to the nearest gram (b) the rate of its decrease after 20 years.

19. If $y = e^{4t}$, show that the rate is given by $R = 4y$.

20. Given $S = 2e^{3t} + 3$, show that the rate of change is $R = 2(S - 3)$. 


Exponential Growth and Decay

Exponential growth or decay are terms that describe a special rate of change that occurs in many situations. Population growth and growth of bacteria in a culture are examples of exponential growth. The decay of radioactive substances, the cooling of a substance and change in pressure are examples of exponential decay.

When a substance grows or decays exponentially, its rate of change is directly proportional to the amount of the substance itself. That is, the more of the substance there is, the faster it grows. For example, in a colony of rabbits, there is fairly slow growth in the numbers at first, but the more rabbits there are, the more rabbits will be born. In other species, such as koalas, the clearing of habitats and other factors may cause the population to decrease, or decay, exponentially.

The exponential function is a function where the rate of change (gradient) is increasing as the function increases.

In mathematical symbols, exponential growth or decay can be written as $\frac{dQ}{dt} = kQ$, where $k$ is the growth or decay constant.
Proof

\[
\frac{dQ}{dt} = kQ
\]

\[\Rightarrow \frac{dt}{dQ} = \frac{1}{kQ}\]

\[t = \int \frac{1}{kQ} dQ\]

\[= \frac{1}{k} \log Q + k_1\]

\[kt = \log Q + k_2\]

\[kt - k_2 = \log Q\]

\[\therefore Q = e^{kt - k}\]

\[= e^{kt} \times e^{-k}\]

\[= Ae^{kt}\]

A is always the initial quantity.

Proof

Given \(Q = Ae^{kt}\), suppose the initial quantity is \(Q_0\), i.e. \(Q = Q_0\) when \(t = 0\).

Substituting into the equation gives

\[Q_0 = Ae^{k \times 0}\]

\[= Ae^0\]

\[= A\]

\[\therefore A\] is the initial quantity.

Sometimes the equation is written as \(Q = Q_0 e^{kt}\).

DID YOU KNOW?

Thomas Malthus (1766–1834), at the beginning of the Industrial Revolution, developed a theory about population growth that we still use today. His theory states that under ideal conditions, the birth rate is proportional to the size of the population. That is, \(\frac{dN}{dt} = kN\) (Malthusian Law of Population Growth).

Malthus was concerned that the growth rate of populations would be higher than the increase in food supplies, and that people would starve.

Was he right? Is this happening? How could we prove this?
EXAMPLES

1. The population of a colony of rabbits over time \( t \) months is given by \( P = 1500e^{0.046t} \). Find

   (a) the initial population
   (b) the population after 2 years
   (c) when the population reaches 5000.

Solution

(a) Initially, \( t = 0 \)
Substituting:

\[
P = 1500e^{0.046t}
\]
\[
= 1500e^{0.046 \times 0}
\]
\[
= 1500e^0
\]
\[
= 1500 \quad (e^0 = 1)
\]
So the initial population is 1500 rabbits.

(b) 2 years = 24 months
So \( t = 24 \)
Substituting:

\[
P = 1500e^{0.046t}
\]
\[
= 1500e^{0.046 \times 24}
\]
\[
= 1500e^{1.104}
\]
\[
= 4524.31
\]
So the population after 2 years is 4524 rabbits.

(c) \( P = 5000 \)
Substituting:

\[
P = 1500e^{0.046t}
\]
\[
5000 = 1500e^{0.046t}
\]
\[
\frac{5000}{1500} = e^{0.046t}
\]
\[
\ln \frac{5000}{1500} = \ln e^{0.046t}
\]
\[
= 0.046t \ln e
\]
\[
= 0.046t \quad (\text{since } \ln e = 1)
\]
\[
\ln \frac{5000}{1500} = 0.046 \quad t = 26.2 = t
\]
So the population reaches 5000 after 26.2 months.
2. The number of bacteria in a culture is given by \( N = Ae^{kt} \). If 6000 bacteria increase to 9000 after 8 hours, find
(a) \( k \) correct to 3 significant figures
(b) the number of bacteria after 2 days
(c) the rate at which the bacteria will be increasing after 2 days
(d) when the number of bacteria will reach 1 000 000
(e) the growth rate per hour as a percentage.

**Solution**

(a) \( N = Ae^{kt} \)

When \( t = 0, N = 6000 \)
\[ \therefore 6000 = Ae^0 \]
\[ = A \]
So \( N = 6000e^{kt} \)

When \( t = 8, N = 9000 \)
\[ 9000 = 6000e^{8k} \]
\[ \frac{9000}{6000} = e^{8k} \]
\[ 1.5 = e^{8k} \]
\[ \log_e 1.5 = \log_e e^{8k} \]
\[ = 8k \log_e e \]
\[ = 8k \]
\[ \frac{\log_e 1.5}{8} = k \]
\[ 0.0507 \div k \]

So \( N = 6000e^{0.0507t} \).

(b) When \( t = 48 \), (48 hours in 2 days)
\[ N = 6000e^{0.0507 \times 48} \]
\[ = 68344 \]

So there will be 68 344 bacteria after 2 days.

(c) Rate: \( \frac{dN}{dt} = 6000(0.0507e^{0.0507t}) \)

When \( t = 48, \)
\[ \frac{dN}{dt} = 304.1e^{0.0507 \times 48} \]
\[ = 3464 \]

So after 2 days the rate of growth will be 3464 bacteria per hour.

Another method:
\[ \frac{dN}{dt} = kN \]
\[ = 0.0507N \]
When \( t = 48, N = 68344 \)
\[ \frac{dN}{dt} = 0.0507 \times 68344 = 3464 \]
(d) When \( N = 1\,000\,000 \)
\[
1\,000\,000 = 6000e^{0.0507t}
\]
\[
6000 = e^{0.0507t}
\]
\[
166.7 = e^{0.0507t}
\]
\[
\log_e 166.7 = \log_e e^{0.0507t}
\]
\[
= 0.0507t \log_e e
\]
\[
= 0.0507t
\]
\[
\log_e 166.7 = t
\]
\[
100.9 \div t
\]
So the number of bacteria will be 1\,000\,000 after 100.9 hours.

(e) \[
\frac{dN}{dt} = kN \text{ where } k = \text{growth constant}
\]
\[
k = 0.0507
\]
\[
= 5.07\%
\]
So the growth rate is 5.07% per hour.

3. A 50 g mass of uranium decays to 35 g after 2 years. If the rate of decay of its mass is proportional to the mass itself, find the amount of uranium left after 25 years.

**Solution**

\[
Q = Ae^{kt}
\]

When \( t = 0, Q = 50 \)
\[
\therefore 50 = Ae^0
\]
\[
= A
\]
So \( Q = 50e^{kt} \)

When \( t = 2, Q = 35 \)
\[
35 = 50e^{2k}
\]
\[
\frac{35}{50} = e^{2k}
\]
\[
0.7 = e^{2k}
\]
\[
\log_e 0.7 = \log_e e^{2k}
\]
\[
= 2k \log_e e
\]
\[
= 2k
\]
\[
\log_e 0.7 = \frac{k}{2}
\]
\[
-0.1783 \div k
\]
So \( Q = 50e^{-0.1783t} \)

When \( t = 25 \)
\[
Q = 50e^{-0.1783 \times 25}
\]
\[
= 0.579
\]
So there will be 0.579 grams, or 579 mg, left after 25 years.
6.2 Exercises

1. The number of birds in a colony is given by \( N = 80e^{0.02t} \) where \( t \) is in days.
   (a) How many birds are there in the colony initially?
   (b) How many birds will there be after 30 days?
   (c) After how many days will there be 500 birds?
   (d) Sketch the curve of the population over time.

2. The number of bacteria in a culture is given by \( N = N_0 e^{0.32t} \), where \( t \) is time in hours.
   (a) If there are initially 20000 bacteria, how many will there be after 5 hours?
   (b) How many hours, to the nearest hour, would it take for the number of bacteria to reach 200000?

3. The decay of radium is proportional to its mass. If 100 kg of radium takes 5 years to decay to 95 kg
   (a) show that the mass of radium is given by \( M = 100 e^{-0.01t} \)
   (b) find its mass after 10 years
   (c) find its half-life (the time taken for the radium to halve its mass).

4. A chemical reaction causes the amount of chlorine to be reduced at a rate proportional to the amount of chlorine present at any one time. If the amount of chlorine is given by the formula \( A = A_0 e^{-kt} \), and 100 L reduces to 65 L after 5 minutes, find
   (a) the amount of chlorine after 12 minutes
   (b) how long it will take for the chlorine to reduce to 10 L.

5. The production output in a factory increases according to the equation \( P = P_0 e^{kt} \), where \( t \) is in years.
   (a) Find \( P_0 \) if the initial output is 5000 units.
   (b) The factory produces 8000 units after 3 years. Find the value of \( k \), to 3 decimal places.
   (c) How many units will the factory produce after 6 years?
   (d) The factory needs to produce 20 000 units to make a maximum profit. After how many years, correct to 1 decimal place, will this happen?

6. The rate of depletion of rainforests can be estimated as proportional to the amount of rainforests. If 3 million m² of rainforest is reduced to 2.7 million m² after 20 years, find how much rainforest there will be after 50 years.
7. The annual population of a country is increasing at a rate of 6.9%; that is, \( \frac{dP}{dt} = 0.069P \). If the population is 50000 in 2015, find (a) a formula for the population growth (b) the population in the year 2020 (c) the rate at which the population will be growing in the year 2020 (d) in which year the population will reach 300000.

8. An object is cooling down according to the formula \( T = T_o e^{-kt} \), where \( T \) is temperature in degrees Celsius and \( t \) is time in minutes. If the temperature is initially 90°C and the object cools down to 81°C after 10 minutes, find (a) its temperature after half an hour (b) how long (in hours and minutes) it will take to cool down to 30°C.

9. In the process of the inversion of sugar, the amount of sugar present is given by the formula \( S = Ae^{kt} \). If 150 kg of sugar is reduced to 125 kg after 3 hours, find (a) the amount of sugar after 8 hours, to the nearest kilogram (b) the rate at which the sugar will be reducing after 8 hours (c) how long it will take to reduce to 50 kg.

10. The mass, in grams, of a radioactive substance is given by \( M = M_o e^{-kt} \), where \( t \) is time in years. Find (a) \( M_o \) and \( k \) if a mass of 200 kg decays to 195 kg after 10 years (b) mass after 15 years (c) the rate of decay after 15 years (d) the half-life of the substance (time taken to decay to half its mass).

11. The number of bacteria in a culture increases from 15000 to 25000 in 7 hours. If the rate of bacterial growth is proportional to the number of bacteria at that time, find (a) a formula for the number of bacteria (b) the number of bacteria after 12 hours (c) how long it will take for the culture to produce 500000 bacteria.

12. A population in a certain city is growing at a rate proportional to the population itself. After 3 years the population increases by 20%. How long will it take for the population to double?
13. The half-life of radium is 1600 years.
(a) Find the percentage of radium that will be decayed after 500 years.
(b) Find the number of years that it will take for 75% of the radium to decay.

14. The population of a city is \( P(t) \) at any one time. The rate of decline in population is proportional to the population \( P(t) \), that is, \( \frac{dP(t)}{dt} = -kP(t) \).
(a) Show that \( P(t) = P(t_0)e^{-kt} \) is a solution of the differential equation \( \frac{dP(t)}{dt} = -kP(t) \).
(b) What percentage decline in population will there be after 10 years, given a 10% decline in 4 years?
(c) What will the percentage rate of decline in population be after 10 years?
(d) When will the population fall by 20%?

15. The rate of leakage of water out of a container is proportional to the amount of water in the container at any one time. If the container is 60% empty after 5 minutes, find how long it will take for the container to be 90% empty.

16. Numbers of sheep in a certain district are dropping exponentially due to drought. A survey found that numbers had declined by 15% after 3 years. If the drought continues, how long would it take to halve the number of sheep in that district?

17. Anthony has a blood alcohol level of 150 mg/dL. The amount of alcohol in the bloodstream decays exponentially. If it decreases by 20% in the first hour, find
(a) the amount of alcohol in Anthony’s blood after 3 hours
(b) when the blood alcohol level reaches 20 mg/dL.

18. The current \( C \) flowing in a conductor dissipates according to the formula \( \frac{dC}{dt} = -kC \). If it dissipates by 40% in 5 seconds, how long will it take to dissipate to 20% of the original current?

19. Pollution levels in a city have been rising exponentially with a 10% increase in pollution levels in the past two years. At this rate, how long will it take for pollution levels to increase by 50%?

20. If \( \frac{dQ}{dt} = kQ \), prove that \( Q = Ae^{kt} \) satisfies this equation by
(a) differentiating \( Q = Ae^{kt} \)
(b) integrating \( \frac{dQ}{dt} = kQ \).
Class Investigation

Research some environmental issues. For example:

1. It is estimated that some animals, such as pandas and koalas, will be extinct soon. How soon will pandas be extinct? Can we do anything to stop this extinction?
2. How do the greenhouse effect and global warming affect the earth? How soon will they have a noticeable effect on us?
3. How long do radioactive substances such as radium and plutonium take to decay? What are some of the issues concerning the storage of radioactive waste?
4. The erosion and salination of Australian soil are problems that may affect our farming in the future. Find out about this issue, and some possible solutions.
5. The effect of blue-green algae in some of our rivers is becoming a major problem. What steps have been taken to remedy this situation?

Look at the mathematical aspects of these issues. For example, what formulae are used to make predictions? What sorts of time scales are involved in these issues?

Research other issues, relating to growth and decay, that affect our environment.

Motion of a Particle in a Straight Line

In this study of the motion of a particle moving along a straight line, we ignore friction, gravity and other influences on the motion. We can find the particle’s displacement, velocity and acceleration at any one time. The term ‘particle’, or ‘body’, can refer to something quite large (e.g. a car). It describes any moving object.

DID YOU KNOW?

Calculus was developed in the 17th century as a solution to problems about the study of motion. Some problems of the time included finding the speed and acceleration of planets, calculating the lengths of their orbits, finding maximum and minimum values of functions, finding the direction in which an object is moving at any one time and calculating the areas and volumes of certain figures.
Displacement

Displacement \( (x) \) measures the distance of a particle from a fixed point (origin). Unlike distance, displacement can be positive or negative, according to which side of the origin it is on. Usually, on the right-hand side it will be positive and on the left it will be negative.

When the particle is at the origin, its displacement is zero. That is, \( x = 0 \).

Velocity

Velocity \( (v) \) is a measure of the rate of change of displacement with respect to time. Since the rate of change is the gradient of the graph of displacement against time,

\[
\frac{dx}{dt} = v
\]

Velocity can also be written as \( \dot{x} \).

Velocity (unlike speed) can be positive or negative, according to which direction the object is travelling in. If the particle is moving to the right, velocity is positive. If it is moving to the left, velocity is negative.

When the object is not moving, we say that it is at rest. That is, \( v = 0 \).

Acceleration

Acceleration is the measure of the rate of change of velocity with respect to time. This gives the gradient of the graph of velocity against time. That is,

\[
\frac{dv}{dt} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d^2x}{dt^2}
\]

Acceleration can also be written as \( \ddot{x} \).

Acceleration can be positive or negative, according to which direction it is in. If the acceleration is to the right, it is positive. If the acceleration is to the left, it is negative.

If the acceleration is in the same direction as the velocity, the particle is speeding up (accelerating). If the acceleration is in the opposite direction from the velocity, the particle is slowing down (decelerating).

If the object is travelling at a constant velocity, there is no acceleration. That is, \( a = 0 \).

Motion graphs

You can describe the velocity of a particle by looking at the gradient function of a displacement graph. The acceleration is the gradient function of a velocity graph.
EXAMPLES

1. The graph below shows the displacement of a particle from the origin as it moves in a straight line.

(a) When is the particle
   (i) at rest?
   (ii) at the origin?
   (b) When is the particle moving at its greatest velocity?

Solution

(a) (i) When the particle is at rest, velocity is zero. i.e. \( \frac{dx}{dt} = 0 \).

   So the particle is at rest at the stationary points \( t_3 \) and \( t_5 \).

   (ii) The particle is at the origin when \( x = 0 \), i.e. on the \( t \)-axis. So the particle is at the origin at \( t_2 \), \( t_4 \) and \( t_6 \).

(b) The greatest velocity is at \( t_4 \) (the curve is at its steepest). Notice that this is where there is a point of inflexion.

2. The graph below shows the displacement \( x \) of a particle over time \( t \).
(a) Draw a sketch of its velocity  
(b) Sketch its acceleration  
(c) Find values of $t$ for which the particle is  
   (i) at the origin and  
   (ii) at rest.  

**Solution**  

(a) The velocity is the rate of change of displacement, or $\frac{dx}{dt}$. By noting where the gradient of the tangent is positive, negative and zero, we draw the velocity graph.  

(b) Acceleration is the rate of change of velocity, or $\frac{d^2x}{dt^2}$. By noting where the gradient of the tangent is positive, negative and zero, we draw the acceleration graph.
(c) (i) The particle is at the origin when \( x = 0 \). This is at 0 and \( t_2 \).
(ii) The particle is at rest when \( v = 0 \). This is at \( t_1 \) (on the \( t\)-axis on the velocity graph or the stationary point on the displacement graph).

3. The graph below is the velocity of a particle.
(a) Draw a sketch showing the displacement of the particle.
(b) Find the times when the particle is
   (i) at rest
   (ii) at maximum velocity.
(c) Explain why you can’t state accurately when the particle is at the origin.

**Solution**

(a) For $0 - t_1$ and between $t_3$ and $t_5$, the velocity is negative (below the $t$-axis), so the displacement has $\frac{dx}{dt} < 0$.

   Between $t_1$ and $t_3$, and all points $> t_5$, the velocity is positive (above the $t$-axis), so the displacement has $\frac{dx}{dt} > 0$.

   At $t_1$, $t_3$ and $t_5$, the velocity is zero, so $\frac{dx}{dt} = 0$ and there is a stationary point.

![Graph showing displacement](image)

Notice that at $t_1$ and $t_3$, LHS $< 0$ and RHS $> 0$ so they are minimum turning points.

At $t_5$, LHS $> 0$ and RHS $< 0$ so it is a maximum turning point.

Here is one graph that could describe the shape of the displacement.

![Graph showing displacement](image)

(b) (i) At rest, $v = 0$.
   This occurs at $t_1$, $t_3$ and $t_5$. 
(ii) Maximum velocity is at $t_4$ since it is furthest from the $t$-axis. Notice also that there is a point of inflexion at $t_4$ on the displacement graph.

(c) You can’t tell when the particle is at the origin as the graph is not accurate. For example, the graph in (a) is at the origin at $t_4$ and another point to the right of $t_4$. However, when sketching displacement from the velocity (or the original function given the gradient or derivative function) there are many possible graphs, forming a family of graphs.

Notice that these cross the $t$-axis at various places, so we can’t tell where the displacement graph shows the particle at the origin.

6.3 Exercises

1. The graphs at right and overleaf show the displacement of an object. Sketch the graphs for velocity and acceleration.
2. The graph below shows the displacement of a particle as it moves along a straight line.

When is the particle (a) at the origin? (b) at rest? (c) travelling with constant acceleration? (d) furthest from the origin?

3. The graph below shows a particle travelling at a constant acceleration.

Sketch the graphs that could represent (a) its velocity and (b) its displacement.

4. The graph below shows the velocity of a particle.

(a) When is the particle at rest? (b) When is the acceleration zero? (c) When is the velocity the greatest? (d) Describe the motion of the particle at (i) $t_2$ and (ii) $t_3$. 
5. The graph below shows the displacement of a pendulum.

(a) When is the pendulum at rest?
(b) When is the pendulum in its equilibrium position (at the origin)?

6. Describe the motion of the particle at \( t_1 \) for each displacement graph.

(a) 
(b) 
(c) 
(d) 
(e) 

**Motion and Differentiation**

If displacement is given, then the velocity is the first derivative and the acceleration is the second derivative.

\[
\begin{align*}
\text{Displacement} \quad x \\
\text{Velocity} \quad \dot{x} &= \frac{dx}{dt} \\
\text{Acceleration} \quad \ddot{x} &= \frac{d^2x}{dt^2}
\end{align*}
\]
EXAMPLES

1. The displacement $x$ cm of a particle over time $t$ seconds is given by $x = 4t - t^2$.
   
   (a) Find any times when the particle is at rest.
   
   (b) How far does the particle move in the first 3 seconds?

Solution

(a) At rest means that the velocity \( \frac{dx}{dt} \) is zero.

\[
\frac{dx}{dt} = 4 - 2t
\]

When \( \frac{dx}{dt} = 0 \)

\[
4 - 2t = 0
\]

\[
4 = 2t
\]

\[
t = 2
\]

So the particle is at rest after 2 seconds.

Notice that there is a stationary point at \( \frac{dx}{dt} = 0 \), so the particle turns around at this time.

(b) Initially, when \( t = 0 \):

\[
x = 4(0) - 0^2
\]

\[
= 0
\]

So the particle is at the origin.

After 3 seconds, \( t = 3 \):

\[
x = 4(3) - 3^2
\]

\[
= 3
\]

So the particle is 3 cm from the origin.

However, after 2 seconds, the particle turns around, so it hasn’t simply moved from the origin to 3 cm away. We need to find where it is when it turns around.

When \( t = 2 \):

\[
x = 4(2) - 2^2
\]

\[
= 4
\]

So the particle is 4 cm from the origin.

The particle moves from the origin \( (x = 0) \) to 4 cm away in the first 2 seconds, so it has travelled 4 cm.

It then turns and goes back to 3 cm from the origin. So in the 3rd second it travels 1 cm back.
2. The displacement of a particle is given by $x = -t^2 + 2t + 3 \text{ cm}$, where $t$ is in seconds.
   (a) Find the initial velocity of the particle.
   (b) Show that the particle has constant acceleration.
   (c) Find when the particle will be at the origin.
   (d) Find the particle’s maximum displacement from the origin.
   (e) Sketch the graph showing the particle’s motion.

**Solution**

(a) 
\[
v = \frac{dx}{dt} = -2t + 2
\]

Initially, $t = 0$
\[
\therefore v = -2(0) + 2 = 2
\]

So the initial velocity is $2 \text{ cm/s}$.

(b) 
\[
a = \frac{d^2x}{dt^2} = -2
\]

\[
\therefore \text{ the particle has a constant acceleration of } -2 \text{ cm/s}^2.
\]

(c) At the origin $x = 0$
   i.e. $-t^2 + 2t + 3 = 0$
   
   \[
   -(t + 1)(t - 3) = 0
   \]
   \[
   \therefore t = -1 \text{ or } 3
   \]

So the particle will be at the origin after $3$ s.

(d) For maximum displacement,
\[
\frac{dx}{dt} = 0
\]
\[
-2t + 2 = 0
\]
\[
2 = 2t
\]
\[
t = 1
\]

(Maximum displacement occurs when $t = 1$.)

When $t = 1$,
\[
x = -(1)^2 + 2(1) + 3
\]
\[
= 4
\]

So maximum displacement is $4 \text{ cm}$. 

Total distance travelled in the first 3 seconds is $4 + 1$ or $5 \text{ cm}$. 

A time of $-1$ does not exist.
3. The displacement of a particle is given by the formula \( x = 1 + 3 \sin(2t) \), where \( x \) is in metres and \( t \) is in seconds.

(a) Sketch the graph of \( x \) as a function of \( t \).

(b) Find the times when the particle will be at rest.

(c) Find the position of the particle at those times.

**Solution**

(a) Graph has period \( \pi \) and amplitude 3.

(b) \( v = \frac{ds}{dt} = 6 \cos 2t \)

When the particle is at rest

\[ v = 0 \]

i.e. \( 6 \cos 2t = 0 \)

\( \cos 2t = 0 \)

\[ 2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \]

\[ \therefore t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots \]
(c) \( x = 1 + 3 \sin 2t \)

When \( t = \frac{\pi}{4} \)

\[
x = 1 + 3 \sin \frac{\pi}{2} = 1 + 3 (1) = 4
\]

When \( t = \frac{3\pi}{4} \)

\[
x = 1 + 3 \sin \frac{3\pi}{2} = 1 + 3 (-1) = -2
\]

Similarly, when \( t = \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots \), \( x \) moves between 4 m and -2 m from the origin.

**Investigation**

In physics, equations for displacement and velocity are given by

\[
s = ut + \frac{1}{2}at^2 \quad \text{and} \quad v = u + at,
\]

where \( u \) is the initial velocity and \( a \) is the acceleration. Differentiate the displacement formula to show the formula for velocity. What happens when you differentiate again?

**6.4 Exercises**

1. The displacement of a particle is given by \( x = t^3 - 9t \text{ cm} \), where \( t \) is time in seconds.
   (a) Find the velocity of the particle after 3 s.
   (b) Find the acceleration after 2 s.
   (c) Show that the particle is initially at the origin, and find any other times that the particle will be at the origin.
   (d) Find after what time the acceleration will be 30 \( \text{cms}^{-2} \).

2. A particle is moving such that its displacement is given by \( s = 2t^2 - 8t + 3 \), where \( s \) is in metres and \( t \) is in seconds.
   (a) Find the initial velocity.
   (b) Show that acceleration is constant and find its value.
   (c) Find the displacement after 5 s.
   (d) Find when the particle will be at rest.
   (e) What will the particle's displacement be at that time?
   (f) Sketch the graph of the displacement against time.
3. A projectile is fired into the air and its height in metres is given by \( h = 40t - 5t^2 + 4 \), where \( t \) is in seconds.
   (a) Find the initial height.
   (b) Find the initial velocity.
   (c) Find the height after 1 s.
   (d) What is the maximum height of the projectile?
   (e) Sketch the graph of the height against time.

4. The displacement in cm after time \( t \) s of a particle moving in a straight line is given by \( x = 2 - t - t^2 \).
   (a) Find the initial displacement.
   (b) Find when the particle will be at the origin.
   (c) Find the displacement after 2 s.
   (d) How far will the particle move in the first 2 seconds?
   (e) Find its velocity after 3 s.

5. The equation for displacement of a particle is given by \( x = e^{2t} + 1 \) m after time \( t \) seconds.
   (a) Find the initial velocity.
   (b) Find the exact acceleration after 1 s.
   (c) Show that the acceleration is always double the velocity.
   (d) Sketch the graph of velocity over time.

6. The displacement of a pendulum is given by \( x = \cos 2t \) cm after time \( t \) seconds.
   (a) Find the equation for the velocity of the pendulum.
   (b) Find the equation for its acceleration.
   (c) Find the initial displacement.
   (d) Find the times when the pendulum will be at rest.
   (e) What will the displacement be at these times?
   (f) When will there be zero displacement?
   (g) Show that acceleration \( a = -4x \).

7. An object is travelling along a straight line over time \( t \) seconds, with displacement according to the formula \( x = t^3 + 6t^2 - 2t + 1 \) m.
   (a) Find the equations of its velocity and acceleration.
   (b) What will its displacement be after 5 s?
   (c) What will its velocity be after 5 s?
   (d) Find its acceleration after 5 s.

8. The displacement, in centimetres, of a body is given by \( x = (4t - 3)^3 \), where \( t \) is time in seconds.
   (a) Find equations for velocity \( \dot{x} \) and acceleration \( \ddot{x} \).
   (b) Find the values of \( x, \dot{x} \) and \( \ddot{x} \) after 1 s.
   (c) Describe the motion of the body after 1 s.

9. The displacement of a particle, in metres, over time \( t \) seconds is \( s = ut + \frac{1}{2} gt^2 \), where \( u = 5 \) and \( g = -10 \).
   (a) Find the equation of the velocity of the particle.
   (b) Find the velocity after 10 s.
   (c) Show that the acceleration is equal to \( g \).

10. The displacement in metres after \( t \) seconds is given by \( s = \frac{2t - 5}{3t + 1} \).
    Find the equations for velocity and acceleration.
11. The displacement of a particle is given by \( x = \log(t + 1) \), where \( x \) is in centimetres and \( t \) is in seconds.
   (a) Find the initial position of the particle.
   (b) Find the velocity after 5 s.
   (c) Find the acceleration after 5 s.
   (d) Describe the motion of the particle at 5 s.
   (e) Find the exact time when the particle will be 3 cm to the right of the origin.

12. Displacement of a particle is given by \( x = t^3 - 4t^2 + 3t \), where \( s \) is in metres and \( t \) is in seconds.
   (a) Find the initial velocity.
   (b) Find the times when the particle will be at the origin.
   (c) Find the acceleration after 3 s.

13. The displacement of a particle moving in simple harmonic motion is given by \( x = 3\sin 2t \), where \( x \) is in centimetres and \( t \) is in seconds.
   (a) Sketch the graph of the displacement of the particle from \( t = 0 \) to \( t = \frac{2\pi}{3} \).
   (b) Find equations for the velocity \( \dot{x} \) and acceleration \( \ddot{x} \).
   (c) Find the exact acceleration after \( \frac{\pi}{3} \) s.
   (d) Show that \( \ddot{x} = -4x \).

14. The height of a projectile is given by \( h = 7 + 6t - t^2 \), where height is in metres and time is in seconds.
   (a) Find the initial height.
   (b) Find the maximum height reached.
   (c) When will the projectile reach the ground?
   (d) Sketch the graph showing the height of the projectile over time \( t \).
   (e) How far will the projectile travel in the first 4 s?

15. A ball is rolled up a slope at a distance from the base of the slope, after time \( t \) seconds, given by \( x = 15t - 3t^2 \) metres.
   (a) How far up the slope will the ball roll before it starts to roll back down?
   (b) What will its velocity be when it reaches the base of the slope?
   (c) How long will the motion of the ball take altogether?

16. The displacement of a particle is given by \( x = 2t^3 - 3t^2 + 42t \).
   (a) Show that the particle is initially at the origin but never returns to the origin.
   (b) Show that the particle is never at rest.

17. A weight on the end of a spring has a height given by \( h = 3\sin 2t \) cm where \( t \) is time in seconds.
   (a) Find its initial height.
   (b) Find when the spring is at its maximum distance from the origin.
   (c) What is the acceleration at these times?
18. A particle moves so that its displacement \( x \) cm is given by \( x = e^t \) at \( t \) seconds.
   (a) Find the exact velocity of the particle after 4 seconds.
   (b) When is the particle at rest?
   (c) Where is the particle at that time?

19. A particle is moving in a straight line so that its displacement \( x \) cm over time \( t \) seconds is given by \( x = t\sqrt{49 - t^2} \).
   (a) For how many seconds does the particle travel?
   (b) Find the exact time at which the particle comes to rest.
   (c) How far does the particle move altogether?

Motion and Integration

If the first derivative of the displacement of a particle gives its velocity, then the reverse is true. That is, the primitive function (indefinite integral) of the velocity gives the displacement. Similarly, the integral of the acceleration gives the velocity.

\[
x = \int v \, dt
\]
\[
v = \int a \, dt
\]

EXAMPLES

1. The velocity of a particle is given by \( v = 3t^2 + 2t + 1 \). If initially the particle is 2 cm to the left of the origin, find the displacement after 5 s.

Solution

\[
x = \int v \, dt
\]
\[
= \int (3t^2 + 2t + 1) \, dt
\]
\[
= t^3 + t^2 + t + C
\]

When \( t = 0 \), \( x = -2 \)

\[
\therefore -2 = 0^3 + 0^2 + 0 + C
\]
\[
= C
\]

\[
\therefore x = t^3 + t^2 + t - 2
\]

When \( t = 5 \),

\[
x = 5^3 + 5^2 + 5 - 2
\]
\[
= 125 + 25 + 5 - 2
\]
\[
= 153
\]

So after 5 s the particle will be 153 cm to the right of the origin.
2. The acceleration of a particle is given by \( a = 6 - \frac{2}{(t + 1)^2} \) ms\(^{-2}\). If the particle is initially at rest 1 m to the right of the origin, find its exact displacement after 9 s.

**Solution**

\[
v = \int a \, dt
\]

\[
= \int \left( 6 - \frac{2}{(t + 1)^2} \right) dt
\]

\[
= \int \left[ 6 - 2(t + 1)^{-2} \right] dt
\]

\[
= 6t - 2(t + 1)^{-1} + C
\]

\[
= 6t + \frac{2}{t+1} + C
\]

When \( t = 0 \), \( v = 0 \)

\[
\therefore \quad 0 = 6(0) + \frac{2}{0+1} + C
\]

\[
= 0 + 2 + C
\]

\[
\therefore \quad C = -2
\]

So \( v = 6t + \frac{2}{t+1} - 2 \)

\[
x = \int v \, dt
\]

\[
= \int \left( 6t + \frac{2}{t+1} - 2 \right) dt
\]

\[
= 3t^2 + 2 \log_e (t + 1) - 2t + C
\]

When \( t = 0 \), \( x = 1 \)

\[
\therefore \quad 1 = 3(0)^2 + 2 \log_e (0 + 1) - 2(0) + C
\]

\[
= 0 + 2\log_e 1 - 0 + C
\]

\[
\therefore \quad x = 3t^2 + 2 \log_e (t + 1) - 2t + 1
\]

When \( t = 9 \)

\[
x = 3(9)^2 + 2 \log_e (9 + 1) - 2(9) + 1
\]

\[
= 243 + 2 \log_e 10 - 18 + 1
\]

\[
= 226 + 2 \log_e 10
\]

\[
= 2(113 + \log_e 10)
\]

So after 9 s the displacement will be \( 2(113 + \log_e 10) \) m.
6.5 Exercises

1. The velocity of a particle is given by \( v = 3t^2 - 5 \text{ cm s}^{-1} \). If the particle is at the origin initially, find its displacement after 3 s.

2. A particle has velocity given by \( \frac{dx}{dt} = 2t - 3 \text{ ms}^{-1} \). After 3 s the particle is at the origin. Find its displacement after 7 s.

3. The velocity of a particle is given by \( v = 4 - 3t \text{ cm s}^{-1} \). If the particle is 5 cm from the origin after 2 s, find the displacement after 7 s.

4. The velocity of a particle is given by \( v = 8t^3 - 3t^2 \text{ cm s}^{-1} \). If the particle is initially at the origin, find (a) its acceleration after 5 s (b) its displacement after 3 s (c) when it will be at the origin again.

5. The velocity of a particle is given by \( \dot{x} = 3e^t \text{ cm s}^{-1} \). Given that the particle is initially 2 cm to the right of the origin, find its exact displacement after 1 s.

6. A particle has a constant acceleration of 12 ms\(^{-2}\). If the particle has a velocity of 2 ms\(^{-1}\) and is 3 m from the origin after 5 s, find its displacement after 10 s.

7. The acceleration of an object is given by \( 6t + 4 \text{ cm s}^{-2} \). The particle is initially at rest at the origin. Find (a) its velocity after 5 s (b) its displacement after 5 s.

8. A projectile is accelerating at a constant rate of \(-9.8 \text{ ms}^{-2}\). If it is initially 2 m high and has a velocity of \(4 \text{ ms}^{-1}\), find the equation of the height of the projectile.

9. A particle is accelerating according to the equation \( a = (3t + 1)^2 \text{ ms}^{-2} \). If the particle is initially at rest 2 m to the left of 0, find its displacement after 4 s.

10. The velocity of a particle is given by \( v = e^t - 1 \text{ ms}^{-1} \). If the particle is initially 3 m to the right of the origin, find its exact position after 5 s.

11. A particle accelerates according to the equation \( a = -e^t \text{ ms}^{-2} \). If the particle has initial velocity of zero \( \text{cm s}^{-1} \) and initial displacement of \(-1 \text{ cm}\), find its displacement after 4 s, to the nearest centimetre.

12. The acceleration of a particle is given by \( a = -9 \sin 3t \text{ cm s}^{-2} \). If the initial velocity is 5 \( \text{cm s}^{-1} \) and the particle is 3 cm to the left of the origin, find the exact displacement after \( \pi \text{ s} \).

13. The velocity of a particle is given by \( \dot{x} = \frac{t}{t^2 + 3} \text{ ms}^{-1} \). If the particle is initially at the origin, find its displacement after 10 s, correct to 2 decimal places.

14. The acceleration of an object is given by \( a = e^t \text{ ms}^{-2} \). If initially the object is at the origin with velocity \(-2 \text{ ms}^{-1}\), find its displacement after 3 seconds, correct to 3 significant figures.
15. The velocity of a particle is given by \( v = 4 \cos 2t \text{ ms}^{-1} \). If the particle is 3 m to the right of the origin after \( \pi \) s, find the exact (a) displacement after \( \frac{\pi}{6} \) s. (b) acceleration after \( \frac{\pi}{6} \) s.

16. The acceleration of a particle is given by \( a = \frac{2}{(t + 3)^2} \text{ ms}^{-2} \) and the particle is initially at rest 2 ln 3 m to the left of the origin. (a) Evaluate its velocity after 2 seconds. (b) Find its displacement after \( n \) seconds. (c) Show that \( 0 \leq v < 1 \) for all \( t \).

17. The acceleration of a particle is \( \ddot{x} = 25 e^{t} \text{ ms}^{-2} \) and its velocity is 5 \text{ ms}^{-1} initially. Its displacement is initially 1 m. (a) Find its velocity after 9 seconds. (b) Find its displacement after 6 seconds. (c) Show that the acceleration \( \ddot{x} = 25x \). (d) Find the acceleration when the particle is 2 m to the right of the origin.
Test Yourself 6

1. A particle moves so that its displacement after \( t \) seconds is \( x = 4t^2 - 5t^3 \) metres. Find (a) its initial displacement, velocity and acceleration, (b) when the particle is at the origin, and (c) its maximum displacement.

2. The velocity of a particle is given by \( V = 6t - 12t^2 \) \( \text{ms}^{-1} \). Find its displacement and acceleration after 3 seconds if it is initially 5 m to the right of the origin.

3. A city doubles its population in 25 years. If it is growing exponentially, when will it triple its population?

4. The displacement of a particle after \( t \) seconds is \( x = 2e^{3t} \) cm. Find (a) its initial velocity, (b) its acceleration after 3 seconds, and (c) show that \( \ddot{x} = 9x \).

5. A particle accelerates at \( \ddot{x} = 8 \cos 2t \) \( \text{ms}^{-2} \). The particle is initially at rest at the origin. Find its displacement after \( \frac{\pi}{6} \) seconds.

6. If a particle has displacement \( x = 2 \sin 3t \), show that its acceleration is given by \( \ddot{x} = -9x \).

7. A particle has displacement \( x = t^4 - 12t^2 + 36t - 9 \) cm at time \( t \) seconds. Find (a) when the particle is at rest, (b) what is the (i) displacement, (ii) velocity, and (iii) acceleration after 1 s, and (c) describe the motion of the particle after 1 s.

8. The acceleration of a particle is \( \ddot{x} = 6t - 12 \) \( \text{ms}^{-2} \). The particle is initially at rest 3 metres to the left of the origin. Find (a) its acceleration, (b) its velocity, and (c) its maximum displacement after 5 seconds.

9. The graph below shows the displacement of a particle.

(a) When is the particle at the origin? (b) When is it at rest? (c) When is it travelling at its greatest speed? (d) Sketch the graph of its (i) velocity and (ii) acceleration.

10. A radioactive substance decays by 10% after 80 years. Find (a) by how much will it decay after 500 years, (b) when will it decay to a quarter of its mass?

11. The equation for displacement of a particle over time \( t \) seconds is \( x = \sin t \) metres. Find (a) the displacement of 0.5 m, (b) when is the velocity 0.5 \( \text{ms}^{-1} \)? (c) find the acceleration after \( \frac{\pi}{4} \) s.
12. The height of a ball is \( h = 20t - 5t^2 \) metres after \( t \) seconds.
   (a) Find the height after 1 s.
   (b) What is the maximum height of the ball?
   (c) What is the time of flight of the ball?

13. A bird population of 8500 increases to 12 000 after 5 years. Find
   (a) the population after 10 years
   (b) the rate at which the population is increasing after 10 years
   (c) when the population reaches 30 000.

14. The graph below shows the velocity of a particle.

   (a) Sketch a graph that shows
      (i) displacement
      (ii) acceleration
   (b) When is the particle at rest?

15. A particle is moving with acceleration
   \( a = 10 \cos 2t \) ms\(^{-2}\). If it is initially 4 m to the right of the origin and has a velocity of \(-2\) ms\(^{-1}\), find its displacement after 5 seconds, to the nearest metre.

---

**Challenge Exercise 6**

1. (a) Find an equation for the displacement of a particle from the origin if its acceleration is given by
   \[ \frac{dv}{dt} = 6 \cos 3t \text{ cm/s}^2 \]
   and initially the particle is at rest 2 cm to the right of the origin.
   (b) Write the acceleration of the particle in terms of \( x \).

2. The displacement of a particle is given by
   \( x = (t^3 + 1)^6 \), where \( x \) is in metres and \( t \) is in seconds.
   (a) Find its initial displacement and velocity.
   (b) Find its acceleration after 2 s in scientific notation, correct to 3 significant figures.
   (c) Show that the particle is never at the origin.

3. (a) Find an equation for the displacement of a particle from the origin if its acceleration is given by
   \[ \frac{dv}{dt} = -16 \cos 4t \text{ cm/s}^2 \]
   and initially the particle is at rest 1 cm to the right of the origin.
   (b) Find the exact velocity of the particle when it is 0.5 cm from the origin.

4. The velocity, in ms\(^{-1}\), of an object is given by
   \[ v = 5 \cos 5t \]
   (a) Show that if the object is initially at the origin, its acceleration will always be \(-25\) times its displacement.
   (b) Find the maximum acceleration of the object.
   (c) Find the acceleration when the object is 0.3 m to the right of the origin.
5. The population of a flock of birds over \( t \) years is given by the formula 
\[ P = P_0 e^{0.015t} \].
(a) How long will it take, correct to 1 decimal place, to increase the population by 35%?
(b) What will be the percentage increase in population after 10 years, to the nearest per cent?

6. The Logistic Law of Population Growth, first proposed by Verhulst in 1837, is given by 
\[ \frac{dN}{dt} = kN - bN^2 \], where \( k \) and \( b \) are constants. Show that the equation 
\[ N = \frac{kN_0}{bN_0 + (k - bN_0)e^{-kt}} \] is a solution of this differential equation (\( N_0 \) is a constant).

7. A particle moves so that its velocity is given by 
\[ \dot{x} = te^t \text{ cm s}^{-1} \].
Find its exact acceleration after 1 s.
**TERMINOLOGY**

- **Annuity**: An annuity is a fixed sum of money invested every year that accumulates interest over a number of years.

- **Arithmetic sequence**: A set of numbers that form a pattern where each successive term is a constant amount (positive or negative) more than the previous one.

- **Common difference**: The constant amount in an arithmetic sequence that is added to each term to get the next term.

- **Common ratio**: The constant amount in a geometric sequence that is multiplied to each term to get the next term.

- **Compound interest**: Interest is added to the balance of a bank account so that the interest and balance both earn interest.

- **Geometric sequence**: A set of numbers that form a pattern where each successive term is multiplied by a constant amount to get the next term.

- **Limiting sum**: (or sum to infinity) The sum of infinite terms of a geometric series where the common ratio $r$ obeys the condition $-1 < r < 1$.

- **Partial sum**: The sum of a certain finite number of terms of a sequence.

- **Sigma notation**: The Greek letter stands for the sum of a sequence of a certain number of terms, called a partial sum.

- **Sequence**: A set of numbers that form a pattern or obey a fixed rule.

- **Series**: The sum of a sequence of $n$ terms.

- **Superannuation**: A sum of money that is invested every year (or more frequently) as part of a salary to provide a large amount of money when a person retires from the paid workforce.

- **Term**: A term refers to the position of the number in a sequence. For example, the first term is the first number in the sequence.
INTRODUCTION

THE INFINITE SUM OF a sequence of numbers (or terms) is called a series. Many series occur in real life—think of the way plants grow, or the way money accumulates as a certain amount earns interest in a bank.

You will look at series in general, arithmetic, geometric series and their applications.

General Series

A sequence forms a pattern. Some patterns are easy to see and some are difficult. People are sometimes asked to identify sequences and their patterns in tests of intelligence.

DID YOU KNOW?

Number patterns and series of numbers have been known since the very beginning of civilisation. The Rhind Papyrus, an Egyptian book written about 1650 BC, is one of the oldest books on mathematics. It was discovered and deciphered by Eisenlohr in 1877, and it showed that the ancient Egyptians had a vast knowledge of mathematics. The Rhind Papyrus describes the problem of dividing 100 loaves among 5 people in a way that shows the Egyptians were exploring arithmetic series.

Around 500 BC the Pythagoreans explored different polygonal numbers:

- **triangular numbers**: 1 + 2 + 3 + 4 + …

  ![Triangular numbers diagram]

  1 1 + 2 1 + 2 + 3 1 + 2 + 3 + 4

- **square numbers**: 1 + 3 + 5 + 7 + …

  ![Square numbers diagram]

  1 1 + 3 1 + 3 + 5 1 + 3 + 5 + 7

Could you find a series for pentagonal or hexagonal numbers?
7.1 Exercises

Find the next 3 terms in each sequence or series of numbers.

1. 5, 8, 11, ...
2. 8, 13, 18, ...
3. 11 + 22 + 33 + ...
4. 100, 95, 90, ...
5. 7 + 5 + 3 + ...
6. 99, 95, 91, ...
7. \(\frac{1}{2}, 1, \frac{3}{2}, \ldots\)
8. 1.3 + 1.9 + 2.5 + ...
9. 2, 4, 8, ...
10. 4 + 12 + 36 + ...

11. 1, –2, 4, –8, ...
12. 3, –6, 12, –24, ...
13. \(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\)
14. \(\frac{2}{5}, \frac{4}{15}, \frac{8}{45}, \ldots\)
15. 1 + 4 + 9 + 16 + 25 + ...
16. 1, 8, 27, 64, ...
17. 0, 3, 8, 15, 24, ...
18. 3 + 6 + 11 + 18 + 27 + ...
19. 2 + 9 + 28 + 65 + ...
20. 1, 1, 2, 3, 5, 8, 13, ...

DID YOU KNOW?

The numbers 1, 1, 2, 3, 5, 8, … are called Fibonacci numbers after Leonardo Fibonacci (1170–1250). The Fibonacci numbers occur in natural situations.

For example, when new leaves grow on a plant’s stem, they spiral around the stem. The ratio of the number of turns to the number of spaces between successive leaves gives the series of fractions \(1 \frac{1}{2}, 2 \frac{1}{2}, 3 \frac{1}{3}, 5 \frac{1}{5}, 8 \frac{1}{8}, 13 \frac{1}{13}, 21 \frac{1}{21}, 34 \frac{1}{34}, 55 \frac{1}{55}, 89 \frac{1}{89}, 144 \frac{1}{144}, \ldots\)
The Fibonacci ratio is the number of turns divided by the number of spaces.

Research Fibonacci numbers and find out where else they appear in nature.

**Formula for the \( n \)th term of a series**

When a series follows a mathematical pattern, its terms can be described by a formula. We call any general or \( n \)th term of a series \( T_n \) where \( n \) stands for the number of the term and must be a positive integer.

**EXAMPLE**

For the series 6 + 13 + 20 + ... find

(a) \( T_1 \)
(b) \( T_2 \)
(c) \( T_3 \)
(d) \( T_n \)

**Solution**

(a) The first term is 6, so when \( n = 1 \): \( T_1 = 6 \)
(b) The 2nd term is 13 so \( T_2 = 13 \)
(c) The 3rd term is 20 so \( T_3 = 20 \)
(d) Each term is 7 more than the previous term.
   The 1st term is 6
   The 2nd term is 13 = 6 + 7
   The 3rd term is 20 = 6 + 7 + 7
   = 6 + 2 \times 7

**CONTINUED**
Following this pattern:
The 4th term = 6 + 7 + 7 + 7
       = 6 + 3 \times 7
and so on
\[ T_n = 6 + (n - 1) \times 7 \]
\[ T_n = 6 + 7n - 7 \]
\[ T_n = 7n - 1 \]

If we are given a formula for the \( n \)th term, we can find out much more about the series.

**EXAMPLES**

1. The \( n \)th term of a series is given by the formula \( T_n = 3n + 4 \). Find the first 3 terms and hence write down the series.

**Solution**

\[ T_1 = 3(1) + 4 \]
\[ = 7 \]
\[ T_2 = 3(2) + 4 \]
\[ = 10 \]
\[ T_3 = 3(3) + 4 \]
\[ = 13 \]
So the series is \( 7 + 10 + 13 + \ldots \)

2. The \( n \)th term of a series is given by \( t_n = 5n - 1 \). Which term of the series is equal to 104?

**Solution**

\[ t_n = 5n - 1 = 104 \]
\[ 5n = 105 \]
\[ n = 21 \]
So 104 is the 21st term of the series.

3. The \( n \)th term of a series is given by \( u_n = 103 - 3n \).
   (a) Find the first 3 terms.
   (b) Find the value of \( n \) for the first negative term in the series.
Chapter 7

Series

Solution

(a) \( u_1 = 103 - 3(1) \)
    \[= 100\]
(b) \( u_2 = 103 - 3(2) \)
    \[= 97\]
(c) \( u_3 = 103 - 3(3) \)
    \[= 94\]

So the series is 100 \(+\) 97 \(+\) 94 \(+\) \ldots

(b) For the first negative term, we want

\[u_n < 0\]

i.e.

\[103 - 3n < 0\]
\[-3n < -103\]
\[n > 34 \frac{1}{3}\]

Since \(n\) is an integer, \(n = 35\)

\[\therefore\] the 35th term is the first negative term.

4. The \(n\)th term of a series is given by the formula \(t_n = 2^n - 1\). Which term of the series is equal to 4095?

Solution

\[t_n = 2^n - 1\]

We are given that the \(n\)th term is 4095 and need to find \(n\).

\[4095 = 2^n - 1\]

\[4096 = 2^n\]

\[\log_{10} 4096 = \log_{10} 2^n\]

\[= n \log_{10} 2\]

\[\frac{\log_{10} 4096}{\log_{10} 2} = n\]

\[12 = n\]

So 4096 is the 12th term of the sequence.

7.2 Exercises

1. Find the first 3 terms of the series with \(n\)th term as follows

(a) \(T_n = 8n - 5\)
(b) \(T_n = 2n + 3\)
(c) \(u_n = 6n - 1\)
(d) \(T_n = 8 - 5n\)
(e) \(t_n = 20 - n\)

(f) \(u_n = 3^n\)
(g) \(Q_n = 2^n + 7\)
(h) \(t_n = 4^n - 2n\)
(i) \(T_n = 8n^2 - n + 1\)
(j) \(T_n = n^3 + n\)

You could use base e instead of base 10.

You could use trial and error to guess what power of 2 is equal to 4096 if you don’t want to use logarithms.
2. Find the first 3 terms of each series.
   (a) \( T_n = 3n - 2 \)
   (b) \( t_n = 4^n \)
   (c) \( T_n = n^2 + n \)

3. Find the 50th term of each series.
   (a) \( T_n = 7n - 1 \)
   (b) \( t_n = 2n + 5 \)
   (c) \( T_n = 5n - 2 \)
   (d) \( T_n = 40 - 3n \)
   (e) \( U_n = 8 - 7n \)

4. Find the 10th term of each series.
   (a) \( T_n = 2^n + 5 \)
   (b) \( t_n = 10 - 3^n \)
   (c) \( T_n = 2^n - n \)
   (d) \( u_n = n^2 - 5n + 3 \)
   (e) \( T_n = n^3 + 2 \)

5. Which of these are terms of the series \( T_n = 3n + 5 \)?
   (a) 68
   (b) 158
   (c) 205
   (d) 266
   (e) 300

6. Which of these are terms of the series \( T_n = 5^n - 1 \)?
   (a) 3126
   (b) 124
   (c) 15 634
   (d) 78 124
   (e) 0

7. Which term of the series with \( n \)th term \( t_n = 9n - 15 \) is equal to 129?

8. Is 255 a term of the series with \( n \)th term \( T_n = 2^n - 1 \)?

9. Which term of the series with \( n \)th term \( u_n = n^3 + 5 \) is 348?

10. Which term of \( T_n = n^2 - 3 \) is equal to 526?

11. For the series \( T_n = n^3 \), find
   (a) the 12th term
   (b) \( n \) if the \( n \)th term is 15 625.

12. A series is given by the formula \( T_n = 3 + 2n - n^2 \).
   (a) Find the 25th term.
   (b) Find \( n \) if the \( n \)th term is −252.

13. Find the value of \( n \) that gives the first term of the series \( T_n = 3n + 2 \) greater than 100.

14. Find the value of \( n \) that gives the first term of the series \( T_n = 2^n \) larger than 500.

15. Find the values of \( n \) where the series \( T_n = 4n - 1 \) is greater than 350.

16. Find the values of \( n \) for which the series \( T_n = 400 - 5n \) is less than 200.

17. Find the value of \( n \) that gives the first negative term of the series \( T_n = 1000 - 2n \).

18. Find the value of \( n \) that gives the first positive term of the series \( T_n = 2n - 300 \).

19. Find
   (a) the value of \( n \) that gives the first negative term of the series \( u_n = 80 - 6n \)
   (b) the first negative term.

20. Find the first negative term of the series \( T_n = 50 - 7n \).
Sigma Notation

Series are often written in **sigma notation**. The Greek letter sigma (Σ) is like an English ‘S’. Here it stands for the sum of a series.

**Application**

The mean of a set of scores is given by 

\[ x = \frac{\Sigma fx}{\Sigma f} \]

where \( \Sigma f \) is the sum of frequencies and \( \Sigma fx \) is the sum of the scores \( \times \) frequencies.

**EXAMPLES**

1. Evaluate \( \sum_{r=1}^{5} r^2 \)

   **Solution**

   \( \sum_{r=1}^{5} r^2 \) means the sum of terms where the formula is \( r^2 \) with \( r \) starting at 1 and ending at 5.

   So \( \sum_{r=1}^{5} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \)

   \[ = 1 + 4 + 9 + 16 + 25 \]

   \[ = 55 \]

2. Evaluate \( \sum_{n=3}^{7} (2n + 5) \)

   **Solution**

   \( \sum_{n=3}^{7} (2n + 5) \) means the sum of terms where the formula is \( 2n + 5 \) with \( n \) starting at 3 and ending at 7.

   Can you work out how many terms there are in this series?

   \( \sum_{n=3}^{7} (2n + 5) = (2 \times 3 + 5) + (2 \times 4 + 5) + (2 \times 5 + 5) + (2 \times 6 + 5) + (2 \times 7 + 5) \)

   \[ = 11 + 13 + 15 + 17 + 19 \]

   \[ = 75 \]
3. Write $7 + 11 + 15 + \ldots + (4k + 3)$ in sigma notation.

**Solution**

In this question, the formula is $4n + 3$ where the last term is at $n = k$.
We need to find the number of terms.
The first term is 7 so we guess that $n = 1$ for this term.
When $n = 1$

$$4n + 3 = 4 \times 1 + 3$$

$$= 7$$

So the first term is at $n = 1$ and the last term is at $n = k$.
We can write the series as $\sum_{1}^{k} (4n + 3)$.

4. How many terms are there in the series

(a) $\sum_{1}^{100} (3n - 7)$?

(b) $\sum_{5}^{80} 2^n$?

**Solution**

(a) The first term is when $n = 1$ and the last term is when $n = 100$.
There are 100 terms.

(b) The first term is when $n = 5$ and the last term is when $n = 50$.
This is harder to work out. Look at example 2 where there are 5 terms.
We can work out the number of terms by $50 - 5 + 1$.

There are 46 terms.

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7.3 **Exercises**

1. Evaluate

(a) $\sum_{n=1}^{4} (3^n + 2)$

(b) $\sum_{n=2}^{6} n^2$

(c) $\sum_{2}^{6} (5n - 6)$

(d) $\sum_{r=1}^{10} (3r + 1)$

(e) $\sum_{x=2}^{5} (k^3 - 1)$

(f) $\sum_{3}^{5} \frac{1}{n}$

(g) $\sum_{n=1}^{5} (n^2 - n)$

(h) $\sum_{2}^{6} |3p - 2|$

(i) $\sum_{1}^{6} (2^n)$

(j) $\sum_{3}^{6} (3^n - 2n - 5)$
2. How many terms are there in each series?
   (a) \( \sum_{n=1}^{48} (7^n) \)
   (b) \( \sum_{n=2}^{100} (n^2 + 1) \)
   (c) \( \sum_{n=3}^{80} (3n + 4) \)
   (d) \( \sum_{r=1}^{200} (3r^2) \)
   (e) \( \sum_{n=7}^{20} (7n + 1) \)
   (f) \( \sum_{n=7}^{45} (n^2 + 3n) \)
   (g) \( \sum_{n=12}^{108} n^3 \)
   (h) \( \sum_{n=9}^{34} 4^n \)

(i) \( \sum_{n=1}^{72} (2n - 3) \)
(j) \( \sum_{n=11}^{55} (7^n + n) \)

3. Write these series in sigma notation.
   (a) \( 1 + 3 + 5 + 7 + \ldots + 11 \)
   (b) \( 7 + 14 + 21 + \ldots + 70 \)
   (c) \( 1 + 8 + 27 + 64 + 125 \)
   (d) \( 2 + 8 + 14 + \ldots + (6n - 4) \)
   (e) \( 9 + 16 + 25 + 36 + \ldots + n^2 \)
   (f) \( -1 - 2 - 3 - \ldots - 50 \)
   (g) \( 3 + 6 + 12 + \ldots + 3 \times 2^n \)
   (h) \( 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{512} \)
   (i) \( a + (a + d) + (a + 2d) + \ldots + (a + [n - 1]d) \)
   (j) \( a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \)

**Arithmetic Series**

In an arithmetic series each term is a **constant amount more** than the previous term. The constant is called the **common difference**.

**EXAMPLES**

1. Find the common difference of the series \( 5 + 9 + 13 + 17 + 21 + \ldots \)

**Solution**

We can see that the common difference between the terms is 4. To check this, we notice that \( 9 - 5 = 4, 13 - 9 = 4, 17 - 13 = 4 \) and \( 21 - 17 = 4 \).

2. Find the common difference of the series \( 85 + 80 + 75 + 70 + 65 + \ldots \)

**Solution**

We can see that the common difference between the terms is \(-5\). To check this, we notice that \( 80 - 85 = -5, 75 - 80 = -5, 70 - 75 = -5 \) and \( 65 - 70 = -5 \).
If $T_1$, $T_2$ and $T_3$ are consecutive terms of an arithmetic series then
\[ d = T_2 - T_1 = T_3 - T_2 \]

Generally, if $T_1, T_2, T_3, \ldots, T_{n-1}, T_n$ are consecutive terms of an arithmetic series then $d = T_2 - T_1 = T_3 - T_2 = \ldots = T_n - T_{n-1}$.

**EXAMPLES**

1. If $5 + x + 31 + \ldots$ is an arithmetic series, find $x$.

**Solution**

For an arithmetic series,
\[ T_2 - T_1 = T_3 - T_2 \]
i.e. \[ x - 5 = 31 - x \]
\[ 2x - 5 = 31 \]
\[ 2x = 36 \]
\[ x = 18 \]

2. (a) Evaluate $k$ if $(k + 2) + (3k + 2) + (6k - 1) + \ldots$ is an arithmetic series.
   (b) Write down the first 3 terms of the series.
   (c) Find the common difference $d$.

**Solution**

(a) For an arithmetic series, $T_2 - T_1 = T_3 - T_2$
   So \[ (3k + 2) - (k + 2) = (6k - 1) - (3k + 2) \]
   \[ 3k + 2 - k - 2 = 6k - 1 - 3k - 2 \]
   \[ 2k = 3k - 3 \]
   \[ 0 = k - 3 \]
   \[ 3 = k \]

(b) The series is $(k + 2) + (3k + 2) + (6k - 1) + \ldots$
   Substituting $k = 3$:
   \[ T_1 = k + 2 \]
   \[ = 3 + 2 \]
   \[ = 5 \]
   \[ T_2 = 3k + 2 \]
   \[ = 3 \times 3 + 2 \]
   \[ = 11 \]
   \[ T_3 = 6k - 1 \]
   \[ = 6 \times 3 - 1 \]
   \[ = 17 \]

(c) The sequence is $5 + 11 + 17 + \ldots$
   \[ 11 - 5 = 17 - 11 = 6 \]
   So common difference $d = 6$. 

\[ \text{Notice that } x = \frac{5 + 31}{2} \text{ which is the average of } T_1 \text{ and } T_2. \text{ We call } x \text{ the arithmetic mean.} \]
Terms of an arithmetic series

\[a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 1)d] + \ldots\] is an arithmetic series with first term \(a\), common difference \(d\) and \(n\)th term given by

\[T_n = a + (n - 1)d\]

Proof

Let the first term of an arithmetic series be \(a\) and the common difference \(d\).

Then first term is \(a\)
- second term is \(a + d\)
- third term is \(a + 2d\)
- fourth term is \(a + 3d\), and so on.

The \(n\)th term is \(a + (n - 1)d\).

**EXAMPLES**

1. Find the 20th term of the series \(3 + 10 + 17 + \ldots\).

**Solution**

\(a = 3,\ d = 7,\ n = 20\)

\[T_n = a + (n - 1)d\]

\[T_{20} = 3 + (20 - 1)7\]

\[= 3 + 19 \times 7\]

\[= 136\]

2. Find an expression for the \(n\)th term of the series \(2 + 8 + 14 + \ldots\).

**Solution**

\(a = 2,\ d = 6\)

\[T_n = a + (n - 1)d\]

\[= 2 + (n - 1)6\]

\[= 2 + 6n - 6\]

\[= 6n - 4\]

3. Find the first positive term of the series \(-50 - 47 - 44 - \ldots\).

**Solution**

\(a = -50,\ d = 3\)

For the first positive term, \(T_n > 0\)
i.e. \( a + (n - 1)d > 0 \)
\(-50 + (n - 1)\ 3 > 0\)
\(-50 + 3n - 3 > 0\)
\(3n - 53 > 0\)
\(3n > 53\)
\(n > 17\frac{2}{3}\)
\(\therefore n = 18\) gives the first positive term
\(T_{18} = -50 + (18 - 1)\ 3\)
\(= -50 + 17 \times 3\)
\(= 1\)
So the first positive term is 1.

4. The 5th term of an arithmetic series is 37 and the 8th term is 55. Find the common difference and the first term of the series.

Solution

\[ T_n = a + (n - 1)d \]
\[ T_5 = a + (5 - 1)d = 37 \]
\[ \text{i.e. } a + 4d = 37 \] (1)

\[ T_8 = a + (8 - 1)d = 55 \]
\[ \text{i.e. } a + 7d = 55 \] (2)

(2) - (1):
\[ a + 4d = 37 \]
\[ 3d = 18 \]
\[ d = 6 \]

Put \( d = 6 \) in (1):
\[ a + 4(6) = 37 \]
\[ a + 24 = 37 \]
\[ a = 13 \]

So \( d = 6 \) and \( a = 13 \).

7.4 Exercises

1. The following series are arithmetic. Evaluate all pronumerals.

(a) \( 5 + 9 + y + ... \)
(b) \( 8 + 2 + x + ... \)
(c) \( 45 + x + 99 + ... \)
(d) \( 16 + b + 6 + ... \)
(e) \( x + 14 + 21 + ... \)

(f) \( 32 + (x - 1) + 51 + ... \)
(g) \( 3 + (2k + 3) + 21 + ... \)
(h) \( x + (x + 3) + (2x + 5) + ... \)
(i) \( (t - 5) + 3t + (3t + 1) + ... \)
(j) \( (2t - 3) + (3t + 1) + (5t + 2) + ... \)
2. Find the 15th term of each series.
   (a) \[4 + 7 + 10 + \ldots\]
   (b) \[8 + 13 + 18 + \ldots\]
   (c) \[10 + 16 + 22 + \ldots\]
   (d) \[120 + 111 + 102 + \ldots\]
   (e) \[-3 + 2 + 7 + \ldots\]
   
3. Find the 100th term of each series.
   (a) \[-4 + 2 + 8 + \ldots\]
   (b) \[41 + 32 + 23 + \ldots\]
   (c) \[18 + 22 + 26 + \ldots\]
   (d) \[125 + 140 + 155 + \ldots\]
   (e) \[-1 + 5 + 9 + \ldots\]
   
4. What is the 25th term of each series?
   (a) \[-14 - 18 - 22 - \ldots\]
   (b) \[0.4 + 0.9 + 1.4 + \ldots\]
   (c) \[1.3 + 0.9 + 0.5 + \ldots\]
   (d) \[1 + 2\frac{1}{2} + 4 + \ldots\]
   (e) \[1\frac{2}{5} + 2 + 2\frac{3}{5} + \ldots\]
   
5. For the series \[3 + 5 + 7 + \ldots\], write an expression for the \(n\)th term.
   
6. Write an expression for the \(n\)th term of the following series.
   (a) \[9 + 17 + 25 + \ldots\]
   (b) \[100 + 102 + 104 + \ldots\]
   (c) \[6 + 9 + 12 + \ldots\]
   (d) \[80 + 86 + 92 + \ldots\]
   (e) \[-21 - 17 - 13 - \ldots\]
   (f) \[15 + 10 + 5 + \ldots\]
   (g) \[\frac{7}{8} + 1 + 1\frac{1}{8} + \ldots\]
   (h) \[-30 - 32 - 34 - \ldots\]
   (i) \[3.2 + 4.4 + 5.6 + \ldots\]
   (j) \[\frac{1}{2} + 1\frac{1}{4} + 2 + \ldots\]
   
7. Find which term of the series \[3 + 7 + 11 + \ldots\] is equal to 111.
8. Which term of the series \[1 + 5 + 9 + \ldots\] is 213?
9. Which term of the series \[15 + 24 + 33 + \ldots\] is 276?
10. Which term of the series \[25 + 18 + 11 + \ldots\] is equal to \(-73\)?
11. Is zero a term of the series \[48 + 45 + 42 + \ldots\]?
12. Is 270 a term of the series \[3 + 11 + 19 + \ldots\]?
13. Is 405 a term of the series \[0 + 3 + 6 + \ldots\]?
14. Find the first value of \(n\) for which the terms of the series \[100 + 93 + 86 + \ldots\] become less than 20.
15. Find the values of \(n\) for which the terms of the series \[-86 - 83 - 80 - \ldots\] are positive.
16. Find the first negative term of \[54 + 50 + 46 + \ldots\].
17. Find the first term that is greater than 100 in the series \[3 + 7 + 11 + \ldots\].
18. The first term of an arithmetic series is \(-7\) and the common difference is 8. Find the 100th term of the series.
19. The first term of an arithmetic series is 15 and the 3rd term is 31. (a) Find the common difference (b) Find the 10th term of the series.
20. The first term of an arithmetic series is 3 and the 5th term is 39. Find its common difference.
21. The 2nd term of an arithmetic series is 19 and the 7th term is 54. Find its first term and common difference.
22. Find the 20th term in an arithmetic series with 4th term 29 and 10th term 83.

23. The common difference of an arithmetic series is 6 and the 5th term is 29. Find the first term of the series.

24. If the 3rd term of an arithmetic series is 45 and the 9th term is 75, find the 50th term of the series.

25. The 7th term of an arithmetic series is 17 and the 10th term is 53. Find the 100th term of the series.

26. (a) Show that \( \log_3 x + \log_3 x^2 + \log_3 x^3 + \ldots \) is an arithmetic series.
   (b) Find the 80th term.
   (c) If \( x = 4 \), evaluate the 10th term correct to 1 decimal place.

27. (a) Show that \( \sqrt{3} + \sqrt{12} + \sqrt{27} + \ldots \) is an arithmetic series.
   (b) Find the 50th term in simplest form.

28. Find the 25th term of \( \log_2 4 + \log_2 8 + \log_2 16 + \ldots \)

29. Find the 40th term of \( 5b + 8b + 11b + \ldots \)

30. Which term is 213\( y \) of the series \( 28y + 33y + 38y + \ldots \)?

**Partial sum of an arithmetic series**

The sum of the first \( n \) terms of an arithmetic series (\( n \)th partial sum) is given by the formula:

\[
S_n = \frac{n}{2}(a + l) \quad \text{where} \quad l = \text{last or} \ n\text{th term}
\]

**Proof**

Let the last or \( n \)th term be \( l \).

\[
\begin{align*}
S_n &= a + (a + d) + (a + 2d) + \ldots + l \\
S_n &= l + (l - d) + (l - 2d) + \ldots + a \\
2S_n &= (a + l) + (a + l) + (a + l) + \ldots + (a + l) \\
&= n(a + l) \\
\therefore \quad S_n &= \frac{n}{2}(a + l)
\end{align*}
\]

In general,

\[
S_n = \frac{n}{2}[2a + (n - 1)d]
\]
Proof

Since $l = n$th term,

$l = a + (n - 1) d$

$\therefore \ S_n = \frac{n}{2} (a + l)$

$= \frac{n}{2} [a + a + (n - 1) d]$

$= \frac{n}{2} [2a + (n - 1) d]$

We use this formula when the $n$th term is unknown.

EXAMPLES

1. Evaluate $9 + 14 + 19 + \ldots + 224$.

Solution

$a = 9, d = 5$

First we find $n$.

$T_n = 224$

$\therefore \ T_n = a + (n - 1) d$

$224 = 9 + (n - 1)5$

$= 9 + 5n - 5$

$= 5n + 4$

$220 = 5n$

$44 = n$

$\therefore \ S_n = \frac{n}{2} (a + l)$

$S_{44} = \frac{44}{2} (9 + 224)$

$= 22 \times 233$

$= 5126$

2. For what value of $n$ is the sum of $n$ terms of $2 + 11 + 20 + \ldots$ equal to 618?

Solution

$a = 2, d = 9, S_n = 618$

$S_n = \frac{n}{2} [2a + (n - 1) d]$

$618 = \frac{n}{2} [2 \times 2 + (n - 1)9]$

$1236 = n(4 + 9n - 9)$

$= n(9n - 5)$

$= 9n^2 - 5n$
3. The 6th term of an arithmetic series is 23 and the sum of the first 10 terms is 210. Find the sum of 20 terms.

**Solution**

\[ T_n = a + (n - 1)d \]
\[ T_6 = a + (6 - 1)d = 23 \]
\[ a + 5d = 23 \]  \hspace{1cm} (1)
\[ S_n = \frac{n}{2}[2a + (n - 1)d] \]
\[ S_{10} = \frac{10}{2}[2a + (10 - 1)d] = 210 \]
\[ 5(2a + 9d) = 210 \]
\[ 2a + 9d = 42 \]  \hspace{1cm} (2)
\[ (1) \times (2): \quad 2a + 10d = 46 \]  \hspace{1cm} (3)
\[ (2) - (3): \quad -d = -4 \]
\[ d = 4 \]

Substitute \( d = 4 \) in (1):
\[ a + 5(4) = 23 \]
\[ a + 20 = 23 \]
\[ a = 3 \]
\[ S_{20} = \frac{20}{2}[2(3) + (20 - 1)4] \]
\[ = 10[6 + 19(4)] \]
\[ = 10 \times 82 \]
\[ = 820 \]

4. Evaluate \( \sum_{r=1}^{50} 3r + 2 \).

**Solution**

\[ \sum_{r=1}^{50} 3r + 2 = (3 \times 1 + 2) + (3 \times 2 + 2) + (3 \times 3 + 2) + \ldots + (3 \times 50 + 2) \]
\[ = 5 + 8 + 11 + \ldots + 152 \]

Arithmetic series with \( a = 5, d = 3, l = 152, n = 50 \)
\[ S_n = \frac{n}{2}(a + l) \]
\[ S_{50} = \frac{50}{2}(5 + 152) \]
\[ = 25 \times 157 \]
\[ = 3925 \]
7.5 Exercises

1. Find the sum of 15 terms of the series.
   (a) \(4 + 7 + 10 + \ldots\)
   (b) \(2 + 7 + 12 + \ldots\)
   (c) \(60 + 56 + 52 + \ldots\)

2. Find the sum of 30 terms of the series.
   (a) \(1 + 7 + 13 + \ldots\)
   (b) \(15 + 24 + 33 + \ldots\)
   (c) \(95 + 89 + 83 + \ldots\)

3. Find the sum of 25 terms of the series.
   (a) \(-2 + 5 + 12 + \ldots\)
   (b) \(5 - 4 - 13 - \ldots\)

4. Find the sum of 50 terms of the series.
   (a) \(50 + 44 + 38 + \ldots\)
   (b) \(11 + 14 + 17 + \ldots\)

5. Evaluate
   (a) \(15 + 20 + 25 + \ldots + 535\)
   (b) \(9 + 17 + 25 + \ldots + 225\)
   (c) \(5 + 2 - 1 - \ldots - 91\)
   (d) \(81 + 92 + 103 + \ldots + 378\)
   (e) \(229 + 225 + 221 + \ldots + 25\)
   (f) \(-2 + 6 + 14 + \ldots + 94\)
   (g) \(0 - 9 - 18 - \ldots - 216\)
   (h) \(79 + 81 + 83 + \ldots + 229\)
   (i) \(14 + 11 + 8 + \ldots - 43\)
   (j) \(1 \frac{1}{2} + 1 \frac{3}{4} + 2 + \ldots + 25 \frac{1}{4}\)

6. Evaluate
   (a) \(\sum_{n=1}^{20} 4n - 7\)
   (b) \(\sum_{r=1}^{15} 5 - 3r\)
   (c) \(\sum_{r=3}^{20} 4 - 6r\)
   (d) \(\sum_{n=1}^{20} 5n + 3\)
   (e) \(\sum_{n=3}^{40} 4 - 3n\)

7. How many terms of the series \(45 + 47 + 49 + \ldots\) give a sum of 1365?

8. For what value of \(n\) is the sum of the arithmetic series \(5 + 9 + 13 + \ldots\) equal to 152?

9. How many terms of the series \(80 + 73 + 66 + \ldots\) give a sum of 495?

10. The sum of the first 5 terms of an arithmetic series is 110 and the sum of the first 10 terms is 320. Find the first term and the common difference.

11. The sum of the first 5 terms of an arithmetic series is 35 and the sum of the next 5 terms is 160. Find the first term and the common difference.

12. Find \(S_{25}\), given an arithmetic series whose 8th term is 16 and whose 13th term is 81.

13. The sum of 12 terms of an arithmetic series is 186 and the 20th term is 83. Find the sum of 40 terms.

14. How many terms of the series \(20 + 18 + 16 + \ldots\) give a sum of 104?

15. The sum of the first 4 terms of an arithmetic series is 42 and the sum of the 3rd and 7th term is 46. Find the sum of the first 20 terms.

16. (a) Show that \((x + 1) + (2x + 4) + (3x + 7) + \ldots\) is an arithmetic series.
   (b) Find the sum of the first 50 terms of the series.
17. The 20th term of an arithmetic series is 131 and the sum of the 6th to 10th terms inclusive is 235. Find the sum of the first 20 terms.

18. The sum of 50 terms of an arithmetic series is 249 and the sum of 49 terms of the series is 233. Find the 50th term of the series.

19. Prove that $T_n = S_n - S_{n-1}$ for any series.

20. Find the sum of all integers between 1 and 100 that are not multiples of 6.

Class Investigation

Look at the working out of question 14 in the previous set of exercises. Why are there two values for $n$?

DID YOU KNOW?

Here is an interesting series:

$$\pi = \frac{1}{4} - \frac{1}{1} + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \ldots$$

Gottfried Wilhelm Leibniz (1646–1716) discovered this result. It is interesting that while $\pi$ is an irrational number, it can be written as the sum of rational numbers.

Geometric Series

In a geometric series each term is formed by multiplying the preceding term by a constant. The constant is called the common ratio.

EXAMPLE

Find the common ratio of the series $3 + 6 + 12 + \ldots$

Solution

By looking at this sequence, each term is multiplied by 2 to get the next term.

If you can’t see this, we divide the terms as follows:

$$\frac{6}{3} = \frac{12}{6} = 2$$
Chapter 7  Series

If \( T_1 + T_2 + T_3 + \ldots \) is a geometric series then
\[
\frac{T_2}{T_1} = \frac{T_3}{T_2}
\]

In general, if \( T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n \) is a geometric series then
\[
\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = \ldots = \frac{T_n}{T_{n-1}}.
\]

EXAMPLES

1. Find \( x \) if \( 5 + x + 45 + \ldots \) is a geometric series.

**Solution**

For a geometric series \( \frac{T_2}{T_1} = \frac{T_3}{T_2} \)

\[
\frac{x}{5} = \frac{45}{x}
\]

\( x^2 = 225 \)

\( x = \pm \sqrt{225} \)

\( = \pm 15 \)

If \( x = 15 \) the series is \( 5 + 15 + 45 + \ldots (r = 3) \).

If \( x = -15 \), the series is \( 5 - 15 + 45 - \ldots (r = -3) \).

2. Is \( \frac{1}{4} + \frac{1}{6} + \frac{1}{18} + \ldots \) a geometric series?

**Solution**

\[
\frac{T_2}{T_1} = \frac{1}{6} + \frac{1}{4} = \frac{1}{6} \times \frac{4}{1} = \frac{2}{3}
\]

\[
\frac{T_3}{T_2} = \frac{1}{18} + \frac{1}{6} = \frac{1}{18} \times \frac{6}{1} = \frac{1}{3}
\]

\( \neq \frac{T_2}{T_1} \)

\( \therefore \) the series is not geometric.
Terms of a geometric series

\[ a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots \] is a geometric series with first term \(a\), common ratio \(r\) and \(n\)th term given by

\[ T_n = ar^{n-1} \]

**Proof**

Let the first term of a geometric series be \(a\) and the common ratio be \(r\).

Then first term is \(a\)  
second term is \(ar\)  
third term is \(ar^2\)  
fourth term is \(ar^3\), and so on  
\(n\)th term is \(ar^{n-1}\)

### EXAMPLES

1.

(a) Find the 10th term of the series \(3 + 6 + 12 + \ldots\)
(b) Write an expression for the \(n\)th term of the series.

**Solution**

(a) This is a geometric series with \(a = 3\) and \(r = 2\).

We want the 10th term, so \(n = 10\).

\[ T_n = ar^{n-1} \]

\[ T_{10} = 3(2)^{10-1} \]

\[ = 3 \times 2^9 \]

\[ = 1536 \]

(b) \(T_n = ar^{n-1}\)

\[ = 3(2)^{n-1} \]

2. Find the common ratio of \(\frac{2}{3} + \frac{4}{15} + \frac{8}{75} + \ldots\) and hence find the 8th term in index form.

**Solution**

\[ r = \frac{\frac{4}{15}}{\frac{2}{3}} \left( = \frac{8}{75} ÷ \frac{4}{15} \right) \]

\[ = \frac{4}{15} \times \frac{3}{2} \]

\[ = \frac{2}{5} \]
3. Find the 10th term of the series −5 + 10 − 20 + ...

**Solution**

\[ T_n = ar^{n-1} \]
\[ T_8 = \frac{2}{3} \left( \frac{2}{5} \right)^{8-1} \]
\[ = \frac{2}{3} \left( \frac{2}{5} \right)^7 \]
\[ = \frac{2^8}{3 \times 5^7} \]

4. Which term of the series 4 + 12 + 36 + ... is equal to 78 732?

**Solution**

This is a geometric series with \( a = 4 \) and \( r = 3 \).

The \( n \)th term is 78 732.

\[ T_n = ar^{n-1} \]
\[ 78 732 = 4(3)^{n-1} \]
\[ 19 683 = 3^{n-1} \]
\[ \log_{10} 19 683 = \log_{10} 3^{n-1} \]
\[ = (n-1) \log 3 \]
\[ \frac{\log_{10} 19 683}{\log_{10} 3} = n - 1 \]
\[ 9 = n - 1 \]
\[ 10 = n \]

So the 10th term is 78 732.
5. The third term of a geometric series is 18 and the 7th term is 1458. Find the first term and the common ratio.

**Solution**

\[ T_n = ar^{n-1} \]

\[ T_3 = ar^{3-1} = 18 \]

\[ ar^2 = 18 \]  (1)

\[ T_7 = ar^{7-1} = 1458 \]

\[ ar^6 = 1458 \]  (2)

\[ (2) ÷ (1): r^4 = 81 \]

\[ r = ±\sqrt[3]{81} \]

\[ r = ±3 \]

Substitute \( r = ±3 \) into (1):

\[ a(±3)^2 = 18 \]

\[ 9a = 18 \]

\[ a = 2 \]

6. Find the first value of \( n \) for which the terms of the series \( \frac{1}{5} + 1 + 5 + ... \) exceed 3000.

**Solution**

\( a = \frac{1}{5}, r = 5 \)

For terms to exceed 3000,

\[ T_n > 3000 \]

\[ ar^{n-1} > 3000 \]

\[ \frac{1}{5}(5^{n-1}) > 3000 \]

\[ 5^{n-1} > 15000 \]

\[ \log 5^{n-1} > \log 15000 \]

\[ (n - 1)\log 5 > \log 15000 \]

\[ n - 1 > \frac{\log 15000}{\log 5} \]

\[ n > \frac{\log 15000}{\log 5} + 1 \]

\[ n > 6.97 \]

\[ n = 7 \]

So the 7th term is the first term to exceed 3000.
7.6 Exercises

1. Are the following series geometric? If they are, find the common ratio.
   (a) $5 + 20 + 60 + ...$
   (b) $-4 + 3 - \frac{1}{4} + ...$
   (c) $\frac{3}{4} \cdot 2 + \frac{3}{14} \cdot 3 + \frac{3}{49} + ...$
   (d) $7 + \frac{5}{6} + \frac{3}{3} + ...$
   (e) $-14 + 42 - 168 + ...$
   (f) $1 + \frac{8}{9} + \frac{8}{27} + ...$
   (g) $5.7 + 1.71 + 0.513 + ...$
   (h) $2 \frac{1}{4} - 1\frac{7}{20} + \frac{81}{100} + ...$
   (i) $63 + 9 + 1\frac{7}{8} + ...$
   (j) $-1\frac{7}{8} + 15 - 120 + ...$

2. Evaluate all pronumerals in these geometric series.
   (a) $4 + 28 + x + ...$
   (b) $-3 + 12 + y + ...$
   (c) $2 + a + 72 + ...$
   (d) $y + 2 + 6 + ...$
   (e) $x + 8 + 32 + ...$
   (f) $5 + p + 20 + ...$
   (g) $7 + y + 63 + ...$
   (h) $-3 + m - 12 + ...$
   (i) $3 + (x - 4) + 15 + ...$
   (j) $3 + (k - 1) + 21 + ...$
   (k) $\frac{1}{4} + t + \frac{1}{9} + ...$
   (l) $\frac{1}{3} + t + \frac{4}{3} + ...$

3. Write an expression for the $n$th term of the following series.
   (a) $1 + 5 + 25 + ...$
   (b) $1 + 1.02 + 1.0404 + ...$
   (c) $1 + 9 + 81 + ...$
   (d) $2 + 10 + 50 + ...$
   (e) $6 + 18 + 54 + ...$
   (f) $8 + 16 + 32 + ...$
   (g) $1 + 4 + 1 + ...$
   (h) $1000 - 100 + 10 - ...$
   (i) $-3 + 9 - 27 + ...$
   (j) $\frac{1}{3} + \frac{2}{15} + \frac{4}{75} + ...$

4. Find the 6th term of each series.
   (a) $8 + 24 + 72 + ...$
   (b) $9 + 36 + 144 + ...$
   (c) $8 - 32 + 128 + ...$
   (d) $-1 + 5 - 25 + ...$
   (e) $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + ...$

5. What is the 9th term of each series?
   (a) $1 + 2 + 4 + ...$
   (b) $4 + 12 + 36 + ...$
   (c) $1 + 1.04 + 1.0816 + ...$
   (d) $-3 + 6 - 12 + ...$
   (e) $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} + ...$

6. Find the 8th term of each series.
   (a) $3 + 15 + 75 + ...$
   (b) $2.1 + 4.2 + 8.4 + ...$
   (c) $5 - 20 + 80 - ...$
   (d) $-\frac{1}{2} + \frac{3}{10} - \frac{9}{50} + ...$
   (e) $1 + \frac{47}{81} + \frac{2}{27} + \frac{3}{9} + ...$

7. Find the 20th term of each series, leaving the answer in index form.
   (a) $3 + 6 + 12 + ...$
   (b) $1 + 7 + 14 + ...$
   (c) $1.04 + 1.04^2 + 1.04^3 + ...$
   (d) $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$
   (e) $\frac{3}{4} + \frac{9}{16} + \frac{27}{64} + ...$
8. Find the 50th term of \(1 + 11 + 121 + \ldots\) in index form.

9. Which term of the series \(4 + 20 + 100 + \ldots\) is equal to 12 500?

10. Which term of \(6 + 36 + 216 + \ldots\) is equal to 7776?

11. Is 1200 a term of the series \(2 + 16 + 128 + \ldots\)?

12. Which term of \(3 + 21 + 147 + \ldots\) is equal to 352 947?

13. Which term of the series \(8 - 4 + 2 - \ldots\) is \(\frac{1}{128}\)?

14. Which term of \(54 + 18 + 6 + \ldots\) is \(\frac{2}{243}\)?

15. Find the value of \(n\) if the \(n\)th term of the series \(-2 + 1 \frac{1}{2} - 1 \frac{1}{8} + \ldots\) is \(-\frac{81}{128}\).

16. The first term of a geometric series is 7 and the 6th term is 1701. Find the common ratio.

17. The 4th term of a geometric series is \(-648\) and the 5th term is \(3888\). (a) Find the common ratio. (b) Find the 2nd term.

18. The 3rd term of a geometric series is \(\frac{2}{5}\) and the 5th term is \(\frac{13}{5}\). Find its first term and common ratio.

19. Find the value of \(n\) for the first term of the series \(5000 + 1000 + 200 + \ldots\) that is less than 1.

20. Find the first term of the series \(\frac{2}{7} + \frac{6}{7} + \frac{4}{7} + \ldots\) that is greater than 100.

---

**Partial sum of a geometric series**

The sum of the first \(n\) terms of a geometric series (\(n\)th partial sum) is given by the formula:

\[
S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{for } |r| > 1
\]

\[
S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{for } |r| < 1
\]

---

**Proof**

The sum of a geometric series can be written

\[
S_n = a + ar + ar^2 + \ldots + ar^{n-1}
\]

\[
rS_n = ar + ar^2 + ar^3 + \ldots + ar^n
\]

(1) \( - \) (2):

\[
S_n (1-r) = a - ar^n
\]

\[
= a (1 - r^n)
\]

\[
S_n = \frac{a (1 - r^n)}{1 - r}
\]

(2) \( - \) (1) gives the formula

\[
S_n = \frac{a (r^n - 1)}{r - 1}
\]
EXAMPLES

1. Find the sum of the first 10 terms of the series $3 + 12 + 48 + ...$

Solution

This is a geometric series with $a = 3$, $r = 4$ and we want $n = 10$.

Since $r > 1$ we use the first formula.

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ = \frac{3(4^{10} - 1)}{4 - 1} \]

\[ = \frac{3(4^{10} - 1)}{3} \]

\[ = 4^{10} - 1 \]

\[ = 1,048,575 \]

2. Evaluate $\sum_{n=1}^{11} 2^n$.

Solution

\[ \sum_{n=1}^{11} 2^n = 2^1 + 2^4 + 2^5 + ... + 2^{11} \]

\[ = 8 + 16 + 32 + ... + 2048 \]

Geometric series with $a = 8$, $r = 2$, $n = 9$

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ S_9 = \frac{8(2^9 - 1)}{2 - 1} \]

\[ = 8 \times 512 \]

\[ = 4088 \]

3. Evaluate $60 + 20 + 6\frac{2}{3} + ... + \frac{20}{81}$.

Solution

\[ a = 60, \ r = \frac{1}{3}, \ T_n = \frac{20}{81} \]

\[ T_n = ar^{n-1} = \frac{20}{81} \]
The sum of \( n \) terms of \( 1 + 4 + 16 + \ldots \) is 21 845. Find the value of \( n \).

**Solution**

\[ a = 1, \ r = 4, S_n = 21845 \]

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ 21845 = \frac{1(4^n - 1)}{4 - 1} \]

\[ 4^n - 1 = 65535 \]

\[ 65536 = 4^n \]

\[ 4^8 = 4^n \]

\[ n = 8 \]

So 8 terms gives a sum of 21 845.
7.7 Exercises

1. Find the sum of 10 terms of the series.
   (a) \(6 + 24 + 96 + \ldots\)
   (b) \(3 + 15 + 75 + \ldots\)

2. Find the sum of 8 terms of the series.
   (a) \(-1 + 7 - 49 + \ldots\)
   (b) \(8 + 24 + 72 + \ldots\)

3. Find the sum of 15 terms of the series.
   (a) \(4 + 8 + 16 + \ldots\)
   (b) \(\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \ldots\)

4. Evaluate
   (a) \(2 + 10 + 50 + \ldots + 6250\)
   (b) \(18 + 9 + 4\frac{1}{2} + \ldots + \frac{9}{64}\)
   (c) \(3 + 21 + 147 + \ldots + 7203\)
   (d) \(\frac{3}{4} + \frac{2\frac{1}{4}}{4} + \frac{6\frac{3}{4}}{4} + \ldots + 182\frac{1}{4}\)
   (e) \(-3 + 6 - 12 + \ldots + 384\)

5. Evaluate
   (a) \(\sum_{n=1}^{8} 2^{n-1}\)
   (b) \(\sum_{n=1}^{8} \frac{1}{3^n}\)
   (c) \(\sum_{n=2}^{10} 5^{n-3}\)
   (d) \(\sum_{r=1}^{8} \frac{1}{2^{r-1}}\)
   (e) \(\sum_{k=1}^{2} 4^{k+1}\)

6. Find the
   (a) 9th term
   (b) sum of 9 terms of the series \(7 + 14 + 28 + \ldots\)

7. Find the sum of 30 terms of the series \(1.09 + 1.09^2 + 1.09^3 + \ldots\), correct to 2 decimal places.

8. Find the sum of 25 terms of the series \(1 + 1.12 + 1.12^2 + \ldots\), correct to 2 decimal places.

9. Find the value of \(n\) if the sum of \(n\) terms of the series \(11 + 33 + 99 + \ldots\) is equal to 108 251.

10. How many terms of the series \(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\) give a sum of \(\frac{1023}{1024}\)?

11. The common ratio of a geometric series is 4 and the sum of the first 5 terms is 3069. Find the first term.

12. Find the number of terms needed to be added for the sum to exceed 1 000 000 in the series \(4 + 16 + 64 + \ldots\)

13. Lucia currently earns \$25 000. Her wage increases by 5% each year. Find
   (a) her wage after 6 years
   (b) her total earnings (before tax) in 6 years.

14. Write down an expression for the series \(2 - 10 + 50 - \ldots + 2(-5^{k-1})\)
   (a) in sigma notation
   (b) as a sum of \(n\) terms.

15. Find the sum of the first 10 terms of the series \(3 + 7 + 13 + \ldots + [2^n + (2n - 1)] + \ldots\)
Limiting sum (sum to infinity)

In some geometric series the sum becomes very large as \( n \) increases. For example,

\[ 2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots \]

We say that this series diverges, or it has an infinite sum.

In some geometric series, the sum does not increase greatly after a few terms. We say this series converges and it has a limiting sum (the sum is limited to a finite number).

**EXAMPLE**

For the series \( 2 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \), notice that after a while the terms are becoming closer and closer to zero and so will not add much to the sum of the whole series.

(a) Evaluate, correct to 4 decimal places, the sum of
   (i) 10 terms
   (ii) 20 terms.

(b) Estimate its limiting sum.

**PUZZLES**

1. A poor girl saved a rich king from drowning one day. The king offered the girl a reward of sums of money in 30 daily payments. He gave the girl a choice of payments:
   - **Choice 1:** $1 the first day, $2 the second day, $3 the third day and so on.
   - **Choice 2:** 1 cent the first day, 2 cents the second day, 4 cents the third day and so on, the payment doubling each day.

   How much money would the girl receive for each choice? Which plan would give the girl more money?

2. Can you solve Fibonacci’s Problem?

   A man entered an orchard through 7 guarded gates and gathered a certain number of apples. As he left the orchard he gave the guard at the first gate half the apples he had and 1 apple more. He repeated this process for each of the remaining 6 guards and eventually left the orchard with 1 apple.

   How many apples did he gather? (He did not give away any half apples.)
Solution

(a) $a = 2, \ r = \frac{1}{2}$

(i) $n = 10$

\[
S_n = \frac{a \left(1 - r^n\right)}{1 - r} = \frac{2 \left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}} = \frac{2 \left(1 - \frac{1}{2^{10}}\right)}{\frac{1}{2}} = 3.9961
\]

(ii) $n = 20$

\[
S_n = \frac{a \left(1 - r^n\right)}{1 - r} = \frac{2 \left(1 - \left(\frac{1}{2}\right)^{20}\right)}{1 - \frac{1}{2}} = \frac{2 \left(1 - \frac{1}{2^{20}}\right)}{\frac{1}{2}} = 4.0000
\]

(b) The limiting sum is 4.

Can you see why the series $2 + 4 + 8 + 16 + \ldots$ diverges and the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots$ converges?

The difference is in the common ratio. Only geometric series with common ratios $|r| < 1$ will converge and have a limiting sum.

\[
S_n = \frac{a}{1 - r}
\]

$|r| < 1$ is the necessary condition for the limiting sum to exist.
**Proof**

\[ S_n = \frac{a(1 - r^n)}{1 - r} \]

For \(|r| < 1\), as \(n\) increases, \(r^n\) decreases and approaches zero.

E.g. When \(r = \frac{1}{2}\):

\[ \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} \text{ and } \left(\frac{1}{2}\right)^{20} = \frac{1}{1048576} \]

\[ \therefore S_n = \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r} \]

**EXAMPLES**

1. Find the limiting sum of \(2 + \frac{1}{2} + \frac{1}{4} + \ldots\)

**Solution**

In the previous example, we guessed that the limiting sum was 4. Here we will use the formula to find the limiting sum.

\(a = 2, r = \frac{1}{2}\)

\[ S_n = \frac{a}{1 - r} = \frac{2}{1 - \frac{1}{2}} = \frac{2}{\frac{1}{2}} = 2 \times 2 = 4 \]

2. Find the sum of the series \(6 + 2 + \frac{2}{3} + \ldots\)

**Solution**

\(a = 6, r = \frac{2}{6} = \frac{3}{2} = \frac{1}{3}\)
Which of the following series have a limiting sum?

(a) \(\frac{3}{4} + \frac{15}{16} + \frac{111}{64} + \ldots\)

(b) \(100 + 50 + 25 + \ldots\)

(c) \(3 + 2\frac{1}{4} + 1\frac{11}{16} + \ldots\)

**Solution**

(a) \(r = \frac{15/16}{3/4} = \frac{11/64}{15/16}\)

\[= \frac{1\frac{1}{4}}{4}\]

\[> 1\]

So this series does not have a limiting sum.

(b) \(r = \frac{50}{100} = \frac{25}{50}\)

\[= \frac{1}{2}\]

\(< 1\)

So this series does have a limiting sum.

(c) \(r = \frac{2\frac{1}{4}}{3} = \frac{1\frac{11}{16}}{2\frac{1}{4}}\)

\[= \frac{3}{4}\]

\(< 1\)

So this series does have a limiting sum.
4. Evaluate $\sum_{n=2}^{\infty} 2 \left(\frac{2}{3}\right)^n$

**Solution**

$$\sum_{n=2}^{\infty} 2 \left(\frac{2}{3}\right)^n = 2 \left(\frac{2}{3}\right)^2 + 2 \left(\frac{2}{3}\right)^3 + 2 \left(\frac{2}{3}\right)^4 + ...$$

$$= 2 \left(\frac{4}{9}\right) + 2 \left(\frac{8}{27}\right) + 2 \left(\frac{16}{81}\right) + ...$$

$$= \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + ...$$

$$a = \frac{8}{9}, \ r = \frac{2}{3}$$

$$S_n = \frac{a}{1-r}$$

$$= \frac{\frac{8}{9}}{1-\frac{2}{3}}$$

$$= \frac{8}{9} \times \frac{3}{1}$$

$$= 2 \frac{2}{3}$$

7.8 Exercises

1. Which of the following series have a limiting sum? Find its limiting sum where it exists.
   (a) $9 + 3 + 1 + ...$
   (b) $\frac{1}{4} + \frac{1}{8} + 1 + ...$
   (c) $16 - 4 + 1 - ...$
   (d) $\frac{2}{3} + \frac{7}{9} + \frac{49}{54} + ...$
   (e) $1 + \frac{2}{3} + \frac{4}{9} + ...$
   (f) $\frac{5}{8} + \frac{1}{8} + \frac{1}{40} + ...$
   (g) $-6 + 36 - 216 + ...$
   (h) $-2\frac{1}{4} + \frac{7}{8} - \frac{127}{48} + ...$
   (i) $\frac{1}{9} + \frac{1}{6} + \frac{1}{4} + ...$
   (j) $2 - \frac{4}{5} + \frac{8}{25} - ...$

2. Find the limiting sum of each series.
   (a) $40 + 20 + 10 + ...$
   (b) $320 + 80 + 20 + ...$
   (c) $100 - 50 + 25 - ...$
   (d) $6 + 3 + 1\frac{1}{2} + ...$
   (e) $\frac{2}{5} + \frac{6}{35} + \frac{18}{245} + ...$
   (f) $72 - 24 + 8 - ...$
   (g) $-12 + 2 - \frac{1}{3} + ...$
   (h) $\frac{3}{4} - \frac{1}{2} + \frac{1}{3} - ...$
   (i) $12 + 9 + 6\frac{3}{4} + ...$
   (j) $\frac{2}{3} + \frac{5}{12} - \frac{25}{96} + ...$
3. Find the difference between the limiting sum and the sum of 6 terms of
(a) 56 − 28 + 14 − ...
(b) 72 + 24 + 8 + ...
(c) 1 + \frac{1}{5} + \frac{1}{25} + ...
(d) \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...
(e) \frac{1}{4} + \frac{15}{16} + \frac{45}{64} + ...

4. Evaluate
(a) \sum_{r=1}^{\infty} \left(\frac{1}{5}\right)^{r-1}
(b) \sum_{n=1}^{\infty} \left(\frac{2}{7}\right)^n
(c) \sum_{k=3}^{\infty} \left(\frac{1}{4}\right)^k
(d) \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^{n-1}
(e) \sum_{p=2}^{\infty} \left(\frac{2}{3}\right)^{p-2}
(f) \sum_{n=1}^{\infty} 5 \left(\frac{1}{2}\right)^n
(g) \sum_{i=1}^{\infty} 2 \left(\frac{1}{6}\right)^i
(h) \sum_{n=3}^{\infty} -4 \left(\frac{2}{3}\right)^{n-1}
(i) \sum_{q=2}^{\infty} 2 \left(-\frac{1}{9}\right)^{q-2}
(j) \sum_{n=0}^{\infty} \frac{1}{2} \left(\frac{2}{5}\right)^n

5. A geometric series has limiting sum 6 and common ratio \frac{1}{3}. Evaluate the first term of the series.

6. A geometric series has a limiting sum of 5 and first term 3. Find the common ratio.

7. The limiting sum of a geometric series is 9\frac{1}{3} and the common ratio is \frac{2}{5}. Find the first term of the series.

8. A geometric series has limiting sum 40 and its first term is 5. Find the common ratio of the series.

9. A geometric series has limiting sum −6\frac{2}{5} and a first term of −8. Find its common ratio.

10. The limiting sum of a geometric series is −\frac{3}{10} and its first term is −\frac{1}{2}. Find the common ratio of the series.

11. The second term of a geometric series is 2 and its limiting sum is 9. Find the values of first term \(a\) and common ratio \(r\).

12. A geometric series has 3rd term 12 and 4th term −3. Find \(a\), \(r\) and the limiting sum.

13. A geometric series has 2nd term \ \frac{2}{3} and 4th term \ \frac{8}{27}. Find \(a\), \(r\) and its limiting sum.

14. The 3rd term of a geometric series is 54 and the 6th term is \ \frac{83}{125}. Evaluate \(a\), \(r\) and the limiting sum.

15. The 2nd term of a geometric series is \ \frac{4}{15} and the 5th term is \ \frac{32}{405}. Find the values of \(a\) and \(r\) and its limiting sum.

16. The limiting sum of a geometric series is 5 and the 2nd term is \ \frac{1}{5}. Find the first term and the common ratio.

17. The series \ \frac{x}{4} + \frac{x}{16} + ... has a limiting sum of \ \frac{7}{8}. Evaluate \(x\).
18. (a) For what values of \( k \) does the limiting sum exist for the series \( k + k^2 + k^3 + \ldots \)?
(b) Find the limiting sum of the series when \( k = -\frac{2}{3} \).
(c) Evaluate \( k \) if the limiting sum of the series is 3.

19. (a) For what values of \( p \) will the limiting sum exist for the series \( 1 - 2p + 4p^2 - \ldots \)?
(b) Find the limiting sum when \( p = \frac{1}{5} \).
(c) Evaluate \( p \) if the limiting sum of the series is \( \frac{7}{8} \).

20. Show that in any geometric series the difference between the limiting sum and the sum of \( n \) terms is \( \frac{ar^n}{1-r} \).

Applications of Series

General applications

Arithmetic and geometric series are useful in solving real life problems involving patterns. It is important to choose the correct type of series and know whether you are asked to find a term or a sum of terms.

EXAMPLES

1. A stack of cans on a display at a supermarket has 5 cans on the top row. The next row down has 2 more cans and the next one has 2 more cans and so on.
   (a) Calculate the number of cans in the 11th row down.
   (b) If there are 320 cans in the display altogether, how many rows are there?

Solution

The first row has 5 cans, the 2nd row has 7 cans, the 3rd row 9 cans and so on. This forms an arithmetic series with \( a = 5 \) and \( d = 2 \).
(a) For the 11th row, we want \( n = 11 \).
\[ T_n = a + (n - 1)d \]
\[ T_{11} = 5 + (11 - 1) \times 2 \]
\[ = 5 + 10 \times 2 \]
\[ = 25 \]
So there are 25 cans in the 11th row.
(b) If there are 320 cans altogether, this is the sum of cans in all rows.

So \( S_n = 320 \)

\[
S_n = \frac{n}{2} [2a + (n - 1)d]
\]

\[
320 = \frac{n}{2} [2 \times 5 + (n - 1) \times 2]
\]

\[
= \frac{n}{2} [10 + 2n - 2]
\]

\[
= \frac{n}{2} [2n + 8]
\]

\[
= n^2 + 4n
\]

\[
0 = n^2 + 4n - 320
\]

\[
= (n - 16)(n + 20)
\]

\[
n - 16 = 0, \quad n + 20 = 0
\]

\[
n = 16, \quad n = -20
\]

Since \( n \) must be a positive integer, then \( n = 16 \).

There are 16 rows of cans.

2. A layer of tinting for a car window lets in 95% of light.

(a) What percentage of light is let in by

(i) 2 layers

(ii) 3 layers

(iii) 10 layers

of tinting?

(b) How many layers will let in 40% of light?

Solution

(a) (i) 1 layer lets in 95% of light.

So 2 layers let in 95% \times 95% of light.

\[
95\% \times 95\% = 0.95 \times 0.95
\]

\[
= 0.9025
\]

\[
= 90.25\%
\]

So 2 layers let in 90.25% of light.

(ii) 1 layer lets in 95% or 0.95 of light.

2 layers let in 0.95 \times 0.95 or 0.95^2 of light.

3 layers let in 0.95^2 \times 0.95 or 0.95^3 of light.

\[
0.95^3 = 0.857
\]

\[
= 85.7\%
\]

So 3 layers let in 85.7% of light.
(iii) The number of layers forms the geometric series
\[ 0.95 + 0.95^2 + 0.95^3 + \ldots \] with \( a = 0.95 \), \( r = 0.95 \).

For 10 layers, \( n = 10 \)
\[ T_n = ar^{n-1} \]
\[ T_{10} = 0.95 (0.95)^{10-1} \]
\[ = 0.95 (0.95)^9 \]
\[ = 0.95^{10} \]
\[ = 0.5987 \]
\[ = 59.87\% \]

So 10 layers let in 59.87\% of light.

(b) We want to find \( n \) when the \( n \)th term is 40\% or 0.4.
\[ T_n = ar^{n-1} \]
\[ 0.4 = 0.95 (0.95)^{n-1} \]
\[ = 0.95^n \]
\[ \log 0.4 = \log 0.95^n \]
\[ = n \log 0.95 \]
\[ \frac{\log 0.4}{\log 0.95} = n \]
\[ 17.9 = n \]

So around 18 layers of tinting will let in 40\% of light.

The limiting sum can also be used to solve various problems.

### EXAMPLES

1. Write 0.\overline{5} as a fraction.

**Solution**

0.\overline{5} = 0.55555555\ldots
\[ = \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \ldots \]

This is a geometric series with \( a = \frac{5}{10} = \frac{1}{2} \) and \( r = \frac{1}{10} \).
\[ S_\infty = \frac{a}{1-r} \]
\[ = \frac{\frac{1}{2}}{1-\frac{1}{10}} \]
\[ = \frac{\frac{1}{2}}{\frac{9}{10}} \]
\[ = \frac{5}{9} \]
2. A ball is dropped from a height of 1 metre and bounces up to \( \frac{1}{3} \) of its height. It continues bouncing, rising \( \frac{1}{3} \) of its height on each bounce until it eventually reaches the ground. What is the total distance through which it travels?

**Solution**

Notice that there is a series for the ball coming downwards and another series upwards.

There is more than one way of calculating the total distance. Here is one way of solving it.

Total distance = \( 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \)

\[ = 1 + 2\left(\frac{\frac{1}{3}}{1 - \frac{1}{3}}\right) \]

\[ = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \]

\( \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \) is a geometric series with \( a = \frac{1}{3} \) and \( r = \frac{1}{3} \)

\[ S = \frac{a}{1 - r} \]

\[ = \frac{\frac{1}{3}}{1 - \frac{1}{3}} \]

\[ = \frac{\frac{1}{3}}{\frac{2}{3}} \]

\[ = \frac{1}{2} \times \frac{3}{2} \]

\[ = \frac{1}{2} \]

Total distance = \( 1 + 2\left(\frac{1}{2}\right) \)

\[ = 1 + 1 \]

\[ = 2 \]

So the ball travels 2 metres altogether.
Investigation

1. In the second example above, in theory will the ball ever stop?

2. Kim owes $1000 on her credit card. If she pays back 10% of the amount owing each month, she will never finish paying it off. Is this true or false?

7.9 Exercises

1. A market gardener plants daffodil bulbs in rows, starting with a row of 45 bulbs. Each successive row has 5 more bulbs.
   (a) Calculate the number of bulbs in the 34th row.
   (b) Which row would be the first to have more than 100 bulbs in it?
   (c) The market gardener plants 10 545 bulbs. How many rows are there?

2. A stack of logs has 1 on the top, then 3 on the next row down, and each successive row has 2 more logs than the one on top of it.
   (a) How many logs are in the 20th row?
   (b) Which row has 57 logs?
   (c) If there are 1024 logs altogether, how many rows are in the stack?

3. A set of books is stacked in layers, where each layer contains three books fewer than the layer below. There are 6 books in the top layer, 9 in the next layer and so on. There are $n$ layers altogether.
   (a) Write down the number of books in the bottom layer.
   (b) Show that there are $\frac{3}{2}n(n + 3)$ books in the stack altogether.

4. A painting appreciates (increases its value) by 16% p.a. It is currently worth $20 000.
   (a) How much will it be worth in
      (i) 1 year?
      (ii) 2 years?
      (iii) 3 years?
   (b) How much will it be worth in 11 years?
   (c) How long will it take for it to be worth $50 000?

5. Water evaporates from a pond at an average rate of 7% each week.
   (a) What percentage of water is left in the pond after
      (i) 1 week?
      (ii) 2 weeks?
      (iii) 3 weeks?
   (b) What percentage is left after 15 weeks?
   (c) If there was no rain, approximately how long would it take for the pond to only have 25% of its water left?
6. A timber fence is to be built on sloping land, with the shortest piece of timber 1.2 m and the longest 1.8 m. There are 61 pieces of timber in the fence.

(a) What is the difference in height between each piece of timber?
(b) What length of timber is needed for the fence altogether?

7. A logo is made with vertical lines equally spaced as shown. The shortest line is 25 mm, the longest is 217 mm and the sum of the lengths of all the lines is 5929 mm.

(a) How many lines are in the logo?
(b) Find the difference in length between adjacent lines.

8. In a game, a child starts at point \( P \) and runs and picks up an apple 3 m away. She then runs back to \( P \) and puts the apple in a bucket. The child then runs to get the next apple 6 m away, and runs back to \( P \) to place it in the bucket. This continues until she has all the apples in the bucket.

(a) How far does the child run to get to the \( k \)th apple?
(b) How far does the child run to fetch all \( k \) apples, including return trips to \( P \)?
(c) The child runs 270 m to fetch all apples and return them to the bucket. How many apples are there?

9. The price of shares in a particular company is falling by an average of 2% each day.
(a) What percentage of their value are they after 2 days?
(b) How many days will it take for the shares to halve in value?
(c) After how many days will the shares be worth 25% of their value?

10. A southern brown bandicoot population in WA is decreasing by 5% each year.
(a) What percentage of the population is left after 5 years?
(b) After how many years will the population be only 50%?
(c) How many years will it take for the population to decrease by 80%?
11. Write each recurring decimal as a fraction.
   (a) 0.4
   (b) 0.7
   (c) 1.2
   (d) 0.25
   (e) 2.8
   (f) 0.2
   (g) 1.4
   (h) 1.01
   (i) 0.13
   (j) 2.3

12. A frog jumps 0.5 metres. It then jumps 0.1 m and on each subsequent jump, travels 0.2 m of the previous distance. Find the total distance through which the frog jumps.

13. A tree 3 m high grows by another $2\frac{2}{5}$ m after 5 years, then by another $1\frac{23}{25}$ m after 5 more years and so on, each year growing $\frac{4}{5}$ more than the previous year’s growth. Find the ultimate height of the tree.

14. An 8 cm seedling grows by $4\frac{4}{5}$ cm in the first week, and then keeps growing by $\frac{3}{5}$ of its previous week’s growth. How tall will it grow?

15. An object rolls 0.5 m in the first second. Then each second after, it rolls by $\frac{5}{6}$ of its previous roll. Find how far it will roll altogether.

16. A lamb grows by $\frac{2}{5}$ of its previous growth each month. If a lamb is 45 cm tall, (a) how tall will it be after 6 months? (b) what will its final height be?

17. A 100 m cliff erodes by $\frac{2}{7}$ of its height each year. (a) What will the height of the cliff be after 10 years? (b) After how many years will the cliff be less than 50 m high?

18. An elastic string drops down 60 cm and then bounces back to $\frac{2}{3}$ of its initial height. It keeps bouncing, each time rising back to $\frac{2}{3}$ of its previous height. What is the total distance through which the string travels?

19. Mary bounces a ball, dropping it from 1.5 m on its first bounce. It then rises up to $\frac{2}{5}$ of its height on each bounce. Find the distance through which the ball travels.
20. A roadside wall has a zig zag pattern on it as shown.

The two longest lines are each 2 m long, then the next two lines are $1\frac{3}{4}$ m long and each subsequent pair of lines are $\frac{7}{8}$ of the length of the previous lines. Find the total distance of the lines.

21. Kate receives an email chain letter that she is asked to send to 8 friends. If Kate forwards this email on to 8 friends and each of them sends it on to 8 friends, and so on,

(a) describe the number of people receiving the email as a sequence (including Kate's email)

(b) how many people would receive the email when it is sent for the 9th time?

(c) how many people would have received the email altogether if it is sent 9 times?

### Compound interest

Compound interest refers to when the interest earned on an investment is then added to the amount of the investment. This means that each amount of interest earned is calculated on the previous interest as well as the investment.

Money that earns compound interest grows according to a geometric series.

#### EXAMPLE

An amount of $2000 is invested at the rate of 6% p.a. Find the amount in the bank at the end of $n$ years.

**Solution**

After 1 year: The amount in the bank is the original amount plus the interest of 6% earned on the amount.

Amount 1 = $2000 + 6% of $2000

= $2000(1 + 0.06)

= $2000(1.06)$

**CONTINUED**
After 2 years: The amount in the bank is the previous amount plus the interest of 6% earned on this amount.

Amount 2 = Amount 1 + 6% of amount 1

\[ \begin{align*}
21 & = 2000(1.06) + 6\% \text{ of } 2000(1.06) \\
2000 & = 2000(1.06)(1 + 6\%) \\
2000 & = 2000(1.06)(1 + 0.06) \\
2000 & = 2000(1.06)(1.06) \\
& = 2000(1.06)^2 \\
\end{align*} \]

Similarly after 3 years the amount is \$2000(1.06)^3 and so on.
The amount after \( n \) years will be \$2000(1.06)^n.

In general the compound interest formula is

\[ A = P(1 + r)^n \] where
- \( P \) = principal (initial amount)
- \( r \) = interest rate as a decimal
- \( n \) = number of time periods

**Proof**

Let \$P\) be invested at the rate of \( r \) p.a. compound interest for \( n \) years where \( r \) is a decimal.

After 1 year:
Amount = \( P + r \text{ of } P \)
\[ = P(1 + r) \]

After 2 years:
Amount = \( P(1 + r) + r \text{ of } P(1 + r) \)
\[ = P(1 + r)(1 + r) \]
\[ = P(1 + r)^2 \]

After 3 years:
Amount = \( P(1 + r)^2 + r \text{ of } P(1 + r)^2 \)
\[ = P(1 + r)^2(1 + r) \]
\[ = P(1 + r)^3 \]

Following this pattern, the amount after \( n \) years will be \( P(1 + r)^n \).
EXAMPLES

1. Find the amount that will be in the bank after 6 years if $2000 is invested at 12% p.a. with interest paid
   (a) yearly
   (b) quarterly
   (c) monthly.

Solution

\[ P = 2000 \]

(a) \[ r = 12\% \]
    \[ = 0.12 \]
    \[ n = 6 \]
    \[ A = P(1 + r)^n \]
    \[ = 2000(1 + 0.12)^6 \]
    \[ = 2000(1.12)^6 \]
    \[ = 3947.65 \]

So the amount is $3947.65.

(b) For quarterly interest, it is divided into 4 amounts each year.
    \[ r = 0.12 \text{ p.a.} \]
    \[ = 0.12 \div 4 \text{ quarterly} \]
    \[ = 0.03 \]

Also the interest is paid in 4 times a year.
    \[ n = 6 \times 4 \]
    \[ = 24 \text{ quarters} \]
    \[ A = P(1 + r)^n \]
    \[ = 2000(1 + 0.03)^{24} \]
    \[ = 2000(1.03)^{24} \]
    \[ = 4065.59 \]

So the amount is $4065.59.

(c) For monthly interest, it is divided into 12 amounts each year.
    \[ r = 0.12 \text{ p.a.} \]
    \[ = 0.12 \div 12 \text{ monthly} \]
    \[ = 0.01 \]

Also the interest is paid in 12 times a year.
    \[ n = 6 \times 12 \]
    \[ = 72 \text{ months} \]
    \[ A = P(1 + r)^n \]
    \[ = 2000(1 + 0.01)^{72} \]
    \[ = 2000(1.01)^{72} \]
    \[ = 4094.20 \]

So the amount is $4094.20.

CONTINUED
2. Geoff wants to invest enough money to pay for a $10 000 holiday in 5 years’ time. If interest is 8% p.a., how much does Geoff need to invest now?

Solution

\[ A = 10 000, \ r = 8\% = 0.08 \text{ and } n = 5 \]

We want to find the principal \( P \).

\[ A = P (1 + r)^n \]

\[ 10 000 = P (1 + 0.08)^5 \]

\[ = P (1.08)^5 \]

\[ \frac{10 000}{1.08^5} = P \]

\[ 6805.83 = P \]

So Geoff will need to invest $6805.83 now.

3. An amount of $1800 was invested at 6% p.a. and is now worth $2722.66. For how many years was the money invested if interest is paid twice a year?

Solution

\[ P = 1800, \ A = 2722.66 \]

Interest is paid twice a year.

\[ r = 6\% = 0.06 \text{ p.a.} \]

\[ = 0.06 \div 2 \]

\[ = 0.03 \text{ twice a year} \]

We want to find \( n \)

\[ A = P (1 + r)^n \]

\[ 2722.66 = 1800(1 + 0.03)^n \]

\[ = 1800(1.03)^n \]

\[ \frac{2722.66}{1800} = 1.03^n \]

\[ 1.5126 = 1.03^n \]

\[ \log 1.5126 = \log 1.03^n \]

\[ = n \log 1.03 \]

\[ \frac{\log 1.5126}{\log 1.03} = n \]

\[ 14 = n \]

Since interest is paid in twice a year, the number of years will be \( 14 \div 2 \).

So the money was invested for 7 years.
7.10 Exercises

1. Find the amount of money in the bank after 10 years if
   (a) $500 is invested at 4% p.a.
   (b) $7500 is invested at 7% p.a.
   (c) $8000 is invested at 8% p.a.
   (d) $5000 is invested at 6.5% p.a.
   (e) $2500 is invested at 7.8% p.a.

2. Sam banks $1500 where it earns interest at the rate of 6% p.a. Find the amount after 5 years if interest is paid:
   (a) annually
   (b) twice a year
   (c) quarterly.

3. Chad banks $3000 in an account that earns 5% p.a. Find the amount in the bank after 10 years if interest is paid:
   (a) quarterly
   (b) monthly.

4. I put $350 in the bank where it earns interest of 8% p.a. with interest paid annually. Find the amount there will be in the account after 2 years.

5. How much money will there be in an investment account after 3 years if interest of 4.5% p.a. is paid twice a year on $850?

6. Find the amount of money there will be in a bank after 8 years if $1000 earns interest of 7% p.a. with interest paid twice a year.

7. Abdul left $2500 in a building society account for 4 years, with interest of 5.5% p.a. paid yearly. How much money did he have in the account at the end of that time?

8. Find the amount of money there will be after 15 years if $6000 earns 9% p.a. interest, paid quarterly.

9. How much money will be in a bank account after 5 years if $500 earns 6.5% p.a. with interest paid monthly?

10. Find the amount of interest earned over 4 years if $1400 earns 6% p.a. paid quarterly.

11. How much money will be in a credit union account after 8 years if $8000 earns 7.5% p.a. interest paid monthly?

12. Elva wins a lottery and invests $500 000 in an account that earns 8% p.a. with interest paid monthly. How much will be in the account after 12 years?

13. Calculate the amount deposited 4 years ago at 5% p.a. if the current amount in the account is:
   (a) $5000
   (b) $675
   (c) $12 000
   (d) $289.50
   (e) $12 800.

14. How much was banked 3 years ago if the amount in the bank now is $5400 and interest is 5.8% p.a. paid quarterly?

15. How many years ago was an investment made if $5000 was invested at 6% p.a. paid monthly and is now worth $6352.45?
16. Find the number of years that $10,000 was invested at 8% p.a. with interest paid twice a year if there is now $18,729.81 in the bank.

17. Jude invested $4500 five years ago at x% p.a. Evaluate x if Jude now has an amount of
   (a) $6311.48
   (b) $5743.27
   (c) $6611.98
   (d) $6165.39
   (e) $6766.46 in the bank.

18. Hamish is given the choice of a bank account in which interest is paid annually or quarterly. If he deposits $1200, find the difference in the amount of interest paid over 3 years if interest is 7% p.a.

19. Kate has $4000 in a bank account that pays 5% p.a. with interest paid annually and Rachel has $4000 in a different account paying 4% quarterly. Which person will receive the most interest over 5 years and by how much?

20. A bank offers investment account A at 8% p.a. with interest paid twice a year and account B with interest paid at 6% p.a. at monthly intervals. If Georgie invests $5000 over 6 years, which account pays the most interest? How much more does it pay?

Annuities

An annuity is a fund where a certain amount of money is invested regularly (often annually, which is where the name comes from) for a number of years. Scholarship and trust funds, certain types of insurance and superannuation are examples of annuities.

While superannuation and other investments are affected by changes in the economy, due to their long-term nature, the amount of interest they earn balances out over the years. In this course, we average out these changes, and use a constant amount of interest annually.

EXAMPLES

1. A sum of $1500 is invested at the beginning of each year in a superannuation fund. If interest is paid at 6% p.a., how much money will be available at the end of 25 years?

Solution

It is easier to keep track of each annual amount separately. The first amount earns interest for 25 years, the 2nd amount earns interest for 24 years, the 3rd amount for 23 years and so on.
The 25th amount earns interest for 1 year.

\[ P = 1500 \text{ and } r = 6\% = 0.06 \]

\[ A = P(1 + r)^n \]

\[ = 1500(1 + 0.06)^n \]

\[ = 1500(1.06)^n \]

\[ A_1 = 1500(1.06)^{25} \]

\[ A_2 = 1500(1.06)^{24} \]

\[ A_3 = 1500(1.06)^{23} \]

\[ \ldots \]

\[ A_{25} = 1500(1.06)^1 \]

Total superannuation amount

\[ T = A_1 + A_2 + A_3 + \ldots + A_{25} \]

\[ = 1500(1.06)^{25} + 1500(1.06)^{24} + 1500(1.06)^{23} + \ldots + 1500(1.06)^1 \]

\[ = 1500(1.06^{25} + 1.06^{24} + 1.06^{23} + \ldots + 1.06^1) \]

\[ = 1500(1.06 + 1.06^2 + 1.06^3 + \ldots + 1.06^{25}) \]

\[ 1.06 + 1.06^2 + 1.06^3 + \ldots + 1.06^{25} \text{ is a geometric series with} \]

\[ a = 1.06, \quad r = 1.06 \text{ and } n = 25 \]

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ S_{25} = \frac{1.06(1.06^{25} - 1)}{1.06 - 1} \]

\[ = 58.16 \]

\[ T = 1500 \times 58.16 \]

\[ = 87234.57 \]

So the total amount of superannuation after 25 years is $87,234.57.

2. An amount of $50 is put into an investment account at the end of each month. If interest is paid at 12% p.a. paid monthly, how much is in the account at the end of 10 years?

**Solution**

\[ P = 50 \]

\[ r = 12\% \]

\[ = 0.12 \]

The monthly interest rate is

\[ 0.12 \div 12 = 0.01. \]

Monthly interest over 10 years gives

\[ n = 10 \times 12 \]

\[ = 120 \]
The first amount earns interest for 119 months, since it is deposited at the end of the month.
The 2nd amount earns interest for 118 months, the 3rd amount for 117 months and so on.
The 120th amount earns no interest as it goes in at the end of the last month.

\[ A = P(1 + r)^n \]
\[ = 50(1 + 0.01)^n \]
\[ = 50(1.01)^n \]
\[ A_1 = 50(1.01)^{119} \]
\[ A_2 = 50(1.01)^{118} \]
\[ A_3 = 50(1.01)^{117} \]
\[ \vdots \]
\[ A_{120} = 50(1.01)^0 \]

Total amount
\[ T = A_1 + A_2 + A_3 + \ldots + A_{120} \]
\[ = 50(1.01)^{119} + 50(1.01)^{118} + 50(1.01)^{117} + \ldots + 50(1.01)^0 \]
\[ = 50(1.01^{119} + 1.01^{118} + 1.01^{117} + \ldots + 1.01^0) \]
\[ = 50(1.01^0 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119}) \]
\[ = 50(1 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119}) \]

\[ 1 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119} \] is a geometric series with \( a = 1, r = 1.01 \) and \( n = 120 \)

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]
\[ S_{120} = \frac{1(1.01^{120} - 1)}{1.01 - 1} \]
\[ = 230.04 \]
\[ T = 50 \times 230.04 \]
\[ = 11501.93 \]

So the total amount after 10 years is $11501.93.

### 7.11 Exercises

1. Find the amount of superannuation available at the end of 20 years if $500 is invested at the beginning of each year and earns 9% p.a.

2. Michael works for a company that puts aside $1200 at the beginning of each year for his superannuation. If the money earns interest at the rate of 5% p.a., find the amount of superannuation that Michael will have at the end of 30 years.
3. How much superannuation will there be at the end of 20 years if $800 is invested at 10% p.a. at the beginning of each year?

4. Rachel starts working for a business at the beginning of 2005. If she retires at the end of 2034, how much superannuation will she have if $1000 is invested at the beginning of each year at 9.5% p.a.?

5. A sum of $1500 is invested at the end of each year for 15 years at 8% p.a. Find the amount of superannuation available at the end of the 15 years.

6. How much superannuation will there be at the end of 18 years if $690 is invested at 8.5% p.a. at the beginning of each year?

7. If Phan pays $750 into a superannuation fund at the beginning of each year, how much will she have at the end of 29 years if the interest is 6.8% p.a.?

8. A sum of $1000 is invested at the end of each year for 22 years, at 9% p.a. Find the amount of superannuation available at the end of the 22 years.

9. Matthew starts work at the beginning of 2010. If he retires at the end of 2037, how much superannuation will he have if he invests $700 at the beginning of each year at 12.5% p.a.?

10. An apprentice starts work for a small business. If she invests $400 at the beginning of each year, how much superannuation will she have at the end of 25 years if the money earns 15.5% p.a.?

11. Lliam wants to save up $15 000 for a car in 5 years’ time. He invests $2000 at the end of each year in an account that pays 7.5% p.a. interest. How much more will Lliam have to pay at the end of 5 years to make up the $15 000?

12. A school invests $5000 at the end of each year at 6% p.a. towards a new library. How much will the school have after 10 years?

13. Jacques puts aside $500 at the end of each year for 5 years. If the money is invested at 6.5% p.a., how much will Jacques have at the end of the 5 years?

14. An employee contributes $200 into a superannuation fund at the end of each year. If the interest rate on this fund is 11.5% p.a., how much will the employee have at the end of 20 years?

15. Mohammed’s mother invests $200 for him each birthday up to and including his 18th birthday. The money earns 6% p.a. How much money will Mohammed have on his 18th birthday?

16. Xuan is saving up for a holiday. She invests $800 at the end of each year at 7.5% p.a. How much will she have for her holiday after 5 years’ time?

17. A couple saves $3000 at the end of each year towards a deposit on a house. If the interest rate is 5% p.a., how much will the couple have saved after 6 years?
18. Lucia saves up $2000 each year and at the end of the year she invests it at 6% p.a. 
   (a) She does this for 10 years. What is her investment worth? 
   (b) Lucia continues investing $2000 a year for 5 more years. What is the future value of her investment?

19. Jodie starts work in 2012 and puts $1000 in a superannuation fund at the end of the year. She keeps putting in this same amount at the end of every year until she retires at the end of 2029. If interest is paid at 10% p.a., calculate how much Jodie will have when she retires.

20. Michelle invests $1000 at the end of each year. The interest rate is 8% p.a. 
   (a) How much will her investment be worth after 6 years? 
   (b) How much more would Michelle's investment be worth after 6 years if she had invested $1200 each year?

21. Jack cannot decide whether to invest $1000 at the end of each year for 15 years or $500 for 30 years in a superannuation fund. If the interest rate is 5% p.a., which would be the better investment for Jack?

22. Kate is saving up to go overseas in 8 years' time. She invests $1000 at the end of each year at 7% p.a. and estimates that the trip will cost her around $10 000. Will she have enough? If so, how much over will it be? If she doesn't earn enough, how much will she need to add to this money to make it up to the $10 000?

23. Farmer Brown puts aside part of the farm's earnings at the beginning of each month to buy a new truck in 10 years' time when the old one wears out. He invests $400 each month at 9% p.a. and estimates the cost of a new truck at $80 000. Will the investment earn enough to buy the new truck? What is the difference?

24. Marcus and Rachel want to save up $25 000 for a deposit on an apartment in 6 years' time. They aim to pay around half the deposit each. Marcus invests an inheritance of $9000 in a bank account where it earns 8% p.a. Rachel invests $100 at the beginning of each month where it earns interest of 9% p.a. 
   (a) What is the future value of Marcus's investment after 6 years? 
   (b) How much will Rachel's investment be worth after 6 years? 
   (c) Will they be able to pay the deposit in 6 years' time?

25. Jenny puts aside $20 at the end of each month for 3 years. How much will she have then if the investment earns 8.2% p.a., paid monthly?
Loan repayments

The formula for compound interest and the geometric series can help with working out regular loan repayments.

**EXAMPLES**

1. A sum of $20 000 is borrowed at 12% p.a. and paid back at regular monthly intervals over 4 years. Find the amount of each payment.

**Solution**

Let $M$ stand for the monthly repayment.

The number of payments will be $4 \times 12$ or 48.

Monthly interest will be $12\% \div 12$ or $1\% (0.01)$.

Each month, interest for that month is added to the loan and the repayment amount is taken off.

The interest added to the first month will be $20 000(1 + 0.01)^1$ or $20 000(1.01)^1$.

The interest added to other months will be $A (1 + 0.01)^1$ or $A (1.01)^1$.

The amount owing for the first month:

$$A_1 = 20 000 (1 + 0.01)^1 - M$$

$$= 20 000 (1.01)^1 - M$$

The amount owing for the 2nd month is what was owing from the 1st month, together with that month’s interest, minus the repayment.

$$A_2 = A_1 (1 + 0.01)^1 - M$$

$$= A_1 (1.01)^1 - M$$

$$= [20 000(1.01)^1 - M](1.01)^1 - M \text{ (substituting in } A_1)$$

$$= 20 000(1.01)^2 - M(1.01)^1 - M$$

$$= 20 000(1.01)^2 - M(1.01^1 + 1)$$

Similarly,

$$A_3 = A_2 (1 + 0.01)^1 - M$$

$$= [20 000(1.01)^2 - M(1.01^1 + 1)](1.01)^1 - M$$

$$= 20 000(1.01)^3 - M(1.01^2 + 1)(1.01^1) - M$$

$$= 20 000(1.01)^3 - M(1.01^2 + 1.01^1) - M$$

$$= 20 000(1.01)^3 - M(1.01^2 + 1.01^1 + 1)$$
Continuing this pattern, after 4 years (48 months) the amount owing will be
\[ A_{48} = 20000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01 + 1). \]

But the loan is paid out after 48 months.
So \( A_{48} = 0 \)

\[ 0 = 20000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01 + 1) \]

\[ M = \frac{20000(1.01)^{48}}{1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01 + 1} \]

\[ 1.01^{47} + 1.01^{46} + \ldots + 1.01^2 + 1.01 + 1 = 1 + 1.01 + 1.01^2 + \ldots + 1.01^{46} + 1.01^{47} \]

This is a geometric series with \( a = 1, r = 1.01 \) and \( n = 48 \).

\[ S_n = \frac{a(r^n - 1)}{r - 1} \]

\[ S_{48} = \frac{1(1.01^{48} - 1)}{1.01 - 1} = 61.223 \]

\[ \therefore M = \frac{20000(1.01)^{48}}{61.223} = 526.68 \]

So the monthly repayment is $526.68.

2. A store charges 10% p.a. for loans and repayments do not have to be made until the 4th month. Ivan buys $8000 worth of furniture and pays it off over 3 years.
(a) How much does Ivan owe after 3 months?
(b) What are his monthly repayments?
(c) How much does Ivan pay altogether?

Solution

Let \( P \) stand for the payments each month.

Number of payments = \( 3 \times 12 - 3 \) (3 months of no repayments)

= 33 months

Monthly interest rate
\[ r = \frac{10\%}{12} \]

= \( 0.1 + 12 \)

= \( 0.0083 \)

(a) After 3 months, the amount owing is
\[ A = P(1 + r)^n \]

\[ = 8000(1 + 0.0083)^3 \]

\[ = 8000(1.0083)^3 \]

\[ = 8201.67 \]

So the amount owing after 3 months is $8201.67.
(b) The first repayment is in the 4th month.

\[ A_1 = 8000 \times (1.0083)^1 \]
\[ A_2 = 8000 \times (1.0083)^2 \]
\[ A_3 = 8000 \times (1.0083)^3 \]
\[ A_4 = 8000 \times (1.0083)^4 - P \]
\[ A_5 = \left[ 8000 \times (1.0083)^5 - P \right](1.0083)^1 - P \]
\[ = 8000 \times (1.0083)^5 - P(1.0083)^1 - P \]
\[ = 8000 \times (1.0083)^5 - P(1.0083^1 + 1) \]
\[ A_6 = \left[ 8000 \times (1.0083)^6 - P(1.0083^1 + 1) \right](1.0083)^1 - P \]
\[ = 8000 \times (1.0083)^6 - P(1.0083^1 + 1)(1.0083^1) - P \]
\[ = 8000 \times (1.0083)^6 - P(1.0083^2 + 1.0083^1) - P \]

Continuing this pattern,
\[ A_{36} = 8000 \times (1.0083)^{36} - P(1.0083^{32} + 1.0083^{31} + \ldots + 1.0083^1 + 1) \]

The loan is paid out after 36 months.

So \( A_{36} = 0 \)
\[ 0 = 8000 \times (1.0083)^{36} - P(1.0083^{32} + 1.0083^{31} + \ldots + 1.0083^1 + 1) \]
\[ P(1.0083^{32} + 1.0083^{31} + \ldots + 1.0083^1 + 1) = 8000 \times (1.0083)^{36} \]
\[ P = \frac{8000 \times (1.0083)^{36}}{1.0083^{32} + 1.0083^{31} + \ldots + 1.0083^1 + 1} \]

This is a geometric series with \( a = 1 \), \( r = 1.0083 \) and \( n = 33 \).
\[ S_n = \frac{a(r^n - 1)}{r - 1} \]
\[ S_{33} = \frac{1(1.0083^{33} - 1)}{1.0083 - 1} \]
\[ = 37.804 \]
\[ \therefore \quad P = \frac{8000 \times (1.0083)^{36}}{37.804} \]
\[ = 285.30 \]

So the monthly repayment is $285.30.

(c) Ivan pays $285.30 \times 33$ or $9414.94 altogether.
1. An amount of $3000 is borrowed at 22% p.a. and repayments made yearly for 5 years. How much will each repayment be?

2. The sum of $20,000 is borrowed at 18% p.a. interest over 8 years. How much will the repayments be if they are made monthly?

3. David borrows $5000 from the bank and pays back the loan in monthly instalments over 4 years. If the loan incurs interest of 15% p.a., find the amount of each instalment.

4. Mr and Mrs Nguyen mortgage their house for $150,000. Find the amount of the (a) monthly repayments they will have to make if the mortgage is over 25 years with interest at 6% p.a. (b) If the Nguyens want to pay their mortgage out after 15 years, what monthly repayments would they need to make?

5. A loan of $6000 is paid back in equal annual instalments over 3 years. If the interest is 12.5% p.a., find the amount of each annual instalment.

6. The Smith family buys a car for $38,000, paying a 10% deposit and taking out a loan for the balance. If the loan is over 5 years with interest of 1.5% monthly, find (a) the amount of each monthly loan repayment (b) the total amount that the Smith family paid for the car.

7. A $2000 loan is offered at 18% p.a. with interest charged monthly, over 3 years. (a) If no repayment need be paid for the first 2 months, find the amount of each repayment. (b) How much will be paid back altogether?

8. Bill thinks he can afford a mortgage payment of $800 each month. How much can he borrow, to the nearest $100, over 25 years at 11.5% p.a.?

9. Get Rich Bank offers a mortgage at 7 \(\frac{1}{2}\) % p.a. over 10 years and Capital Bank offers a mortgage at 5 \(\frac{1}{2}\) % p.a. over 25 years. (a) Find the amount of the monthly repayments for each bank on a loan of $80,000. (b) Find the difference in the total amount paid on each mortgage.

10. Majed buys a $35,000 car. He puts down a 5% deposit and pays the balance back in monthly instalments over 4 years at 12% p.a. Find the total amount that Majed pays for the car.

11. Amy borrowed money over 7 years at 15.5% p.a. and she pays $1200 a month. How much did she borrow?
12. NSW Bank offers loans at 9% p.a. with an interest-free period of 3 months, while Sydney Bank offers loans at 7% p.a. Compare these loans on an amount of $5000 over 3 years and state which bank offers the best loan and why.

13. Danny buys a plasma TV for $10 000. He pays a $1500 deposit and borrows the balance at 18% p.a. over 4 years. (a) Find the amount of each monthly repayment. (b) How much did Danny pay altogether?

14. A store offers furniture on hire purchase at 20% p.a. over 5 years, with no repayments for 6 months. Ali buys furniture worth $12 000. (a) How much does Ali owe after 6 months? (b) What are the monthly repayments? (c) How much does Ali pay for the furniture altogether?

15. A loan of $6000 over 5 years at 15% p.a. interest, charged monthly, is paid back in 5 annual instalments. (a) What is the amount of each instalment? (b) How much is paid back altogether?
1. Find a formula for the $n$th term of each sequence.
   (a) 9, 13, 17, ...
   (b) 7, 0, −7, ...
   (c) 2, 6, 18, ...
   (d) 200, 50, $12 \frac{1}{2}$, ...
   (e) −2, 4, −8, ...

2. For the series $156 + 145 + 134 + ...$
   (a) Find the 15th term.
   (b) Find the sum of 15 terms.
   (c) Find the sum of 14 terms.
   (d) Write a relationship between $T_{15}$, $S_{15}$ and $S_{14}$.
   (e) Find the value of $n$ for the first negative term.

3. Evaluate
   (a) $\sum_{r=1}^{50} (9r - 7)$
   (b) $\sum_{n=1}^{7} \left(\frac{1}{n}\right)$ as a fraction
   (c) $\sum_{n=1}^{12} 5(3^n)$
   (d) $\sum_{n=1}^{100} (7n - 6)$
   (e) $\sum_{n=3}^{\infty} 2\left(\frac{1}{5}\right)^{n-2}$

4. A bamboo blind has 30 slats. When the blind is down, each gap between slats, and between the top and bottom of the window, is 3 mm. When the blind is up, the slats have no gaps between them.
   (a) Show that when the blind is up, the bottom slat rises 90 mm.
   (b) How far does the next slat rise?
   (c) Explain briefly why the distances the slats rise when the blind is up form an arithmetic series.
   (d) Find the distance the 17th slat from the bottom rises.
   (e) What is the sum of the distances that all slats rise?

5. Find the amount invested in a bank account at 9.5% p.a. if the balance in the account is $5860.91 after 6 years.

6. State whether each series is
   (i) arithmetic
   (ii) geometric
   (iii) neither.
   (a) $97 + 93 + 89 + ...$
   (b) $\frac{2}{3} + \frac{1}{2} + \frac{3}{8} + ...$
   (c) $\sqrt{5} + \sqrt{20} + \sqrt{45} + ...$
   (d) $-1.6 - 0.4 + 0.6 - ...$
   (e) $3.4 + 7.5 + 11.6 + ...$
   (f) $48 + 24 + 12 + ...$
   (g) $-\frac{1}{5} + 1 - 5 + ...$
   (h) $105 + 100 + 95 + ...$
   (i) $1 \frac{1}{2} + 1 \frac{1}{4} + 1 + ...$
   (j) $\log_{10} x + \log_{10} x^2 + \log_{10} x^3 + ...

7. The $n$th term of the series $8 + 13 + 18 + ...$ is 543. Evaluate $n$.

8. Amanda earned $20 000 in one year. At the beginning of the next year she received a salary increase of $450.
She now receives the same increase each year.
(a) What will her salary be after 10 years?
(b) How much will Amanda earn altogether over the 10 years?

9. The 11th term of an arithmetic series is 97 and the 6th term is 32. Find the first term and common difference.

10. A series has \( n \)th term given by \( T_n = n^3 - 5 \). Find
(a) the 4th term
(b) the sum of 4 terms
(c) which term is 5827.

11. A series has terms 5 + \( x \) + 45 + ... Evaluate \( x \) if the series is
(a) arithmetic
(b) geometric.

12. Convert each recurring decimal to a fraction.
(a) 0.4
d.
(b) 0.72
c.
(c) 1.57
c.

13. If \( x, 2x + 3 \) and \( 5x \) are the first 3 terms of an arithmetic series, calculate the value of \( x \).

14. Find the 20th term of
(a) 3, 10, 17, ...
(b) 101, 98, 95, ...
(c) 0.3, 0.6, 0.9, ...

15. Find the limiting sum (sum to infinity) of \( 81 + 27 + 9 + ... \)

16. Karl puts $300 aside as an annuity for his son at the beginning of each year. If interest is 7% p.a., how much will his son receive at the end of 15 years?

17. For each series, write an expression for the sum of \( n \) terms.
(a) 5 + 9 + 13 + ...
(b) 1 + 1.07 + 1.07^2 + ...

18. (a) For what values of \( x \) does the geometric series \( 1 + x + x^2 + ... \) have a limiting sum?
(b) Find the limiting sum when \( x = \frac{3}{5} \).
(c) Evaluate \( x \) when the limiting sum is \( 1 \frac{1}{2} \).

19. The first term of an arithmetic series is 4 and the sum of 10 terms is 265. Find the common difference.

20. Every week during a typing course, Tony improves his typing speed by 3 words per minute until he reaches 60 words per minute by the end of the course.
(a) If he can type 18 words per minute in the first week of the course, how many words per minute can he type by week 8?
(b) How many weeks does the course run for?

21. A farmer borrows $50 000 for farm machinery at 18% p.a. over 5 years and makes equal yearly repayments on the loan at the end of each year.
(a) How much does he owe at the end of the first year, just before he makes the first repayment?
(b) How much is each yearly repayment?

22. A ball drops from a height of 1.2 metres then bounces back to \( \frac{3}{5} \) of this height.
On the next bounce, it bounces up to \( \frac{3}{5} \) of this height and so on. Through what distance will the ball travel?

23. If \( x + 2, 7x - 2 \) and \( 15x + 6 \) are consecutive terms in a geometric series, evaluate \( x \).

24. (a) If $2000 is invested at 4.5% p.a., how much will it be worth after 4 years?
(b) If interest is paid quarterly, how much would the investment be worth after 4 years?
25. Evaluate \(8 + 14 + 20 + \ldots + 122\).

26. (a) Calculate the sum of all the multiples of 7 between 1 and 100.
(b) Calculate the sum of all numbers between 1 and 100 that are not multiples of 7.

27. Scott borrows $200 000 to buy a house. If the interest is 6% p.a. and the loan is over 20 years,
   (a) how much is each monthly repayment?
   (b) how much does Scott pay altogether?

28. The sum of \(n\) terms of the series \(214 + 206 + 198 + \ldots\) is 2760. Evaluate \(n\).

29. Evaluate \(n\) if the \(n\)th term of the series \(4 + 12 + 36 + \ldots\) is 354 292.

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**Challenge Exercise 7**

1. The \(n\)th term of a sequence is given by 
   \[T_n = \frac{n^2}{n + 1}\].
   (a) What is the 9th term of the sequence?
   (b) Which term is equal to \(18\frac{1}{20}\)?

2. For the sequence \(\frac{3\pi}{4}, \frac{5\pi}{4}, \ldots\) evaluate
   (a) the common difference
   (b) the 7th term
   (c) the sum of 6 terms.

3. Evaluate the sum of the first 20 terms of the series
   (a) \(3 + 5 + 9 + 17 + 33 + 65 + \ldots\)
   (b) \(5 - 2 + 10 - 8 + 15 - 32 + \ldots\)

4. A factory sells shoes at $60 a pair. For 10 pairs of shoes there is a discount, whereby each pair costs $58. For 20 pairs, the cost is $56 a pair and so on. Find
   (a) the price of each pair of shoes on an order of 100 pairs of shoes.
   (b) the total price on an order of 60 pairs of shoes.

5. Which term of the sequence
   \[\frac{7}{9}, \frac{14}{45}, \frac{28}{225}, \ldots\] is equal to \(\frac{224}{28125}\)?

6. Find the sum of all integers between 1 and 200 that are not multiples of 9.

7. Find the least number of terms for which the sum of the series \(20 + 4 + \frac{4}{5} + \ldots\) is greater than 24.99.

8. Find the values of \(n\) for which the sequence given by \(T_n = n^2 - 4n\) is negative.

9. The sum of the first 5 terms of a geometric sequence is 77 and the sum of the next 5 terms is \(-2464\). Find the 4th term of the sequence.

10. Jane’s mother puts $300 into an account at the beginning of each year to pay for Jane’s education in 5 years’ time. If 6% p.a. interest is paid quarterly, how much money will Jane’s mother have at the end of the 5 years?

11. The geometric series \(2x + 4x^2 + 8x^3 + \ldots\) has a limiting sum of 3. Evaluate \(x\).

12. Find the amount of money in a bank account if $5000 earns 8.5% p.a. for 4 years, then 6.5% p.a. for 3 years, with interest paid monthly for all 7 years.
13. Find the value of \( k \) for which 81 920 is the 8th term of the geometric sequence \( 5 + k + 80 + \ldots \)

14. Kim borrows $10 000 over 3 years at a rate of 1% interest compounded each month. If she pays off the loan in three equal annual instalments, find
(a) the amount Kim owes after one month
(b) the amount she owes after the first year, just before she pays the first instalment
(c) the amount of each instalment
(d) the total amount of interest Kim pays.

15. (a) Find the limiting sum of the series \( 1 + \cos^2 x + \cos^4 x + \ldots \) in simplest form.
(b) Why does this series have a limiting sum?

16. A superannuation fund paid 6% p.a. for the first 10 years and then 10% p.a. after that. If Thanh put $5000 into this fund at the end of each year, how much would she have at the end of 25 years?
1. \( \angle ABE = 180^\circ - \angle ABD \) (straight angle 180°)  
\( \angle CBE = 180^\circ - \angle CBD \) (straight angle)  
\( = 180^\circ - \angle ABD \) (\( \angle ABD = \angle CBD \) — given)  
\( = \angle ABE \)

2. \( \angle DBF = 180^\circ - (180 - x)^\circ \) (\( \angle AFB \) is a straight angle)  
\( \therefore \angle AFC = x \) (vertically opposite angles)  
\( \angle CFE = 180^\circ - (x + 180^\circ - 2x) \) (\( \angle AFB \) is a straight angle)  
\( = x \)  
\( \therefore \angle AFC = \angle CFE \)  
\( \therefore CD \) bisects \( \angle AFE \)

3. \( \angle WBC + \angle BCY = 2x + 115 + 65 - 2x = 180^\circ \)  
These are supplementary cointerior angles.  
\( \therefore VW \parallel XY \)

4. \( x + y = 180^\circ \) (given)  
\( \therefore \angle A + \angle D = 180^\circ \)  
These are supplementary cointerior angles.  
\( \therefore AB \parallel DC \)  
Also \( \angle A = \angle B \) (similarly)  
These are supplementary cointerior angles.  
\( \therefore AD \parallel BC \)  
\( \therefore ABCD \) is a parallelogram.

5. \( \angle ADB = \angle CDB = 110^\circ \) (given)  
\( \angle ABD = \angle CBD \) (\( BD \) bisects \( \angle ABC \))  
\( BD \) is common  
\( \therefore \) by AAS,  
\( \triangle ABD \equiv \triangle CBD \)

6. (a) \( AB = AE \) (given)  
\( \angle B = \angle E \) (base angles of isosceles \( \triangle \))  
\( BC = DE \) (given)  
\( \therefore \) by SAS, \( \triangle ABC \equiv \triangle AED \)  
(b) \( \angle DCA = \angle EDA \)  
(corresponding angles in congruent \( \triangle \))  
\( \angle ACD = 180^\circ - \angle BCA \) (\( BCD \) is a straight angle)  
\( = 180^\circ - \angle EDA \)  
\( = \angle ADC \)  
\( \therefore \) since base angles are equal, \( \triangle ACD \) is isosceles

7. \( DC = BC \) (given)  
\( \angle B = \angle D = 90^\circ \) (given)  
\( DM = \frac{1}{2} AD \) (given)  
and \( \quad BN = \frac{1}{2} AB \)  
\( \therefore DM = BN \)  
\( \therefore \) by SAS, \( \triangle DMBC \equiv \triangle NBC \)  
\( \therefore MC = NC \)  
(corresponding sides in congruent \( \triangle \))

8. \( \angle OCA = \angle OCB = 90^\circ \) (given)  
\( OA = OB \) (equal radii)  
\( OC \) is common  
\( \therefore \) by RHS, \( \triangle OAC \equiv \triangle OBC \)  
\( \therefore AC = BC \) (corresponding sides in congruent \( \triangle \))  
\( \therefore OC \) bisects \( AB \)

9. \( \angle CDB = \angle BDC = 90^\circ \) (altitudes given)  
\( \angle ACB = \angle ABC \) (base angles of isosceles \( \triangle \))  
\( CB \) is common  
\( \therefore \) by AAS, \( \triangle CDB \equiv \triangle BCE \)  
\( \therefore CE = BD \)  
(corresponding sides in congruent \( \triangle \))  
\( \therefore AC \) bisects \( \angle DCB \)

10. \( AB = AD \) (given)  
\( BC = DC \) (given)  
\( AC \) is common  
\( \therefore \) by SSS, \( \triangle ABC \equiv \triangle ADC \)  
\( \therefore \angle DAC = \angle BAC \) (corresponding angles in congruent \( \triangle \))  
So \( AC \) bisects \( \angle DAB \)  
Also \( \angle BCA = \angle DCA \) (corresponding angles in congruent \( \triangle \))  
\( \therefore \) \( AC \) bisects \( \angle DCB \)

11. (a) \( \angle NMO = \angle MOP \) (alternate angles, \( MN \parallel PO \))  
\( \angle PMO = \angle MON \) (alternate angles, \( PM \parallel ON \))  
\( MO \) is common  
\( \therefore \) by SAS,  
\( \triangle NMO \equiv \triangle MPO \)  
(b) \( \angle PMO = \angleMON \) (alternate angles, \( PM \parallel ON \))  
\( MN = NO \) (given)  
\( \therefore \) \( \angle PMQ = \angle NMQ \)  
(corresponding sides in congruent \( \triangle \))  
\( \therefore \) \( PM = NO \) (given)  
\( \therefore \) \( \angle PMQ = \angle NMQ \) (from (b))  
\( MQ \) is common  
\( \therefore \) by SAS, \( \triangle PMQ \equiv \triangle NMQ \)  
(d) \( \angle MQN = \angle MPQ \) (corresponding angles in congruent \( \triangle \))  
But \( \angle MQN + \angle MPQ = 180^\circ \) (\( \angle PQN \) straight angle)  
\( \therefore \) \( \angle MQN = \angle MPQ = 90^\circ \)  

12. A geometric diagram is shown with points A, B, C, D, E, and line segments connecting them. Points A, B, and C are collinear, with B between A and C. Point D is located outside the line segment AC, and line segments AD and DC are drawn from D to A and C, respectively. Point E is located on line segment AB. The diagram includes angles and segments labeled with corresponding relationships and measurements to illustrate geometric properties and theorems.
Let $ABCD$ be a parallelogram with diagonal $AC$.

$\angle DAC = \angle ACB$ (alternate angles, $AD \parallel BC$)

$\angle BAC = \angle ACD$ (alternate angles, $AB \parallel DC$)

$AC$ is common

$\therefore$ by AAS, $\triangle ADC \equiv \triangle ABC$

13. $\triangle ADC \equiv \triangle ABC$ (see question 12)

$\therefore \angle A = \angle C$

$\therefore$ opposite angles are equal

14. $AB = DC$ (opposite sides in || gram)

$RM = DN$ (given)

$\therefore AB - BM = DC - DN$

i.e. $AM = NC$

Also $AM \parallel NC$ ($ABCD$ is a || gram)

Since one pair of sides is both parallel and equal, $AMCN$ is a parallelogram.

15. $AD = BC$ (opposite sides in || gram)

$BC = FE$ (similarly)

$\therefore AD = FE$

Also $AD \parallel BC$ ($ABCD$ is a || gram)

and $BC \parallel FE$ ($BCEF$ is a || gram)

$\therefore AD \parallel FE$

Since one pair of sides is both parallel and equal, $AFED$ is a parallelogram.

16. $\angle DEC = \angle DCE$ (base angles of isosceles $\triangle$)

Also, $\angle DEC = \angle ECB$ (alternate angles, $AD \parallel BC$)

$\therefore \angle DCE = \angle ECB$

$\therefore CE$ bisects $\angle BCD$

17. $AB = CD$ (given)

$\angle BAC = \angle DCA$ (given)

$AC$ is common

$\therefore$ by SAS, $\triangle ABC \equiv \triangle ADC$

$\therefore AD = BC$

(corresponding sides in congruent $\triangle$s)

Since two pairs of opposite sides are equal, $ABCD$ is a parallelogram.

18. (a) $AE = EC$ (diagonals bisect each other in || gram)

$\angle AEB = \angle CEB = 90^\circ$ (given)

$EB$ is common

$\therefore$ by SAS, $\triangle ABE \equiv \triangle CBE$

$\therefore AB = BC$

(corresponding sides in congruent $\triangle$s)

(b) $\angle ABE = \angle CBE$

(corresponding angles in congruent $\triangle$s)

19. Let $ABCD$ be a rectangle

$AD = BC$ (opposite sides in || gram)

$\angle D = \angle C = 90^\circ$

$DC$ is common

$\therefore$ by SAS, $\triangle ADC \equiv \triangle BCD$

$\therefore AC = DB$

(corresponding sides in congruent $\triangle$s)

20. Let $ABCD$ be a rectangle with $\angle D = 90^\circ$

$\therefore \angle C = 180^\circ - 90^\circ$

($\angle D$ and $\angle C$ co-interior angles, $AD \parallel BC$)

$\angle B = 180^\circ - 90^\circ$

($\angle B$ and $\angle C$ co-interior angles, $AB \parallel DC$)

$\angle A = 180^\circ - 90^\circ$

($\angle A$ and $\angle B$ co-interior angles, $AD \parallel BC$)

$\therefore$ all angles are right angles

21. $AD = CD$ (given)

$AD = BC$ (opposite sides of || gram)

Also $AB = CD$ (similarly)

$\therefore AB = AD = BC = CD$

$\therefore$ all sides of the rhombus are equal

22. Construct $BE \parallel AD$

Then $AD = BE$ (opposite sides of || gram)

But $AD = BC$ (given)

Then $BE = BC$

$\therefore \angle BCD = \angle BCE$ (base angles of isosceles $\triangle$)

Also, $\angle ADC = \angle ECB$ (corresponding angles, $AD \parallel BE$)

$\therefore \angle ADC = \angle BCD$

23. $AD = AB$ (given)

$DC = BC$ (given)

$AC$ is common

$\therefore$ by SSS, $\triangle ADC \equiv \triangle ABC$

$\therefore \angle ADC = \angle ABC$

(corresponding angles in congruent $\triangle$s)
24. \[ AD = BC \] (opposite sides of \( \text{gram} \))
\[ \angle D = \angle C = 90^\circ \quad (\text{given}) \]
\[ DE = EC \quad (E \text{ is the midpoint—given}) \]
\[ \therefore \text{by SAS, } \triangle ADE \cong \triangle BCE \]
\[ \therefore AE = BE \]
(corresponding sides in congruent \( \triangle \))

25. (a) \[ AD = BC \] (opposite sides of \( \text{gram} \))
\[ AB = DC \] (similarly)
\[ DB \text{ is common} \]
\[ \therefore \text{by SSS, } \triangle ADB \cong \triangle BCD \]
(b) \( \angle ABE = \angle CBE \)
(corresponding angles in congruent \( \triangle \))
(c) \( \triangle ABE \cong \triangle CBE \)
(adjacent sides in rhombus)
But \( \angle AEB + \angle BEC = 180^\circ \) (AEC is a straight angle)
\[ \therefore \angle AEB = \angle BEC = 90^\circ \]

Exercises 1.2
1. (a) 14 452 mm\(^2\)  (b) 67 200 mm\(^3\)
2. \( 90 \text{ m}^3 \)
3. \( V = 2x^3 + 3x^2 - 2x \)
4. \[ V = \frac{\pi}{3}h \]
\[ 250 = \frac{\pi}{3}h \]
\[ 250 \cdot \frac{3}{\pi} = r^2 \]
\[ \sqrt{\frac{250}{\pi}} = r \]
5. \( V = 5\pi r^3 \)
6. \( A = \frac{3b^2}{2} \)
7. \( V = (x + 2)^3 = x^3 + 6x^2 + 12x + 8 \)
8. \( S = 24\pi r^2 \)
9. \( V = x(3 - 2x)^2 = 4x^3 - 12x^2 + 9x \)
10. \( 262 \text{ cm}^3 \)
11. \( V = 3h^2 + 2h \)
12. \( V = 2h^2 + 5h \)
13. \( V = \frac{1}{3}(6x^3 - 5x^2 - 34x - 15) \)
14. (a) \( V = 18x^3 - 12x^2 + 2x \)  (b) \( S = 54x^2 - 30x + 4 \)
15. \( l = \sqrt{h^2 + r^2} \)
16. \[ V = \frac{x^2y}{4\pi} \]
17. \( h = \frac{400}{\pi} \)
18. \( h = \frac{750 - \pi^2}{\pi} \)
19. \( l = \frac{850 - \pi^2}{\pi} \)
20. \( y = \sqrt{810 000 - x^2} \)

Exercises 1.3
1. Show \( m_{ab} = m_{cd} = 4 \) and \( m_{ab} = m_{cd} = -\frac{4}{7} \)
2. Show \( m_{ac} \times m_{bd} = \frac{7}{4} \times -\frac{4}{7} = -1; \)
\[ \therefore \text{right-angled triangle with } \angle C = 90^\circ \]
3. (a) \( AB = AC = \sqrt{73}, BC = 6 \)  (b) 8 units  (c) 24 units\(^2\)
4. Show \( m_{xy} = m_{yz} = \frac{1}{5} \)
5. (a) Show \( AB = AD = \sqrt{26}, \quad BC = CD = \sqrt{80} \)
(b) Show \( m_{ac} \times m_{bd} = -1 \)
(c) \( E = (-1, 2), \ CE = \sqrt{72} = 6\sqrt{2}, \quad AE = \sqrt{18} = 3\sqrt{2} \)
6. \( r = 1 \)
7. (a) \( 2x - 3y + 13 = 0 \)
(b) Substitute \((7, 9)\) into the equation
(c) Isosceles
8. \( \angle AOB = \angle COD = 90^\circ \)
\[ \frac{OD}{OB} = \frac{4}{2} = 2 \]
\[ \frac{OC}{OA} = \frac{14}{7} = 2 \]
\[ \therefore \frac{OD}{OB} = \frac{OC}{OA} \]
Since 2 pairs of sides are in proportion and their included angles are equal, \( \triangle OAB \parallel \triangle OCD. \)
9. (a) \( OB \text{ is common} \)
\[ OA = BC = 5 \]
\[ AB = OC = \sqrt{20} \]
\[ \therefore \text{by SSS } \triangle ABO \cong \triangle OCB \]
(b) Show \( m_{oa} = m_{bc} = \frac{1}{3} \) and \( m_{ab} = m_{oc} = -2 \)
10. \( \angle ABC = 90^\circ \) and \( AB = BC = 2 \)
So \( ABC \) is isosceles
\[ \therefore \angle CAB = \angle ACB \]
But \( \angle CAB + \angle ACB = 90^\circ \) (angle sum of triangle)
\[ \therefore \angle CAB = \angle ACB = 45^\circ \]
Similarly, other angles are 45°.
11. \( PR = QS = \sqrt{145} \)
Since diagonals are equal, \( PQRS \) is a rectangle.
12. (a) \( X = (-2, 2), \ Y = (-1, 0) \)
(b) \( m_{xy} = m_{bc} = -2 \)
So \( XY \parallel BC \)
(c) \( XY = \sqrt{5}, \ BC = \sqrt{20} = 2\sqrt{5} \)
So \( BC = 2XY \)
13. \( m_{\triangle} \times m_{\triangle} = 1 \times -1 = -1 \)
    So \(AC\) and \(BD\) are perpendicular.
    Midpoint \(AC\) = midpoint \(BD\) = \(-\frac{a}{2}, \frac{b}{2}\)
    So \(AC\) and \(BD\) bisect each other.
    So \(AC\) and \(BD\) are perpendicular bisectors.

14. (a) Distance from \(X\) = distance from \(Y\) = 1 unit
    
    (b) \(Z = \left(\frac{1}{4}, 0\right)\)
    
    (c) \(1\frac{1}{4}\) units

15. Midpoint \(AB\): \(W = \left(2, -1\frac{1}{2}\right)\)
    Midpoint \(BC\): \(X = (-2, -3)\)
    Midpoint \(CD\): \(Y = \left(-4\frac{1}{2}, \frac{1}{2}\right)\)
    Midpoint \(AD\): \(Z = \left(-\frac{1}{2}, \frac{3}{2}\right)\)
    \(m_{WX} = m_{ZY} = \frac{3}{8}\)
    So \(WX\) \(\parallel ZY\)
    \(m_{XY} = m_{WZ} = -\frac{7}{5}\)
    So \(XY\) \(\parallel WZ\)
    \(WXYZ\) is a parallelogram.

Test yourself 1

1. (a) \(AB = AC\) (given)
    So \(BD = EC\) (midpoints)
    \(\angle DBC = \angle ECB\) (base \(\angle\)'s in isosceles \(\Delta\))
    \(BC\) is common
    \(\therefore \triangle ABC \equiv \triangle EBC\) (SAS)
    (b) \(\because BE = DC\) (corresponding sides in \(\triangle\))

2. \(c^2 = a^2 + b^2\)
   
   \[
   c = \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2} = \sqrt{\frac{x^2 + y^2}{4}} = \frac{\sqrt{x^2 + y^2}}{2}
   \]

3. \(AB = \sqrt{(7 - 4)^2 + (-5 - (-1))^2} = 5\)
    \(BC = \sqrt{(7 - (-1))^2 + (-5 - 3)^2} = 10\)
    \(AC = \sqrt{(4 - 1)^2 + (-1 - 3)^2} = 5\)
    Since \(AB = AC \neq BC\), \(\triangle ABC\) is isosceles.

4. \(h = \frac{50 - \pi r^2}{\pi r}\)

5. (a) \(\angle ADE = 45^\circ\) (corresponding \(\angle\)'s \(BC||AD\))
    \(\angle EAD = 90^\circ - 45^\circ\) (\(\angle\)'s of \(\triangle\))
    \(= 45^\circ\)
    So \(\triangle ADE\) is isosceles.
    (b) \(AE = DE = y\) (isosceles \(\triangle\))
    \(CD = y\) (\(CD = DE\), given)
    \(AB = y\) (oppositesides in \(\parallel\) gram)
    \(AB \parallel CE\) (given)
    So \(ABCE\) is a trapezium.
    \(A = \frac{1}{2}h(a + b)\)
    \(= \frac{1}{2} \times y \times (y + 2y)\)
    \(= \frac{1}{2} \times 3y \times 3y\)
    \(3y^2\)
    \(= \frac{3}{2}\)

6. (a) \(\frac{CB}{BA} = \frac{CG}{GF}\) (equal ratio of intercepts)
    \(\frac{CG}{GF} = \frac{CD}{DE}\) (similarly)
    \(\therefore \frac{CB}{BA} = \frac{CD}{DE}\)
    (b) \(20.4\) cm

7. \(m_{AC} = -\frac{-4 - (-4)}{-2 - 5} = 0\)
    \(m_{CD} = -\frac{-4 - 3}{5 - 6} = 7\)
    \(m_{AB} = -\frac{-4 - 3}{2 - 1} = 7\)
    \(BC \parallel \triangle AD, CD \parallel AB\)
    So \(ABCD\) is a parallelogram.

8. \(DC^2 + BC^2 = 12^2 + 5^2\)
    \(= 144 + 25\)
    \(= 169\)
    \(= 13^2\)
    \(= DB^2\)
    \(\because \angle C = 90^\circ\) (Pythagoras)
    So \(ABCD\) is a rectangle.

9. \(\frac{9.18}{3.4} = \frac{5.13}{1.9} = 2.7\)
    \(\because \angle Y = \angle P = 39^\circ\) (given)
    \(\therefore \triangle PQR \parallel \triangle AWXY\)
    (two pairs of sides in proportion, with included \(\angle\)'s equal)

10. Let \(\triangle BAC = x\)
    Then \(\angle DAC = x\) (given—\(AC\) bisects \(\angle BAD\))
    \(\triangle DCA = x\) (base \(\angle\)'s of isosceles \(\triangle DAC\))
    \(\therefore \angle BAC = \angle DCA\)
These are equal alternate angles.

\[ \therefore \angle EDA = 2x \quad (\text{exterior } \angle \text{ of } \triangle DAC) \]

\[ \angle DEA = \angle DAE \quad (\text{base } \angle \text{ of isosceles } \triangle AED) \]

\[ \angle DAE = \frac{180^\circ - 2x}{2} \quad (\angle \text{ sum of isosceles } \triangle) \]

\[ = 90^\circ - x \]

\[ \angle EAC = 90^\circ - x + x \]

\[ = 90^\circ \]

\[ \therefore \angle EAC = \angle ACB \]

These are equal alternate angles.

\[ \therefore AE \parallel CB \]

So \(ABCE\) is a parallelogram.

11. \((-1, 0)\)

12. (a) \(4x - 3y - 1 = 0\)
(b) 2.4 units
(c) 12 units\(^2\)

13. (a) \(AB = AD\) \(\quad\) (given)
\(BC = DC\) \(\quad\) (given)
AC is common
\[ \therefore \triangle ABC \cong \triangle ADC \quad (SSS) \]
(b) \(AB = AD\) \(\quad\) (given)
\[ \angle BAE = \angle DAE \quad (\text{corresponding } \angle \text{s in } \cong \triangle) \]
\(AE\) is common
\[ \therefore \angle ABE \cong \angle ADE \quad (SAS) \]
(c) \(BE = DE\) \(\quad\) (corresponding sides in \(\cong \triangle\))
\[ \therefore AC\text{ bisects } BD \]
\[ \angle BEA + \angle DEA = 180^\circ \quad (\text{straight } \angle) \]
\[ \therefore \angle BEA = \angle DEA = 90^\circ \]
So AC is perpendicular to BD.

14. (a) \(AB: m_1 = \frac{1}{2}\) \(BC: m_2 = -2\)
\[ m_1 m_2 = \frac{1}{2} \times -2 = -1 \]
So \(AB\) and \(BC\) are perpendicular.
(b) \((3, -2)\)
(c) \((3, \frac{1}{2})\)
(d) 5 units

15. (a) \(500 = 4x^2 + 6xh\)
\[ 500 - 4x^2 = 6xh \]
\[ 250 - 2x^2 = 3xh \]
\[ \frac{250 - 2x^2}{3x} = h \]
(b) \(V = 2x^2h\)
\[ = 2x^2 \left( \frac{250 - 2x^2}{3x} \right) \]
\[ = 2x \left( \frac{250 - 2x^2}{3} \right) \]
\[ = \frac{500x - 4x^3}{3} \]

**Challenge exercise 1**

1. \(BD\) is common
\[ \angle ADB = \angle CDB = 90^\circ \quad \text{given} \]
\(AD = DC\) \(\quad\) (\(BD\) bisects \(AC\)---given)
\[ \therefore \triangle ADB \cong \triangle CDB \]
So \(\triangle ABC\) is isosceles.

2. (a) \(AD = \frac{1}{2}AB\)
\[ \therefore \frac{AD}{AB} = \frac{1}{2} \]
\(AE = \frac{1}{2}AC\)
\[ \therefore \frac{AE}{AC} = \frac{1}{2} \]
\(AB = AC\)
\[ \therefore \frac{AB}{AC} = 1 \]
\[ \angle A\text{ is common} \]
Since two pairs of sides are in proportion and their included angles are equal,
\(\triangle ADE \parallel \triangle ABC\)
\[ \therefore \angle ADE = \angle ABC \]
These are equal corresponding angles
\[ \therefore DE \parallel BC \]
(b) Since \(\triangle ADE \parallel \triangle ABC\),
\[ \frac{AD}{DE} = \frac{AB}{BC} = \frac{1}{2} \]
\[ \therefore DE = \frac{1}{2}BC \]

3. Let \(ABCD\) be a rhombus with \(AD = DC\)  
To prove: \(\angle ADE = \angle CDE\)

**Proof**
\[ AD = DC \quad \text{(given)} \]
\[ \therefore \triangle ADC\text{ is isosceles} \]
\[ \therefore \angle DAE = \angle DCE \]
\(AE = EC\) \(\quad\) (diagonals bisect each other)
\[ \therefore \angle ADE = \angle CDE \]
\[ \therefore \angle ADE = \angle CDE \quad \text{(corresponding angles in congruent } \triangle) \]
(Note: We can prove other pairs of angles equal similarly.)

4. 1189 mm \(\times\) 841 mm

5. \(S = 2x^2 + \frac{4000}{x}\)
6. Each angle = \( \frac{(n - 2) \times 180^\circ}{n} \)  
\[ = \frac{180n - 360^\circ}{n} \]
\[ = 180^\circ - \frac{360^\circ}{n} \]

\[ \angle EAF = \angle ACD \] (alternate angles, \( AB \parallel DC \))
\[ \angle ADC = \angle ABC \] (opposite angles in gram)
\[ \angle ABC = \angle AEF \] (corresponding angles, \( BC \parallel EF \))
\[ \therefore \angle ADC = \angle AEF \]
Since 2 pairs of angles equal, third is equal by angle sum of \( \Delta \)
\[ \therefore \Delta AEF \parallel \Delta ADC \]

\[ \frac{CP}{PA} = \frac{GR}{RB} = \frac{1}{1} \] (\( P \) and \( R \) are midpoints)
\[ \therefore PR \parallel AB \] (equal ratios on \( \parallel \) lines)
Similarly \( PQ \parallel CB \) and \( QR \parallel AC \)
\[ \angle QPR = \angle QRC \] (alternate \( \angle s \), \( PQ \parallel CB \))
\[ \angle CPR = \angle PQR \] (alternate \( \angle s \), \( AC \parallel RQ \))
\[ PR \] is common
\[ \therefore \text{by AAS}, \Delta PQR \equiv \Delta CPR \]

10. \[ 188 \text{ mm} \]

12. \[ (P, S \text{ are midpoints}) \]
\[ \angle D \text{ is common} \]
Since 2 pairs of sides are in proportion and the included angles are equal,
\[ \Delta DPS \parallel \Delta DAC \]
\[ \frac{DP}{PA} = \frac{DS}{AC} = \frac{1}{2} \] (\( P \), \( S \) are midpoints)
\[ \therefore PS \parallel AC \] (equal ratios on \( \parallel \) lines)
Similarly, \( \frac{BQ}{QA} = \frac{BR}{RC} \)
\[ \therefore QR \parallel AC \]
\[ \therefore PS \parallel QR \]
\[ \frac{AP}{PD} = \frac{AQ}{QD} = \frac{1}{1} \] (\( P \), \( Q \) are midpoints)
\[ \therefore PQ \parallel DB \] (equal ratios on \( \parallel \) lines)
Similarly \( SR \parallel DB \)
\[ \therefore PQ \parallel SR \]
Since \( PS \parallel QR \), \( PQ \parallel SR \).
\( PQRS \) is a parallelogram.

13. \( 70 \text{ cm} \)

**Chapter 2: Geometrical applications of calculus**

**Problem**

\((-0.25, -1.125)\)

**Exercises 2.1**

1. \[ (a) \]

\[ \frac{AP}{PD} = \frac{AQ}{QD} = \frac{1}{1} \] (\( P \), \( Q \) are midpoints)
\[ \therefore PQ \parallel DB \] (equal ratios on \( \parallel \) lines)
Similarly \( SR \parallel DB \)
\[ \therefore PQ \parallel SR \]
Since \( PS \parallel QR \), \( PQ \parallel SR \).
\( PQRS \) is a parallelogram.
2. \( x < \frac{1}{4} \)  
3. \( x < 0 \)
4. (a) \( x < 1.5 \)  (b) \( x > 1.5 \)  (c) \( x = 1.5 \)
5. \( f'(x) = -2 < 0 \) for all \( x \)
6. \( y' = 3x^2 > 0 \) for all \( x \neq 0 \)
7. \((0, 0)\)  
8. \( x = -3, 2 \)
9. (a) \((1, -4)\)  (b) \((0, 9)\)
    (c) \((1, 1)\) and \((2, 0)\)  (d) \((0, 1), (1, 0)\) and \((-1, 0)\)
10. \((2, 0)\)  
11. \(-1 < x < 1\)  
12. \(x < -3, x > 3\)
13. (a) \(x = 2, 5\)  (b) \(2 < x < 5\)  (c) \(x < 2, x > 5\)
14. \( p = -12 \)
15. \( a = \frac{1}{2}, b = -6 \)
16. (a) \( \frac{dy}{dx} = 3x^2 - 6x + 27 \)
    (b) The quadratic function has \( a > 0 \)
    \( b^2 - 4ac = -288 < 0 \)
    So \( 3x^2 - 6x + 27 > 0 \) for all \( x \)
    The function is monotonic increasing for all \( x \).
22. \( y = \frac{1}{x^2} \) \( \frac{dy}{dx} = -\frac{2}{x^3} \)

23. (2, 0) and \( \left( \frac{2}{3}, \frac{13}{81} \right) \)

24. \( \frac{x}{2\sqrt{x} + 1} + \sqrt{x + 1} = \frac{3x + 2}{2\sqrt{x} + 1} \left( \frac{2}{3} - \frac{2\sqrt{3}}{9} \right) \)

25. \( a = -1.75 \)

26. \( \frac{1}{2\sqrt{x}} \neq 0 \) 27. \( \frac{-3}{x^3} \neq 0 \)

Exercises 2.2
1. (0, -1); \( y' < 0 \) on LHS, \( y' > 0 \) on RHS
2. (0, 0) minimum 3. (0, 2) inflexion
4. (-2, 11); show \( f'(x) > 0 \) on LHS and \( f'(x) < 0 \) on RHS.
5. (-1, -2) minimum 6. (4, 0) minimum
7. (0, 5) maximum, (4, -27) minimum
8. \( f'(0) = 0 \), \( f'(x) > 0 \) on LHS and RHS
9. (0, 5) maximum, (2, 1) minimum
10. (0, -3) maximum, (1, -4) minimum, (-1, -4) minimum
11. (1, 0) minimum, (-1, 4) maximum
12. \( m = -6 \frac{1}{12} \)
13. \( x = -3 \) minimum 14. \( x = 0 \) minimum, \( x = -1 \) maximum
15. \( x = 1 \) inflexion, \( x = 2 \) minimum
16. (a) \( \frac{dP}{dx} = 2 - \frac{50}{x^2} \)
   (b) (5, 20) minimum, (-5, -20) maximum
17. \( \left( 1, \frac{1}{2} \right) \) minimum
18. (2.06, 54.94) maximum, (-20.6, -54.94) minimum
19. (4.37, 54.92) minimum, (-4.37, -54.92) maximum
20. (a) \( \frac{dA}{dx} = \sqrt{3600 - x^2} - \frac{x^2}{\sqrt{3600 - x^2}} \)
   \[ = \frac{3600 - 2x^2}{\sqrt{3600 - x^2}} \]
   (b) (42.4, 1800) maximum, (-42.4, -1800) minimum

Exercises 2.3
1. \( 7x^4 - 10x^4 + 4x^3 - 1 \); \( 42x^4 - 40x^3 + 12x^2 \);
   \( 210x^4 - 120x^2 + 3x^2 \); \( 840x^3 - 240x + 24 \)
2. \( f''(x) = 72x^2 \) 3. \( f'(x) = 10x^4 - 3x^2 \), \( f''(x) = 40x^4 \) - 6x
3. \( f'(1) = 11 \), \( f''(-2) = 168 \)
4. \( f'(x) = -x^2 + 6x - 5 \)
5. \( f''(2) = 20x - 6 \)
6. \( \frac{dy}{dx} = 4x - 3 \), \( \frac{d^2 y}{dx^2} = 4 \)
7. \( f''(-1) = -16 \), \( f''(2) = 40 \)
8. \( -4x^3 - 20x \)
9. \( s'''(4) = \frac{1}{32} \) 10. \( \frac{d^2 h}{dt^2} = 26 \) when \( t = 1 \)
11. \( x = \frac{7}{18} \) 12. \( x > \frac{1}{3} \)
13. 20(4x - 3)^4 320(4x - 3)^3
14. \( f''(x) = -2 \frac{x}{\sqrt{2}} \)
15. \( f''(x) = \frac{1}{4\sqrt{(x-3)^3}} \)
16. \( \frac{d^2 y}{dt^2} = 24t + 16 \)
17. \( b = \frac{2}{3} \)
18. \( f''(2) = \frac{3}{4\sqrt{2}} \)
19. \( f''(1) = 196 \) 20. \( b = -2.7 \)

Exercises 2.4
1. \( x > -\frac{1}{3} \) 2. \( x < 3 \) 3. \( y'' = -8 < 0 \) 4. \( y'' = 2 > 0 \)
5. \( x < 2 \frac{1}{3} \) 6. (1, 9) 7. (1, -17) and (-1, -41)
8. (0, -2); \( y'' < 0 \) on LHS, \( y'' > 0 \) on RHS
9. \( -2 < x < 1 \)
10. (a) No—minimum at (0, 0)
    (b) Yes—inflexion at (0, 0)
    (c) Yes—inflexion at (0, 0)
    (d) Yes—inflexion at (0, 0)
    (e) No—minimum at (0, 0)
11.
12. None: (2, 31) is not an inflexion since concavity does not change.

14. \( f''(x) = \frac{12}{x^4} \)
   \[ x^4 > 0 \text{ for all } x \neq 0 \]
   So \( \frac{12}{x^4} > 0 \text{ for all } x \neq 0 \)
   So the function is concave upward for all \( x \neq 0 \).

15. (a) (0, 7), (1, 0) and \((-1, 14)\) (b) (0, 7)

16. (a) \( \frac{d^2y}{dx^2} = 12x^2 + 24 \)
   \[ x^2 \geq 0 \text{ for all } x \]
   So \( 12x^2 \geq 0 \text{ for all } x \)
   \( 12x^2 + 24 \geq 24 \)
   So \( 12x^2 + 24 \neq 0 \) and there are no points of inflexion.
   (b) The curve is always concave upwards.

17. \( a = 2 \)

18. \( p = 4 \)

19. \( a = 3, b = -3 \)

20. (a) \((-8, 0\), (2, 2)\)
   (b) \( \frac{dy}{dx} = 6x^4 - 15x^4 + 21 \)
   At \((0, -8): \frac{dy}{dx} = 6(0)^4 - 15(0)^4 + 21 = 21 \neq 0 \)
   At \((2, 2): \frac{dy}{dx} = 6(2)^4 - 15(2)^4 + 21 = -27 \neq 0 \)
   So these points are not horizontal points of inflexion.

Exercises 2.5

1. (a) \( \frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0 \) (b) \( \frac{dy}{dx} < 0, \frac{d^2y}{dx^2} < 0 \)
   (c) \( \frac{dy}{dx} > 0, \frac{d^2y}{dx^2} < 0 \) (d) \( \frac{dy}{dx} < 0, \frac{d^2y}{dx^2} > 0 \)
   (e) \( \frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0 \)

2. (a) \( \frac{dp}{dt} > 0, \frac{d^2p}{dt^2} < 0 \) (b) The rate is decreasing.
6. \( M \)

7. \( \frac{dM}{dt} < 0, \frac{d^2 M}{dt^2} > 0 \)

8. (a) The number of fish is decreasing.
    (b) The population rate is increasing.
    (c)

9. The level of education is increasing, but the rate is slowing down.

10. The population is decreasing, and the population rate is decreasing.

Exercises 2.6

1. (1, 0) minimum  2. (0, 1) minimum

3. (2, -5), \( y'' = 6 > 0 \)  4. (0, 0.25), \( y'' < 0 \) so maximum

5. (0, -5); \( f''(x) = 0 \) at (0, -5), \( f''(x) < 0 \) on LHS, \( f''(x) > 0 \) RHS

6. Yes—inflexion at (0, 3)

7. (-2, -78) minimum, (-3, -77) maximum

8. (0, 1) maximum, (-1, -4) minimum, (2, -31) minimum

9. (0, 1) maximum, (0.5, 0) minimum, (-0.5, 0) minimum

10. (a) (4, 176) maximum, (5, 175) minimum
    (b) (4.5, 175.5)

11. (3.67, 0.38) maximum

12. (0, -1) minimum, (-2, 15) maximum, (-4, -1) minimum

13. (a) \( a = \frac{2}{3} \) (b) maximum, as \( y'' < 0 \)

14. \( m = -5 \frac{1}{2} \)  15. \( a = 3, b = -9 \)
5. \[ y = \begin{cases} 2 & \text{for } -2 < x < 1 \\ -2 & \text{for } 1 < x < 3 \end{cases} \]

6. \[ y = \begin{cases} 2 & \text{for } -2 < x < 1 \\ -2 & \text{for } 1 < x < 3 \end{cases} \]

7. \[ y = \begin{cases} 40 & \text{for } -1 < x < 4 \\ -80 & \text{for } 4 < x < 5 \end{cases} \]

8. (a) \((0, -7)\) minimum, \((-4, 25)\) maximum
   (b) \((-2, 9)\)
   (c) \((-4, 25)\)

9. \[ y = \begin{cases} 10 & \text{for } -3 < x < 1 \\ -30 & \text{for } 1 < x < 5 \end{cases} \]

10. \[ y = \begin{cases} 10 & \text{for } -3 < x < 1 \\ -30 & \text{for } 1 < x < 5 \end{cases} \]

11. \[ y = \begin{cases} 10 & \text{for } -3 < x < 1 \\ -30 & \text{for } 1 < x < 5 \end{cases} \]

12. \[ y = \begin{cases} 10 & \text{for } -3 < x < 1 \\ -30 & \text{for } 1 < x < 5 \end{cases} \]
13. 
\[ \frac{dy}{dx} = \frac{-2}{(1 + x)^2} \neq 0 \]

14. 
\[ (0, 16) \]

15. 
\[ (4, 8) \]

Exercises 2.8

1. Maximum value is 4.

2. Maximum value is 9, minimum value is −7.

3. Maximum value is 25.

4. Maximum value is 86, minimum value is −39.

5. Maximum value is −2.

6. Maximum value is 5, minimum value is −16 \frac{1}{3}.

7. Absolute maximum 29, relative maximum −3, absolute minimum −35, relative minimum −35, −8

8. Minimum −25, maximum 29
9. Maximum 3, minimum 1

10. Maximum ∞, minimum −∞

--

Problem

The disc has radius \( \frac{30}{7} \) cm. (This result uses Stewart’s theorem—check this by research.)

Exercises 2.9

1. \( A = xy \)

\[
\begin{align*}
50 &= xy \\
\therefore 50 &= x y \\
P &= 2x + 2y \\
&= 2x + 2 \times \frac{50}{x} \\
&= 2x + \frac{100}{x}
\end{align*}
\]

2. \( 2x + 2y = 120 \)

\[
\begin{align*}
y &= 60 - x \\
A &= xy \\
&= x(60 - x) \\
&= 60x - x^2
\end{align*}
\]

3. \( xy = 20 \)

\[
\begin{align*}
y &= \frac{20}{x} \\
S &= x + y \\
&= x + \frac{20}{x}
\end{align*}
\]

4. \( V = \pi r^2 h \)

\[
\begin{align*}
400 &= \pi r^2 h \\
\frac{400}{\pi r^2} &= h \\
S &= 2\pi r^2 + 2\pi rh \\
&= 2\pi r^2 + 2\pi r \left( \frac{400}{\pi r^2} \right) \\
&= 2\pi r^2 + \frac{800}{r}
\end{align*}
\]

5. (a) \( x + y = 30 \)

\( \therefore \ y = 30 - x \)

(b) The perimeter of one square is \( x \), so its side is \( \frac{1}{4} \). The other square has side \( \frac{1}{4} y \).

\[
\begin{align*}
A &= \left( \frac{1}{4} x \right)^2 + \left( \frac{1}{4} y \right)^2 \\
&= \frac{x^2}{16} + \frac{y^2}{16} \\
&= \frac{x^2 + y^2}{16} \\
&= \frac{x^2 + (30 - x)^2}{16} \\
&= \frac{x^2 + 900 - 60x + x^2}{16} \\
&= \frac{2x^2 - 60x + 900}{16} \\
&= \frac{2(x^2 - 30x + 450)}{16} \\
&= \frac{x^2 - 30x + 450}{8}
\end{align*}
\]

6. (a) \( x^2 + y^2 = 280 \)

\( y = \sqrt{78 \ 400 - x^2} \)

(b) \( A = xy \)

\( y = \sqrt{78 \ 400 - x^2} \)

7. \( 10 - 2x \)

\( V = x(10 - 2x)(7 - 2x) = x(70 - 20x - 14x + 4x^2) = x(70 - 34x + 4x^2) = 70x - 34x^2 + 4x^3 \)

8. Profit per person = Cost – Expenses

\[
\begin{align*}
&= (900 - 100x) - (200 + 400x) \\
&= 900 - 100x - 200 - 400x \\
&= 700 - 500x
\end{align*}
\]

For \( x \) people, \( P = x(700 - 500x) = 700x - 500x^2 \)
9. After \( t \) hours, Joel has travelled 75\( t \) km. He is \( 700 \leq 75t \) km from the town. After \( t \) hours, Nick has travelled 80\( t \) km. He is \( 680 \leq 80t \) km from the town.

\[
d = \sqrt{(700 - 75t)^2 + (680 - 80t)^2} \\
= \sqrt{490000 - 105000t + 5625t^2 + 640000 - 108800t + 6400t^2} \\
= \sqrt{1130000 - 108800t + 7400t^2}
\]

10. The river is 500 m, or 0.5 km, wide.

Distance \( AB \):
\[
\text{Speed} = \frac{\text{distance}}{\text{time}} \\
\therefore \text{Time} = \frac{\text{distance}}{\text{speed}} \\
t = \frac{\sqrt{x^2 + 0.25}}{5}
\]

Distance \( BC \):
\[
d = 7 - x \\
\text{Time} = \frac{\text{distance}}{\text{speed}} \\
t = \frac{7 - x}{4}
\]

So total time taken is:
\[
t = \frac{\sqrt{x^2 + 0.25}}{5} + \frac{7 - x}{4}
\]

Exercises 2.10

1. 2 s, 16 m  2. 7.5 km

3. \( 2x + 2y = 60 \)
   \[
   2y = 60 - 2x \\
y = 30 - x \\
A = xy \\
= x(30 - x) \\
= 30x - x^2
   \]
Max. area 225 \( \text{m}^2 \)

4. (a) \( A = xy = 4000 \)
   \[
   \therefore y = \frac{4000}{x} \\
P = 2x + 2y \\
= 2x + 2 \left( \frac{4000}{x} \right) \\
= 2x + \frac{8000}{x}
   \]
(b) 63.2 m by 63.2 m
(c) \$12 332.89

5. 4 m by 4 m  6. 14 and 14  7. 2.5 and 2.5

8. \( x = 1.25 \) m, \( y = 1.25 \) m

9. (a) \( V = x(30 - 2x)(80 - 2x) \)
   \[
   = x(2400 - 220x - 4x^2) \\
   = 2400x - 220x^2 + 4x^3
   \]
(b) \( x = \frac{2}{3} \) cm
(c) 7407.4 \( \text{cm}^3 \)

10. \( V = \pi r^2 h = \frac{54\pi}{r^2} \)
    \[
    h = \frac{54}{r^2} \\
    = \frac{54}{54} \\
    = r \\
    S = 2\pi(r + h) \\
    = 2\pi \left( \frac{r + \frac{54}{r}}{r} \right) \\
    = 2\pi r^2 + 108\pi
    \]
Radius is 3 m.

11. (a) \( S = 2\pi r^2 + 17\frac{200}{r} \)
    (b) 2323.7 \( \text{m}^2 \)

12. 72 \( \text{cm}^2 \)

13. (a) \( xy = 400 \)
    \[
    \therefore y = \frac{400}{x} \\
    A = (y - 10)(x - 10) \\
    = xy - 10y - 10x + 100 \\
    = x \left( \frac{400}{x} \right) - 10 \left( \frac{400}{x} \right) - 10x + 100 \\
    = 400 - \frac{4000}{x} - 10x + 100 \\
    = 500 - 10x - \frac{4000}{x}
    \]
(b) 100 \( \text{cm}^2 \)

14. 20 cm by 20 cm by 20 cm  15. 1.12 \( \text{m}^3 \)

16. (a) 7.5 m by 7.5 m  (b) 2.4 m

17. 301 \( \text{cm}^2 \)  18. 160\( \frac{1}{6} \) \( \text{cm}^2 \)  19. 1.68 cm, 1.32 cm

20. \( d^2 = (200 - 800)^2 + (120 - 60t)^2 \)
    \[
    = 40000 - 32000t + 6400t^2 + 14400 - 14400t + 3600t^2 \\
    = 10000t^2 - 46400t + 54400 \\
    = 24 \text{ km}
    \]

21. (a) \( d = (x^2 + 2x + 5) - (4x - x^2) \)  (b) \( \frac{1}{2} \) \( \text{unit} \)
    \[
    = x^2 - 2x + 5 - 4x + x^2 \\
    = 2x^2 - 6x + 5
    \]
22. (a) Perimeter = \(2x + y + \frac{1}{2}(2\pi)\) where \(r = \frac{y}{2}\)
\[1200 = 2x + y + \frac{1}{2}(2\pi \times \frac{y}{2})
\]
\[= 2x + y + \frac{\pi y}{2}
\]
\[1200 - y - \frac{\pi y}{2} = 2x
\]
\[600 - \frac{\pi y}{4} = x
\]
\[2400 - 2y - \frac{\pi y}{2} = x
\]
(b) \(A = xy + \frac{1}{2}\pi r^2\)
\[= \left(\frac{2400 - 2y - \pi y}{4}\right) + \frac{1}{2}\pi \left(\frac{y}{2}\right)^2
\]
\[= \frac{2400y - 2y^2 - \pi y^2}{4} + \frac{\pi y^2}{8}
\]
\[= \frac{4800y - 4y^2 - 2\pi y^2}{8} + \frac{\pi y^2}{8}
\]
\[= \frac{4800y - 4y^2 - \pi y^2}{8}
\]
(c) \(x = 168\) m, \(y = 336\) m

23. (a) Equation \(AB:\)
\[y = mx + b
\]
\[= \frac{b}{a}x + b
\]
Substitute \((-1, 2)\)
\[2 = \frac{b}{a}(-1) + b
\]
\[= \frac{b}{a} + b
\]
\[2a = -b + ab
\]
\[= b(-1 + a)
\]
\[= b(a - 1)
\]
\[\frac{2a}{a - 1} = b
\]
(b) \(a = 2, b = 4\)

24. 26 m

25. (a) \(s = \frac{d}{t}\)
So \(t = \frac{d}{s}\)
\[= \frac{1500}{s}
\]
Cost of trip taking \(t\) hours:
\(C = (s^2 + 9000)t\)
\[= (s^2 + 9000) \times \frac{1500}{s}
\]
\[= 1500s + \frac{9000 \times 1500}{s}
\]
\[= 1500\left(s + \frac{9000}{s}\right)
\]
(b) 95 km/h
(c) $2846

Exercises 2.11
1. (a) \(x^2 - 3x + C\)  (b) \(\frac{x^3}{3} + 4x^2 + x + C\)

2. (a) \(f(x) = 2x^3 - \frac{x^2}{2} + C\)  (b) \(f(x) = \frac{x^4}{5} - x^3 + 7x + C\)

3. (a) \(y = x^6 - 9x + C\)  (b) \(y = \frac{-x^3}{3} + 2x^{-1} + C\)

4. (a) \(\frac{2\sqrt{x^3}}{3} + C\)  (b) \(-\frac{x^2}{2} + C\)

5. \(y = \frac{x^4}{4} - x^3 + 5x - \frac{1}{4}\)
6. \(f(x) = 2x^2 - 7x + 11\)

7. \(f(1) = 8\)  8. \(y = 2x - 3x^2 + 19\)  9. \(x = 16\frac{1}{3}\)

10. \(y = 4x^2 - 8x + 7\)  11. \(y = 2x^3 + 3x^2 - 2\)
12. \(f(x) = x^3 - x^2 - x + 5\)
13. \(f(2) = 20.5\)
14. \(\frac{x^3}{3} + \frac{x^2}{2} - 12x + 24\frac{1}{2}\)
15. \(\frac{4x^3}{3} - 15x - 14\frac{1}{3}\)
16. \(\frac{x^3}{3} - 2x^2 + 3x - \frac{2}{3}\)
17. \(f(x) = x^3 - x^2 + 2x^2 - 4x - 2\)
18. \(y = 3x^2 + 8x + 8\)
19. \(f(-2) = 77\)
20. \(y = 0\)

Test yourself 2
1. \((-3, -11)\) maximum, \((-1, -15)\) minimum
2. \(x > -\frac{1}{6}\)
3. \(y = 2x^3 + 6x^2 - 5x - 33\)
4. (a) \(-8\) (b) 26 (c) \(-90\) 5. 50 m
6. \((0, 0)\) minimum 7. \(x > -1\)
8. (a) \((0, 1)\) maximum, \((-4, -511)\) minimum, 
\((2, -79)\) minimum

(b) 
\[
\begin{align*}
\text{Minimum} & : (2, -79) \\
\text{Maximum} & : (0, 1)
\end{align*}
\]

9. \(\left(\frac{1}{2}, -1\right)\) 

10. \(f(x) = \frac{5x^3}{2} + 6x^2 - 49x + 59\)

11. (a) 
\[V = \pi \frac{2}{3} h\]
\[\frac{375}{\pi^2} = h\]

(b) \(3.9\) cm

12. (a) \((0, 0)\) and \((-1, 1)\)
(b) \((0, 0)\) minimum, \((-1, 1)\) point of inflexion
(c) 
\[
\begin{align*}
\text{Minimum} & : (-1, 1) \\
\text{Maximum} & : (0, 0)
\end{align*}
\]

13. (a) 
\[
\begin{align*}
S &= 2x^2 + 4xh \\
250 &= 2x^2 + 4xh \\
250 - 2x^2 &= 4xh \\
\frac{4x}{2x} &= h \\
2(125 - x^2) &= h \\
\frac{125 - x^2}{2x} &= h \\
V &= x^2 \left( \frac{125 - x^2}{2x} \right) \\
&= x(125 - x^2) \\
&= 125x - x^3 \\
&= \frac{125x - x^3}{2}
\end{align*}
\]
(b) \(6.45\) cm \(\times 6.45\) cm \(\times 6.45\) cm

14. \(x < 3\)

15. \(y = x^3 + 3x^2 + 3\)

16. 150 products

17. For decreasing curve, \(\frac{dy}{dx} < 0\)
\[
\frac{dy}{dx} = -3x^2
\]
\(< 0 \quad \text{(since } x^2 > 0 \text{ for all } x \neq 0)\)

\(\therefore\) monotonic decreasing function

18. (a) 
\[
\begin{align*}
\text{Minimum} & : (0, 0) \\
\text{Point of inflexion} & : (-1, 1)
\end{align*}
\]

(b) 
\[
\begin{align*}
\text{Minimum} & : (0, 0) \\
\text{Point of inflexion} & : (-1, 1)
\end{align*}
\]

(c) 
\[
\begin{align*}
\text{Minimum} & : (0, 0) \\
\text{Point of inflexion} & : (-1, 1)
\end{align*}
\]

19. (a) \(x^2 + y^2 = 5^2 = 25\)
\(y^2 = 25 - x^2\)
\(y = \sqrt{25 - x^2}\)
\(A = \frac{1}{2} xy\)
\(= \frac{1}{2} x \sqrt{25 - x^2}\)
(b) \(6.25\) m\(^2\)

20. (a) \((4, -171)\) minimum, \((-6, 329)\) maximum 
(b) \((-1, 79)\)
(c) 
\[
\begin{align*}
\text{Minimum} & : (-6, 329) \\
\text{Maximum} & : (4, -171)
\end{align*}
\]

21. \(x < 1\)

22. \(f(x) = 2x^3 - 3x^2 - 31x + 68\)

23. \((0, 1)\) and \((3, -74)\)
24. 179

25.

![Graph](image)

**Challenge exercise 2**

1. \(\frac{20x^2 - 120x - 1}{(4x^2 + 1)^3}; \quad -\frac{8(60x^3 - 420x^2 - 9x + 15)}{(4x^2 + 1)^3}\)

2. 

![Graph](image)

3. \(x < -\frac{1}{2}; \quad x > 4\)

4. 16 m^2

5. 27; -20.25

6. \(f'(0.6) = f''(0.6) = 0\) and concavity changes

7. Show sum of areas is least when \(r = s = 12.5\)

8. \(25 \frac{5}{6}\)

9. (a) \(\frac{dy}{dx} = \frac{1}{2\sqrt{x - 1}} \neq 0\)

(b) Domain: \(x \geq 1\); range: \(y \geq 0\)

(c) 

![Graph](image)

10. \(r = 3.17\) cm, \(h = 6.34\) cm

11. \(y = \frac{x^3}{3} - x^2 - 15x - 1\)

12. 110 km/h

13. \(y = x^2 + 2x + 3\) (There may be other solution.)

14. (a) \(f(x) = \frac{x^4}{4} - \frac{2x^3}{3} - \frac{3x^2}{2}\)

(b) \((0, 0); \quad (3, -11 \frac{1}{4}); \quad (-1, -\frac{7}{12})\)

(c) 

![Graph](image)

15. 4 m \times 4 m \times 4 m

16. \(f(3) = -22 \frac{1}{6}\) 17. (a) -2  (b) -1

18. \(y' = 0\) at \((0, 0)\);

(a) \(y'' > 0\) on LHS and RHS

(b) \(y'' < 0\) on LHS, \(y'' > 0\) on RHS

19. \(21 \frac{1}{3}\) cm\(^3\)  20. (a) \((0, 1)\)  (b) \(k = 2, 4, 6, 8, \ldots\)

21. minimum \(-1\); maximum \(-\frac{1}{5}\)

22. 87 km/h\(^{-1}\)

---

**Chapter 3: Integration**

**Exercises 3.1**

1. 2.5  2. 10  3. 2.4  4. 0.225  5. (a) 28  (b) 22

6. 0.39  7. 0.41  8. 1.08  9. 0.75  10. 0.65

11. 0.94  12. 0.92  13. 7.51  14. 16.5  15. 650.2

**Exercises 3.2**

1. 48.7  2. 30.7  3. 1.1  4. 0.41  5. (a) 3.4475  (b) 3.4477

6. 2.75  7. 0.693  8. 1.93  9. 72

10. 5.25  11. 0.558  12. 0.347  13. 3.63  14. 7.87  15. 175.8

**Exercises 3.3**

1. 8  2. 10  3. 125  4. -1  5. 10  6. 54  7. \(3 \frac{1}{3}\)

8. 16  9. 50  10. 52 \(\frac{2}{3}\)  11. \(\frac{2}{3}\)  12. 21 \(\frac{1}{4}\)  13. 0

14. 4 \(\frac{2}{3}\)  15. 1 \(\frac{1}{4}\)  16. 4 \(\frac{1}{3}\)  17. 0  18. 2 \(\frac{1}{3}\)  19. 0

20. 6 \(\frac{2}{9}\)  21. 101 \(\frac{1}{4}\)  22. -12 \(\frac{3}{4}\)  23. 22 \(\frac{2}{3}\)  24. 2 \(\frac{1}{3}\)

25. 0.0126
Exercises 3.4

1. \( \frac{x^3}{3} + C \) 2. \( \frac{x^5}{2} + C \) 3. \( \frac{2x^6}{5} + C \) 4. \( \frac{m^2}{2} + m + C \)

5. \( \frac{t^3}{3} - 7t + C \) 6. \( \frac{h^8}{8} + 5h + C \) 7. \( \frac{y^2}{2} - 3y + C \)

8. \( x^2 + 4x + C \) 9. \( \frac{b^4}{3} + \frac{b^2}{2} + C \) 10. \( \frac{a^4}{4} - \frac{a^2}{2} - a + C \)

11. \( \frac{x^3}{3} + x^2 + 5x + C \) 12. \( x^4 - x^3 + 4x^2 - x + C \)

13. \( x^4 + \frac{x^7}{5} + \frac{x^2}{2} + C \) 14. \( \frac{3x^3}{8} - \frac{3x^7}{7} - 9x + C \)

15. \( \frac{x^4}{2} - \frac{x^2}{2} - 2x + C \) 16. \( \frac{x^4}{6} + \frac{x^7}{4} + 4x + C \)

17. \( \frac{4x^3}{3} - \frac{5x^2}{8} - 8x + C \) 18. \( \frac{3x^3}{5} - \frac{x^4}{2} + \frac{x^7}{2} + C \)

19. \( \frac{3x^3}{2} + \frac{5x^5}{3} - 4x + C \) 20. \( -x^3 - \frac{x^2}{2} - 2x^2 + C \)

21. \( \frac{1}{7x^2} + C \) 22. \( \frac{3x^3}{4} + C \) 23. \( \frac{x^4}{4} - x^3 + x^3 + C \)

24. \( x - 2x^2 + \frac{4x^4}{3} + C \) 25. \( \frac{x^3}{3} + \frac{3x^2}{2} - 10x + C \) 26. \( \frac{3}{x} + C \)

27. \( -\frac{1}{2x^2} + C \) 28. \( -\frac{4}{x} - x + \frac{3}{2x^2} - \frac{7}{4x^4} + C \)

29. \( \frac{y^4}{3} + \frac{y^6}{6} + 5y + C \) 30. \( \frac{t^4}{4} - \frac{t^3}{3} - 2t^2 + 4t + C \)

31. \( \frac{2\sqrt{x^3}}{3} + C \) 32. \( \frac{1}{2t^n} + C \) 33. \( \frac{3\sqrt{x^3}}{4} + C \)

34. \( \frac{2\sqrt{x^3}}{5} + C \) 35. \( \frac{2\sqrt{x^3}}{3} + x + C \)

Exercises 3.5

1. (a) (i) \( 3x^3 - 12x^2 + 16x + C \) (ii) \( \frac{(3x - 4)^3}{9} + C \)

(b) \( \frac{(x + 1)^5}{5} + C \) (c) \( \frac{(5x - 1)^9}{50} + C \)

(d) \( \frac{(3y - 2)^8}{24} + C \) (e) \( \frac{(4 + 3x)^3}{15} + C \)

(f) \( \frac{(7x + 8)^{11}}{91} + C \) (g) \( \frac{(1 - x)^7}{7} + C \)

(h) \( \frac{\sqrt{(2x - 5)^3}}{3} + C \) (i) \( \frac{2(3x + 1)^3}{9} + C \)

(j) \( -3(x + 7)^{-1} + C \) (k) \( \frac{1}{16(4x - 5)^2} + C \)

(l) \( \frac{3\sqrt{(4x + 3)^3}}{16} + C \) (m) \( -2(2 - x)^{\frac{3}{2}} + C \)

(n) \( \frac{2\sqrt{t + 3)^3}}{5} + C \) (o) \( \frac{2\sqrt{(5x + 2)^3}}{35} + C \)

2. (a) 288.2 (b) -1\( \frac{1}{4} \) (c) -\( \frac{1}{8} \) (d) 6\( \frac{2}{3} \) (e) \( \frac{1}{6} \) (f) \( \frac{1}{7} \)

(g) 4\( \frac{2}{3} \) (h) -\( \frac{1}{8} \) (i) \( \frac{1}{5} \) (j) \( \frac{3}{5} \)

Exercises 3.6

1. \( \frac{1}{3} \) units\(^2 \) 2. 36 units\(^2 \) 3. 4.5 units\(^2 \) 4. \( \frac{2}{3} \) units\(^2 \)

5. \( \frac{1}{6} \) units\(^2 \) 6. 14.3 units\(^2 \) 7. \( \frac{4}{3} \) units\(^2 \) 8. 0.4 units\(^2 \)

9. 8 units\(^2 \) 10. 24.25 units\(^2 \) 11. 2 units\(^2 \) 12. \( \frac{1}{3} \) units\(^2 \)

13. \( \frac{11}{3} \) units\(^2 \) 14. \( \frac{1}{6} \) units\(^2 \) 15. \( \frac{2}{3} \) units\(^2 \) 16. \( \frac{1}{3} \) units\(^2 \)

17. \( \frac{5}{3} \) units\(^2 \) 18. 18 units\(^2 \) 19. \( \pi = 3.14 \) units\(^2 \)

20. \( \frac{a^4}{2} \) units\(^2 \)

Exercises 3.7

1. \( 21\frac{1}{3} \) units\(^2 \) 2. 20 units\(^2 \) 3. \( \frac{4}{3} \) units\(^2 \)

4. 1.5 units\(^2 \) 5. \( \frac{1}{4} \) units\(^2 \) 6. \( \frac{2}{3} \) units\(^2 \)

7. \( \frac{10}{3} \) units\(^2 \) 8. \( \frac{1}{6} \) units\(^2 \) 9. \( \frac{7}{9} \) units\(^2 \)

10. 2 units\(^2 \) 11. \( \frac{11}{4} \) units\(^2 \) 12. 60 units\(^2 \)

13. 4.5 units\(^2 \) 14. \( \frac{1}{3} \) units\(^2 \) 15. 1.9 units\(^2 \)

Exercises 3.8

1. \( \frac{1}{3} \) units\(^2 \) 2. \( \frac{1}{3} \) units\(^2 \) 3. \( \frac{1}{6} \) units\(^2 \)

4. \( \frac{10}{3} \) units\(^2 \) 5. \( \frac{20}{3} \) units\(^2 \) 6. 8 units\(^2 \)

7. \( \frac{2}{3} \) units\(^2 \) 8. \( \frac{166}{3} \) units\(^2 \) 9. 0.42 units\(^2 \)

10. \( \frac{2}{3} \) units\(^2 \) 11. \( \frac{11}{12} \) units\(^2 \) 12. \( \frac{1}{3} \) units\(^2 \)

13. 36 units\(^2 \) 14. \(\frac{2}{3} \) units\(^2 \) 15. \( \pi - 2 \) units\(^2 \)

Problem

\( \frac{206\pi}{15} \) units\(^3 \)
Exercises 3.9

1. \( \frac{243\pi}{5} \) units\(^3\) 2. \( \frac{485\pi}{3} \) units\(^3\)
3. \( \frac{376\pi}{15} \) units\(^3\) 4. \( \frac{\pi}{2} \) units\(^3\) 5. \( \frac{39\pi}{2} \) units\(^3\) 6. \( \frac{758\pi}{3} \) units\(^3\)
7. \( \frac{2\pi}{3} \) units\(^3\) 8. \( \frac{992\pi}{5} \) units\(^3\) 9. \( \frac{5\pi}{3} \) units\(^3\) 10. \( \frac{9\pi}{2} \) units\(^3\)
11. \( \frac{27\pi}{2} \) units\(^3\) 12. \( \frac{64\pi}{3} \) units\(^3\) 13. \( \frac{16385\pi}{7} \) units\(^3\)
14. \( \frac{25\pi}{2} \) units\(^3\) 15. \( \frac{65\pi}{2} \) units\(^3\) 16. \( \frac{1023\pi}{5} \) units\(^3\)
17. \( \frac{5\pi}{3} \) units\(^3\) 18. \( 13\pi \) units\(^3\) 19. \( \frac{344\pi}{27} \) units\(^3\)
20. \( \frac{3\pi}{5} \) units\(^3\) 21. \( \frac{2\pi}{5} \) units\(^3\) 22. \( \frac{72\pi}{5} \) units\(^3\)

23. \( y = \sqrt{r^2 - x^2} \)
\( y^2 = r^2 - x^2 \)
\( V = \pi \int_0^r y^2 \, dx \)
\( = \pi \int (r^2 - x^2) \, dx \)
\( = \pi \left[ r^2 x - \frac{x^3}{3} \right] \)
\( = \pi \left[ r^2 \left( -\frac{r}{3} \right) \right] \)
\( = \pi \left( \frac{2r^3}{3} \right) \)
\( = \frac{4\pi r^3}{3} \) units\(^3\)

Test yourself 3

1. (a) 0.535 (b) 0.5
2. (a) \( \frac{3x^2}{2} + x + C \) (b) \( \frac{5x^2}{2} - x + C \) (c) \( \frac{2\sqrt{x^3}}{3} + C \)
   (d) \( \frac{(2x + 5)^3}{16} + C \)
3. 14.83 4. (a) 2 (b) 0 (c) \( 2\frac{1}{5} \)
5. (a)
(b)
6. 3 units\(^2\) 7. 1.1 units\(^2\)
8. \( \frac{2\pi}{3} \) units\(^2\) 9. \( 9\pi \) units\(^3\) 10. \( 4\frac{1}{2} \) units\(^2\) 11. \( \frac{3}{4} \) units\(^2\)
12. \( \frac{(7x + 3)^3}{84} + C \) 13. 3 units\(^2\) 14. (a) \( \frac{206\pi}{15} \) units\(^3\)
   (b) \( \frac{\pi}{2} \) units\(^3\) 15. (a) \( x = \pm \sqrt{y} - 3 \) (b) \( 3\frac{2}{3} \) units\(^2\)
   (c) \( \frac{5\pi}{2} \) units\(^3\)
16. 36 17. \( 85\frac{1}{3} \) units\(^2\) 18. \( \frac{3\pi}{5} \) units\(^3\)
19. (a) \( \frac{(2x - 1)^3}{2} + C \) (b) \( x^\frac{3}{8} + C \)

Challenge exercise 3

1. (a) \( \frac{1}{12} \) units\(^2\) (b) \( \frac{2\pi}{35} \) units\(^3\)
2. (a) Show \( f(-x) = -f(x) \) (b) 0 (c) 12 units\(^2\)
3. 27.2 units\(^3\) 4.9 units\(^3\) 5. (a) \( 36x^3(x^4 - 1)^8 \)
   (b) \( \frac{(x^4 - 1)^9}{36} + C \)
6. (a) \( \frac{-22x}{(3x^2 - 4)^2} \) (b) \( \frac{1}{8} \) 7. 7.35 units\(^2\) 8. \( \frac{2\pi}{3} \) units\(^3\)
9. \( f(0) = \frac{1}{0} = \infty \)
10. (a)
(b) 3.08 units\(^2\)
11. \( 17\sqrt{17} \) units\(^2\) 12. \( \frac{215\pi}{6} \) units\(^3\)
13. (a) \( \frac{3x + 6}{2\sqrt{x + 3}} = \frac{3(x + 2)}{2\sqrt{x + 3}} + C \) (b) \( \frac{2x\sqrt{x + 3}}{3} + C \)
14. (a) \( 6\frac{2}{3} \) (b) \( 6\frac{2}{3} \)
15. 5 \( \frac{12}{12} \) units\(^2\)
16. (a) \( 8a^2 \) units\(^2\) (b) \( 2\pi a^3 \) units\(^3\)
1. Let $ABCD$ be a parallelogram with diagonal $AC$.

\[ \angle ACD = \angle BAC \quad \text{(alternate } \angle A, AB || DC) \]

\[ \angle DAC = \angle BCA \quad \text{(alternate } \angle A, AD || BC) \]

$AC$ is common

\[ \therefore \angle ACD = \angle BAC \quad \text{(by AAS)} \]

\[ \therefore \angle DAC = \angle BCA \quad \text{(corresponding sides in congruent } \Delta \text{s)} \]

\[ AC \text{ is common} \]

\[ \therefore \angle ACD = \angle BAC \quad \text{(by AAS)} \]

\[ \therefore \angle DAC = \angle BCA \quad \text{(corresponding sides in congruent } \Delta \text{s)} \]

Also $AB || ED$ (since $ACDF$ is a parallelogram)

\[ \therefore AB = ED \quad \text{(since } ACDF \text{ is a parallelogram)} \]

2. $x < \frac{1}{2}$

3. $x^2 - x^2 + 1 + 4.2458 \text{ m}$

4. $AC = FD$ (opposite sides of a parallelogram)

5. $BC = FE$ (given)

6. $AB = AC - BC$

7. $x^2 + 2x + C$

8. $x^0 + 2x^0 + C$

9. (a) $\frac{198\pi}{7} \text{ units}^3$ (b) $\frac{96\pi}{5} \text{ units}^3$

10. $1\frac{1}{3} \text{ units}^2$

11. $f'(3) = 20, f''(-2) = -16$

12. $8 \text{ units}^3$

13. $AB = AC$ (given)

$BD = CD$ (given)

$AD$ is common.

\[ \therefore \triangle ABD = \triangle ACD \]

(corresponding $\angle$s in congruent $\Delta$s)

But $\angle ADB + \angle ADC = 180^\circ$

\[ \therefore \angle ADB = \angle ADC = 90^\circ \]

\[ \therefore AD \perp BC \]

14. (a) $78.7 \text{ units}^3$ (b) $1.57 \text{ units}^3$

15. $x > -\frac{7}{9}$

16. $f(1) = 3, f'(1) = -2, f''(1) = 18$; curve is decreasing and concave upwards at $(1, 3)$

17. $P = 8x + 4y = 4$

$\therefore 4y = 4 - 8x$

$\therefore y = 1 - 2x$

\[ A = 3x^2 + y^2 \]

\[ = 3x^2 + (1 - 2x)^2 \]

\[ = 3x^2 + 1 - 4x + 4x^2 \]

\[ = 7x^2 - 4x + 1 \]

Rectangle $\frac{2}{7} \text{ m} \times \frac{6}{7} \text{ m}$, square with sides $\frac{3}{7} \text{ m}$

18. $f(-1) = 0$

19. $12$

20. $0.837$

21. $AB^2 = 24^2$

\[ = 576 \]

$BC^2 = 32^2$

\[ = 1024 \]

$AC^2 = 40^2$

\[ = 1600 \]

\[ AB^2 + BC^2 = 576 + 1024 \]

\[ = 1600 \]

\[ AC^2 \]

\[ \therefore \Delta ABC \text{ is right angled at } \angle B \text{ (Pythagoras' theorem)} \]

22. $\frac{(3x + 5)^2}{24} + C$

23. $-5\frac{1}{3}$

24. $(1, 1)$

25. (a) $1.11$ (b) $1.17$

26. $(0, 3)$ maximum, $(1, 2)$ minimum, $(-1, 2)$ minimum

27. $2\frac{8}{13}$

28. (a) $1.58 \text{ units}^2$ (b) $\frac{5\pi}{2} \text{ units}^3$
29. $12 \frac{2}{3}$ 30. $10 \frac{2}{3}$ units$^2$

31. $\angle ABC$ is common

\[
\frac{\angle BDC}{\angle ACB} = \frac{90^\circ}{90^\circ} = 1
\]
\[\therefore \Delta ABC \parallel \Delta CBD \text{ (AAA)}
\]

32. \[f'(x) = f''(x) = 0 \text{ and } f'''(x) > 0 \text{ on both LHS and RHS of } (0, 0)
\]

33. $2 \frac{2}{3} m^3$ 34. $f(2) = -16$ 35. $9 \text{ units}^2$

36. $119.3 \text{ m}^2$ 37. \[f(x) = x^3 - 4x^2 - 3x + 20
\]

38. \[
\begin{align*}
\text{(a)} & \quad y = 2x + 10 \\
\text{(b)} & \quad y = \frac{x}{4}
\end{align*}
\]

39. \[
\begin{align*}
\text{(a)} & \quad \text{Let } ABCD \text{ be a rhombus with } AC = x \text{ and } BD = y. \\
& \quad \angle AEB = 90^\circ \quad \text{(diagonals perpendicular in rhombus)} \\
& \quad DE = BE = \frac{1}{2}y \quad \text{(diagonals bisect each other)} \\
& \quad \Delta ACB \text{ has area } \frac{1}{2} \times x \times \frac{1}{2}y = \frac{1}{4}xy \\
& \quad \Delta ADC \text{ has area } \frac{1}{2} \times x \times \frac{1}{2}y = \frac{1}{4}xy \\
\end{align*}
\]
\[\therefore ABCD \text{ has area } \frac{1}{4}xy + \frac{1}{4}xy = \frac{1}{2}xy
\]

40. \[
\frac{AB}{AC} = \frac{AG}{AD} \quad \text{(equal ratios of intercepts, } BG \parallel CD)
\]
\[
\frac{AG}{AD} = \frac{AF}{AE} \quad \text{(equal ratios of intercepts, } GF \parallel DE)
\]
\[\therefore \quad \frac{AB}{AC} = \frac{AF}{AE}
\]

41. \[f(2) = \frac{17}{3}
\]

42. \[x^{n+1} + C
\]

(b) Since \[\frac{d}{dx}(C) = 0, \] the primitive function could include $C$.

43. (c), (d) 44. (a), (b) 45. (c)

46. (d) 47. (b) 48. (a) 49. (b) 50. (d)

Chapter 4: Exponential and logarithmic functions

Exercises 4.1

1. (a) 4.48 (b) 0.14 (c) 2.70 (d) 0.05 (e) −0.14

2. (a)

\[
\begin{align*}
y & = 2e^x \\
x & \geq 0
\end{align*}
\]

3. (a) $9e^x$ (b) $-e^x$ (c) $e^x + 2x$ (d) $6x^2 - 6x + 5 - e^x$

(e) $3e^x (e^x + 1)^2$ (f) $7e^x (e^x + 5)^x$ (g) $4e^x (2e^x - 3)$

(h) $e^x (x + 1)$ (i) $\frac{e^x (x - 1)}{x^2}$ (j) $xe^x (x + 2)$

(k) $(2x + 1)e^x + 2e^x = e^x (2x + 3)$ (l) $\frac{e^x (7x - 10)}{(7x - 3)^2}$

(m) $\frac{5e^x - 5xe^x}{e^{3x}} = \frac{5(1 - x)}{e^x}$

4. $f'(1) = 6 - e$; $f''(1) = 6 - e$ 5. $e^x$ 6. $-e^x = -\frac{1}{e^x}$

7. 19.81 8. $e^x + y = 0$ 9. $x + e^y - 3 - e^y = 0$
10. \((-1, -\frac{1}{e})\) min

\[
\begin{align*}
\frac{dy}{dx} &= 7e^x; \quad \frac{d^2y}{dx^2} = 7e^x = y \\
\frac{dy}{dx} &= 2e^x; \quad \frac{d^2y}{dx^2} = 2e^x \\
\therefore y &= 2e^x + 1 \\
\therefore \frac{d^2y}{dx^2} &= y - 1
\end{align*}
\]

Exercises 4.2

1. (a) \(7e^{2x}\)  (b) \(-e^x\)  (c) \(6e^{x-2}\)  (d) \(2xe^{x-1}\)  
   (e) \((3x^2 + 5)e^{x-3x+1}\)  (f) \(5e^{2x}\)  (g) \(-2e^{-2x}\)  (h) \(10e^{2x}\)  
   (i) \(2e^{2x} + 1\)  (j) \(2x + 2 - e^{-x}\)  (k) \(5(1 + 4e^x)(x + e^x)^4\)  
   (l) \(e^{2x}(2x + 1)\)  (m) \(\frac{e^{2x}(3x - 2)}{x^3}\)  (n) \(x^2e^{3x}(5x + 3)\)  
   (o) \(4e^{2x+1}(x + 2)\)  (p) \(28e^{3x}(e^{2x} + 1)\)  (q) \(7e^{2x} + 1\)

2. \(28e^{3x}(e^{3x} + 1)^y(7e^{2x} + 1)\)

3. \(f(1) = 3e; f'(0) = 9e^{-2}\)  4. 5

5. \(x + y - 1 = 0\)  6. \(\frac{1}{3e^2}\)

7. \(y = 2ex - e\)  8. \(f'(-1) = -18 - 4e^2\)

9. \((0, 0)\) min; \((-1, e^{-2})\) max

11. \(y = 3e^{2x}\)  
   \[
   \begin{align*}
   \frac{dy}{dx} &= 6e^{2x} \\
   \frac{d^2y}{dx^2} &= 12e^{2x} \\
   \text{LHS} &= \frac{d^2y}{dx^2} - \frac{3}{2} \frac{dy}{dx} + 2y \\
   &= 12e^{2x} - 3(6e^{2x}) + 2(3e^{2x}) \\
   &= 12e^{2x} - 18e^{2x} + 6e^{2x} \\
   &= 0 \\
   &= \text{RHS} \\
   \therefore \frac{d^2y}{dx^2} &= \frac{3}{2} \frac{dy}{dx} + 2y = 0
   \end{align*}
   \]

Exercises 4.3

1. \(\frac{1}{2}e^{3x} + C\)  (b) \(\frac{1}{4}e^{3x} + C\)  (c) \(-e^x + C\)  (d) \(\frac{1}{5}e^{3x} + C\)  
   (e) \(-\frac{1}{2}e^{2x} + C\)  (f) \(\frac{1}{4}e^{x+1} + C\)  (g) \(\frac{3}{5}e^{3x} + C\)  
   (b) \(\frac{1}{2}e^{2x} + C\)  (c) \(\frac{1}{7}e^{2x} - 2x + C\)  (i) \(e^{x-3} + \frac{x^2}{2} + C\)

2. \(\frac{1}{5}(e^x - 1)\)  (b) \(e^{-2} - 1 \text{ or } \frac{1}{e^2} - 1\)  (c) \(\frac{2}{3}e^t(e^t - 1)\)  
   (d) \(19 - \frac{1}{2}e^t(e^t - 1)\)  (e) \(\frac{1}{2}e^t + 1\)  (f) \(e^t - e - \frac{1}{2}\)  
   (g) \(\frac{1}{2}e^t + e^{-3} - \frac{1}{2}\)
3. (a) 0.32 (b) 268.29 (c) 37 855.68 (d) 346.85 (e) 755.19
4. \(e^t - e^c = e^c(e^t - 1)\) units²
5. \(\frac{1}{4}(e - e^{-3})\) units²
6. 2.86 units² 
7. 29.5 units² 
8. \(\frac{\pi}{2}(e^t - 1)\) units²
9. 4.8 units³ 
10. 7.4 
11. (a) \(x(2 + x)e^t\) (b) \(x^2e^t + C\)
12. \(\pi\) units³
13. \(\frac{1}{2}(e^t - 5)\) units³

Exercises 4.4

1. (a) 4 (b) 2 (c) 3 (d) 1 (e) 2 (f) 1 (g) 0 (h) 7
2. (a) 9 (b) 3 (c) −1 (d) 12 (e) 8 (f) 4 (g) 14 (h) 14 (i) 1 (j) 2
3. (a) −1 (b) \(\frac{1}{2}\) (c) \(\frac{1}{2}\) (d) −2 (e) \(\frac{1}{4}\) (f) \(\frac{1}{3}\) (g) \(\frac{1}{2}\) (h) \(\frac{1}{3}\) (i) \(1\frac{1}{2}\) (j) \(\frac{1}{2}\)

4. (a) 3.08 (b) 2.94 (c) 3.22 (d) 4.94 (e) 10.40 (f) 7.04 (g) 0.59 (h) 3.51 (i) 0.43 (j) 2.21
5. (a) \(\log_3 y = x\) (b) \(\log_3 z = x\) (c) \(\log_2 y = 2\) (d) \(\log_2 a = b\) (e) \(\log_2 d = 3\) (f) \(\log_4 y = x\) (g) \(\log_5 y = x\) (h) \(\log_7 y = x\) (i) \(\log_9 y = x\) (j) \(\log_4 Q = x\)
6. (a) 3\(^t\) = 5 (b) \(a^t = 7\) (c) 3\(^t\) = \(a\) (d) \(x^t = x\) (e) \(a^t = b\) (f) \(2^t = 6\) (g) \(3^t = x\) (h) \(10^t = 9\) (i) \(e^t = 4\) (j) \(7^t = x\)
7. (a) \(x = 1000000\) (b) \(x = 243\) (c) \(x = 7\) (d) \(x = 2\) (e) \(x = −1\) (f) \(x = 3\) (g) \(x = 44.7\) (h) \(x = 10000\) (i) \(x = 8\) (j) \(x = 64\)
8. \(y = 5\) 
9. \(4.47\) 
10. \(2.44\) 
11. \(10.2\) 
12. \(1.1\)
13. (a) 1 (b) 3 (c) 2 (d) 3 (e) 1 (f) 3 (g) 7 (h) \(\frac{1}{2}\) (i) −2 (j) 4
14. Domain: \(x > 0\); range: all real \(y\)

Exercises 4.5

1. (a) \(\log_x 4y\) (b) \(\log_x 20\) (c) \(\log_x 4\) (d) \(\log_x \frac{b}{5}\) 
   (e) \(\log_y x^3\) (f) \(\log_x y^3\) (g) \(\log_y x^3\) (h) \(\log_{xy} x^3\) 
   (i) \(\log_{49} ab^t c^3\) (j) \(\log_{54} \frac{p^7}{q^4}\)
2. (a) 1.19 (b) −0.47 (c) 1.55 (d) 1.06 (e) 1.08 (f) 1.36 (g) 2.02 (h) 1.83 (i) 2.36 (j) 2.19
3. (a) 2 (b) 6 (c) 2 (d) 3 (e) 1 (f) 3 (g) 7 (h) \(\frac{1}{2}\) (i) −2 (j) 4
4. (a) \(x + y\) (b) \(x − y\) (c) \(3x\) (d) \(2y\) (e) \(2x\) (f) \(f\) \(x + 2y\) (g) \(x + 1\) (h) \(1 − y\) (i) \(2x + 1\) (j) \(3y − 1\)
5. (a) \(p + q\) (b) \(3q\) (c) \(q − p\) (d) \(2p\) (e) \(c + 5q\) (f) \(2p − q\) (g) \(p + 1\) (h) \(1 − 2q\) (i) \(3 + q\) (j) \(p − 1 − q\)
6. (a) 1.3 (b) 12.8 (c) 16.2 (d) 9.1 (e) 6.7 (f) 23.8 (g) −3.7 (h) 3 (i) 22.2 (j) 23
7. (a) \(x = 4\) (b) \(y = 28\) (c) \(x = 48\) (d) \(x = 3\) (e) \(k = 6\)

Exercises 4.6

1. (a) 1.58 (b) 1.80 (c) 2.41 (d) 3.58 (e) 2.85 (f) 2.66 (g) 1.40 (h) 4.55 (i) 4.59 (j) 7.29
2. (a) \(x = 1.6\) (b) \(x = 1.5\) (c) \(x = 1.4\) (d) \(x = 3.9\) (e) \(x = 2.2\) (f) \(x = 2.3\) (g) \(x = 6.2\) (h) \(x = 2.8\) (i) \(x = 2.9\) (j) \(x = 2.4\)
3. (a) \(x = 2.58\) (b) \(y = 1.68\) (c) \(x = 2.73\) (d) \(m = 1.78\) (e) \(k = 2.82\) (f) \(t = 1.26\) (g) \(x = 1.15\) (h) \(p = 5.83\) (i) \(x = 3.17\) (j) \(n = 2.58\)
4. (a) \(x = 0.9\) (b) \(n = 0.9\) (c) \(x = 6.6\) (d) \(n = 1.2\) (e) \(x = −0.2\) (f) \(n = 2.2\) (g) \(x = 2.2\) (h) \(k = 0.9\) (i) \(x = 3.6\) (j) \(y = 0.6\)
5. (a) \(x = 5.30\) (b) \(t = 0.536\) (c) \(t = 3.62\) (d) \(x = 3.81\) (e) \(n = 3.40\) (f) \(t = 0.536\) (g) \(t = 24.6\) (h) \(k = 67.2\) (i) \(t = 54.9\) (j) \(k = −43.3\)
Exercises 4.7

1. (a) \(1 + \frac{1}{x}\) (b) \(-\frac{1}{x}\) (c) \(\frac{3}{3x+1}\) (d) \(\frac{2x}{x^3-4}\) (e) \(\frac{15x^2+3}{5x^2+3x-9}\) (f) \(\frac{5}{5x+1}\) (g) \(6x + 5 + \frac{1}{x}\) (h) \(\frac{8}{8x-9}\) (i) \(6x + 5 + \frac{1}{x} + \frac{1}{x(x+2)(3x-1)}\) (j) \(\frac{4}{4x+1} - \frac{2}{2x-7} = \frac{-30}{(4x+1)(2x-7)}\) (k) \(\frac{5}{x}(1 + \log x)^4\) (l) \(\frac{9}{x}(\frac{1}{x} - 1)(\ln x - x)^3\) (m) \(\frac{4}{x}(\log x)^3\) (n) \(6(2x + \frac{1}{x})(x^2 + \log x)^3\) (o) \(1 + \log x\) (p) \(\frac{1 - \log x}{x^2}\) (q) \(\frac{2x + 1}{x} + 2\log x\) (r) \(\frac{x^3 + 3x^2\log(x+1)}{x+1}\) (s) \(\frac{1}{x \log x}\) (t) \(\frac{x-2 - x \log x}{x(x-2)}\) (u) \(e^x(2x \log x - 1)\) (v) \(e^x\left(\frac{1}{x} + \log x\right)^3\) (w) \(\frac{10 \log x}{x}\)

2. \(f'(1) = \frac{1}{2}\) \(3. \frac{1}{x \log 10}\) 4. \(x - 2y - 2 + 2\log 2 = 0\)

5. \(y = x - 2\) 6. \(-\frac{2}{5}\) 7. \(5x + y - \log 5 - 25 = 0\)

8. \(5x - 19y + 19 \log 19 + 15 = 0\) 9. \(\left(\frac{1}{2}, \frac{1}{2}\log(1 + \frac{1}{4})\right)\)

10. \((e, \frac{1}{7})\) maximum

11. (a)

12. (b) \(\frac{2}{x+3} + \frac{2}{x-3}\)

(b) \(\frac{1}{(x-3)/(x+3)}\) (c) \(\frac{2}{(x+3)(x-3)}\)

(c) \(\frac{2(x+3)}{x^2 - 9}\) \(\frac{3x+3}{x^2 - 9}\) \(= \text{LHS}\)

13. (a) \(3\ln 3\) (b) \(10^3 \ln 10\)

14. \(4\ln 4 \cdot x - y + 4 = 0\) 15. \(3\log_3 x + y - 1 - 9\log_3 x = 0\)

Exercises 4.8

1. (a) \(\log_3 (2x + 5) + C\) (b) \(\log_3 (2x + 1) + C\)

(c) \(\ln (x^3 - 2) + C\) (d) \(\frac{1}{2}\ln x + C\) or \(\frac{1}{2}\log_2 x + C\)

(e) \(2\ln x + C\) (f) \(\frac{5}{3}\log_3 x + C\) (g) \(\log_3 (x^2 - 3x) + C\)

(h) \(\frac{1}{2}\ln x^2 + C\) (i) \(\frac{3}{2}\log_3 (x^2 + 7) + C\)

(j) \(\frac{1}{2}\log_3 (x^2 + 2x - 5) + C\)

2. (a) \(\ln (4x - 1) + C\) (b) \(\log_3 (x + 3) + C\)

(c) \(\frac{1}{6}\ln (2x^3 - 7) + C\) (d) \(\frac{1}{12}\log_2 (2x^6 + 5) + C\)

(e) \(\frac{1}{2}\log_2 (x^2 + 6x + 2) + C\)

3. (a) 0.5 (b) 0.7 (c) 1.6 (d) 3.1 (e) 0.5

4. \(\log_3 3 - \log_2 2 = \log_3 1.5 \text{ units}^2\) 5. \(\log_2 2 \text{ units}^2\)

6. \(0.5 + \log_2 2 \text{ units}^2\) 7. 0.61 \(\text{ units}^2\)

8. \(\pi \log_3 3 \text{ units}^1\) 9. \(2\pi \log_3 9 \text{ units}^3\)

10. 47.2 \(\text{ units}^3\) 11. \(\frac{\pi}{2} \text{ units}^3\) (a) \(\pi^3 (e^4 - 1) \text{ units}^3\)

12. (a) \(\text{RHS} = \frac{1}{x + 3} + \frac{2}{x - 3}\)

(b) \(\frac{1}{(x-3)/(x+3)}\) (c) \(\frac{2(x+3)}{x^2 - 9}\)

(c) \(\frac{3x+3}{x^2 - 9}\) \(\frac{1}{x + 3} + \frac{2}{x - 3}\) \(\text{LHS}\)

(b) \(\log_3 (x + 3) + 2\log_3 (x - 3) + C\)
13. (a) RHS = $1 - \frac{5}{x - 1}$
    
    $\frac{x - 1}{x - 1} - \frac{5}{x - 1}$
    
    $\frac{x - 6}{x - 1}$
    
    LHS

    (b) $x - 5 \log_e (x - 1) + C$

14. $\frac{3^{x-1}}{2 \log_e 3} + C$  15. 1.86 units$^2$

Test yourself 4

1. (a) 6.39  (b) 1.98  (c) 3.26  (d) 1.40  (e) 0.792
    (f) 3.91  (g) 5.72  (h) 72.4  (i) 6  (j) 2

2. (a) $5e^{x+1}$  (b) $-2e^{1-x}$  (c) $\frac{1}{x}$  (d) $\frac{4}{4x + 5}$  (e) $e^t (x + 1)$
    (f) $1 - \ln \frac{x}{x^2}$  (g) $10e^{(e^t - 1)}$

3. (a) $\frac{1}{4} e^{x+1} + C$  (b) $\frac{1}{2} \ln (x^2 - 9) + C$  (c) $-e^t + C$
    (d) $\ln (x + 4) + C$

4. $3x - y + 3 = 0$  5. $-\frac{e^t}{e^t + 1}$  6. $\frac{1}{2} e^t (e^t - 1)$ units$^2$

7. $\frac{\pi}{6} e^{(e^t - 1)}$ units$^3$

8. (a) 0.92  (b) 1.08  (c) 0.2  (d) 1.36  (e) 0.64

9. $e (e^t - 1)$ units$^2$

10. (a) 2.16 units$^2$  (b) $x = e^t$  (c) 2.16 units$^2$

11. (a) $x = 1.9$
    (b) $x = 1.9$
    (c) $x = 3$
    (d) $x = 36$
    (e) $t = 18.2$

12. (a) $\frac{3}{2} (e^t - 1)$
    (b) $\frac{1}{3} \ln 10$
    (c) $\frac{1}{6} + 3 \ln 2$

13. $e^t x - y - 3e^t = 0$

14. 0.9

15. (a) $e(e^t - 1)$ units$^2$
    (b) $\frac{\pi}{2} e^{(e^t - 1)}$ units$^3$

16. (a) $\log_e x^3 y^3$
    (b) $\log_e \frac{k^2 p}{3}$

17. $2x + y - \ln 2 - 4 = 0$

18. $(0, 0)$ point of inflexion, $(-3, -27e^{-t})$ minimum

19. 5.36 units$^2$

20. (a) 0.65
    (b) 1.3

Challenge exercise 4

1. $\frac{(e^{2t} + x) \frac{1}{x} - (2e^{2t} + 1) \log_e x}{(e^{2t} + x)^2}$

2. $2e$

3. (a) 2.8  (b) 1.8  (c) 2.6

4. $9 \left(4e^{2t} + \frac{1}{x}\right)(e^{2t} + \log_e x)^2$  5. 0.42 units$^2$

6. $-\frac{2}{2x - 3}$  7. 12 units$^3$  8. $5 \log_e 5$

9. $\frac{d}{dx} (x^2 \log_e x) = x(1 + 2 \log_e x); 18 \log_e 3$  10. $\frac{3t^2}{\log_e 3} + C$

11. (a) $(1, 0)$  (b) $x - y - 1 = 0; x - \log_e 10 \cdot y - 1 = 0$
    (c) $\left(1 + \frac{1}{\log_e 10}\right)$ units  12. 0.645 units$^2$

13. $e(t + \log_e x) - x \log_e x$

14. $\frac{dy}{dx} = e^t + e^{-t}$
    $\frac{dy}{dx} = e^t - e^{-t}$
    $\frac{d^2 y}{dx^2} = e^t - (-e^{-t})$
    $\frac{d^2 y}{dx^2} = e^t + e^{-t}$
    $y$

15. $y = 3e^{3t} - 2$
    $\frac{dy}{dx} = 15e^{3t}$
    $\frac{d^2 y}{dx^2} = 75e^{3t}$

LHS $= \frac{d^2 y}{dx^2} - \frac{4}{dx} = 5y - 10$

$= 75e^{3t} - 4(15e^{3t}) - 5(3e^{3t} - 2) - 10$

$= 75e^{3t} - 60e^{3t} - 15e^{3t} + 10 - 10$

$= 0$

RHS

16. $f(x) = 3e^{3t} - 6x$

17. 

![Graph](image-url)
Chapter 5: Trigonometric functions

Exercises 5.1

1. (a) 36° (b) 120° (c) 225° (d) 210° (e) 540° (f) 140° (g) 240° (h) 420° (i) 20° (j) 50°

2. (a) \( \frac{3\pi}{4} \) (b) \( \frac{\pi}{6} \) (c) \( \frac{5\pi}{6} \) (d) \( \frac{4\pi}{3} \) (e) \( \frac{5\pi}{3} \) (f) \( \frac{7\pi}{20} \) (g) \( \frac{2\pi}{3} \) (h) \( \frac{5\pi}{4} \) (i) \( \frac{2\pi}{3} \)

3. (a) 0.98 (b) 1.19 (c) 1.78 (d) 1.54 (e) 0.88

4. (a) 0.32 (b) 0.61 (c) 0.17 (d) 1.54 (e) 0.88

5. (a) 62° 27’ (b) 44° 0’ (c) 66° 28’ (d) 56° 43’ (e) 18° 20’ (f) 183° 21’ (g) 154° 42’ (h) 246° 57’ (i) 320° 51’ (j) 6° 18’

6. (a) 0.34 (b) 0.07 (c) 0.06 (d) 0.83 (e) \(-1.14\) (f) 0.33 (g) \(-1.50\) (h) 0.06 (i) \(-0.73\) (j) 0.16

Exercises 5.2

1.

<table>
<thead>
<tr>
<th>( \frac{\pi}{3} )</th>
<th>( \frac{\pi}{4} )</th>
<th>( \frac{\pi}{6} )</th>
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<tr>
<td>( \sin )</td>
<td>( \sqrt{3}/2 )</td>
<td>( 1/\sqrt{2} )</td>
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<tr>
<td>( \cos )</td>
<td>( 1/2 )</td>
<td>( 1/\sqrt{2} )</td>
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<td>( \tan )</td>
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<td>( \csc )</td>
<td>( 2/\sqrt{3} )</td>
<td>( \sqrt{2} )</td>
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<td>( \sec )</td>
<td>2</td>
<td>( \sqrt{2} )</td>
</tr>
<tr>
<td>( \cot )</td>
<td>( 1/\sqrt{3} )</td>
<td>1</td>
</tr>
</tbody>
</table>

2. (a) \( \frac{1}{3} \) (b) \( \frac{1}{2} \) (c) \( \frac{\sqrt{3}}{8} \) (d) \( \frac{\sqrt{3}}{3} \) (e) \( \frac{4\sqrt{3}}{3} \) (f) \( \frac{2\sqrt{3} + 1}{2} \) (g) \( \sqrt{2} - \sqrt{3} \) (h) \( \sqrt{2 + \sqrt{2}} \) (i) \( \frac{3 - \sqrt{2}}{2} \) (j) \( \frac{2 + \sqrt{3}}{2} \)

3. (a) \( 1/4 \) (b) \( \sqrt{6} - \sqrt{2}/4 \) (c) \( \sqrt{3}/2 \) (d) 1 (e) \( 4\sqrt{1}/4 \)

4. (a) \( \sqrt{6} + \sqrt{2}/4 \) (b) \( \sqrt{3} - 2 \)

5. LHS = \( \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} \)
   \[ = \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} \]
   \[ = \frac{1}{2} + \frac{\sqrt{3}}{2} \]
   RHS = \( \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6} \)
   \[ = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} \]
   \[ = \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2}} \]
   LHS = RHS

So \( \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4} = \sin \frac{\pi}{4} \cos \frac{\pi}{6} + \cos \frac{\pi}{4} \sin \frac{\pi}{6} \)

6. (a) \( \frac{3\pi}{4} - \frac{\pi}{6} \) (b) 2nd (c) \( -1/\sqrt{2} \)
   \[ = \pi - \frac{\pi}{6} \]

7. (a) \( \frac{5\pi}{6} - \frac{6\pi}{6} \) (b) 2nd (c) \( 1/2 \)
   \[ = \pi - \frac{\pi}{6} \]

8. (a) \( \frac{7\pi}{4} - \frac{8\pi}{4} \) (b) 4th (c) \( -1 \)
   \[ = 2\pi - \frac{\pi}{4} \]

9. (a) \( \frac{4\pi}{3} - \frac{3\pi}{3} + \frac{\pi}{3} \) (b) 3rd (c) \( -1/2 \)
   \[ = \pi + \frac{\pi}{3} \]

10. (a) \( \frac{5\pi}{6} - \frac{6\pi}{3} \) (b) 4th (c) \( -\sqrt{3}/2 \)
    \[ = 2\pi - \frac{\pi}{3} \]

11. (a) \( -1 \) (b) \( \sqrt{3}/2 \) (c) \( -\sqrt{3} \) (d) \( -1/\sqrt{2} \) (e) \( 1/\sqrt{3} \)

12. (a) \( \frac{13\pi}{6} = \frac{12\pi}{6} + \frac{\pi}{6} \) (b) 1st (c) \( \sqrt{3}/2 \)
    \[ = 2\pi + \frac{\pi}{6} \]
    (b) (i) \( \frac{1}{\sqrt{2}} \) (ii) \( \sqrt{2} \) (iii) \( -\frac{1}{\sqrt{2}} \) (iv) \( \frac{1}{\sqrt{3}} \) (v) \( -\sqrt{3}/2 \)

13. (a) \( \frac{5\pi}{3} \) (b) \( \frac{5\pi}{4} \) (c) \( \frac{5\pi}{4} \) (d) \( \frac{4\pi}{3} \) (e) \( \frac{7\pi}{6} \)

14. (a) \( \sin \theta \) (b) \( -\tan x \) (c) \( -\cos \alpha \) (d) \( \sin x \) (e) \( \cot \theta \)
15. $\sin^2 \theta$

16. $\cos x = -\frac{4}{3}; \sin x = \frac{3}{5}$

17. $x = \pi \frac{3\pi}{4} \frac{5\pi}{4} \frac{7\pi}{4}$

**Exercises 5.3**

1. (a) $4\pi$ cm (b) $\pi$ m (c) $\frac{25\pi}{3}$ cm (d) $\frac{\pi}{2}$ cm (e) $\frac{7\pi}{4}$ mm

2. (a) 0.65 m (b) 3.92 cm (c) 6.91 mm (d) 2.39 cm (e) 3.03 m

3. 1.8 m

4. 7.5 m

5. $\pi \frac{21}{2}$

6. 25 mm

7. 1.83

8. $13 \frac{7}{9}$ mm

9. $25.3$ mm

10. $\frac{175\pi}{36}$ cm$^3$

**Exercises 5.4**

1. (a) $8\pi$ cm$^2$ (b) $\frac{3\pi}{2}$ m$^2$ (c) $\frac{125\pi}{3}$ cm$^2$ (d) $\frac{3\pi}{4}$ cm$^2$ (e) $\frac{49\pi}{8}$ mm$^2$

2. (a) 0.48 m$^2$ (b) 6.29 cm$^2$ (c) 24.88 mm$^2$ (d) 7.05 cm$^2$ (e) 3.18 m$^2$

3. 16.6 m$^2$

4. $\theta = 4 \frac{4}{9}$ m

6. (a) $\frac{7\pi}{6}$ cm (b) $\frac{49\pi}{12}$ cm$^2$

7. $\frac{6845}{8\pi}$ mm$^2$

8. 7.5 cm$^2$

9. 9.197 cm$^2$

10. $\theta = \frac{\pi}{15}, r = 3$ cm

**Exercises 5.5**

1. (a) $8\pi$ cm$^2$ (b) $\frac{6\pi - 9\sqrt{3}}{4}$ m$^2$ (c) $\frac{125\pi - 75}{3}$ cm$^2$ (d) $\frac{3(\pi - 3)}{4}$ cm$^2$ (e) $\frac{49(\pi - 2\sqrt{2})}{8}$ mm$^2$

2. (a) 0.01 m$^2$ (b) 1.45 cm$^2$ (c) 3.65 mm$^2$ (d) 0.19 cm$^2$ (e) 0.99 m$^2$

3. 0.22 cm$^2$

4. (a) $\frac{3\pi}{7}$ cm (b) $\frac{9\pi}{14}$ cm$^2$ (c) 0.07 cm$^2$

5. 134.4 cm$^2$

6. (a) 2.6 cm (b) $\frac{5\pi}{6}$ cm (c) 0.29 cm$^2$

7. (a) 10.5 mm (b) 4.3 mm$^2$

8. (a) $\frac{25\pi}{4}$ cm$^2$ (b) 0.5 cm$^2$

9. (a) $77^\circ 22'$ (b) 70.3 cm$^2$ (c) 26.96 cm$^2$ (d) 425.43 cm$^2$

10. 9.4 cm$^2$

11. (a) $\frac{11\pi}{9}$ cm (b) $22 - \frac{121\pi}{18} = \frac{396 - 121\pi}{18}$ cm$^2$

(c) $22 + \frac{11\pi}{9} = \frac{11(18 + \pi)}{9}$ cm

12. (a) 5 cm$^2$ (b) 0.3% (c) 15.6 cm

13. (a) 10$\pi$ cm (b) 24$\pi$ cm$^2$

14. (a) $8 + \frac{20\pi}{9} = \frac{4(18 + 5\pi)}{9}$ cm (b) 3.7

15. (a) $\frac{225\pi}{2}$ cm$^3$ (b) $\frac{105\pi}{2} + 180 = \frac{15(7\pi + 24)}{2}$ cm$^2$

**Exercises 5.6**

1. (a) 0.045 (b) 0.003 (c) 0.999 (d) 0.065 (e) 0.005

2. $\frac{1}{4}, \frac{3}{3}$

4. $1343622$ km $5.7367$ m

**Exercises 5.7**

1. (a) $y = \pi x$ for $x = -\pi, \pi, 2\pi$

(b) $y = \pi x$ for $x = -\pi, \pi, 2\pi$
3. (a) 

(b) 

(c) 

(d) 

(e) 

4. 

5. (a) 

(b) 

(c) 

(d)
(e) \( y = \sin x \)

(f) \( y = \sin 2x \)

(g) \( y = \sin x \) with a graph showing a period of \( \frac{\pi}{2} \) and amplitude of 1.

(h) \( y = \sin x \) with a graph showing a period of \( \pi \) and amplitude of 1.

(i) \( y = \sin x \) with a graph showing a period of \( \frac{\pi}{2} \) and amplitude of 1.

(j) \( y = \sin 2x \) with a graph showing a period of \( \frac{\pi}{2} \) and amplitude of 1.
8. (a) \[ y = 2 \cos x \]

(b) \[ y = 2 \cos x + 3 \sin x \]

9. \[ y = \cos 2x - \cos x \]

10. (a) \[ y = \cos x + \sin x \]

(b) \[ y = \sin 2x - \sin x \]

(c) \[ y = \sin x + 2 \cos 2x \]

(d) \[ y = 3 \cos x - \cos 2x \]
Exercises 5.8

1. (a) 

There are 2 points of intersection, so there are 2 solutions to the equation. 

(b) 

There are 3 points of intersection, so there are 3 solutions to the equation. 

2. \( x = 0 \) 3. \( x = 1.5 \) 4. \( x = 0, 4.5 \) 5. \( x = 0, 1 \)

6. \( x = 0.8, 4 \)

7. (a) Period 12 months, amplitude 1.5  (b) 5.30 p.m. 

8. (a) 1300  (b) (i) 1600  (ii) 1010  
    (c) Amplitude 300, period 10 years 

9. (a) 

(b) It may be periodic – hard to tell from this data. 
    Period would be about 10 months. 
    (c) Amplitude is 1.5 

10. (a) 

(b) Period 24 hours, amplitude 1.25  (c) 2.5 m 

Exercises 5.9 

1. (a) \( 4 \cos 4x \)  (b) \(-3 \sin 3x \)  (c) \( 5 \sec^2 5x \)  
   (d) \( 3 \sec^2 (3x + 1) \)  (e) \( \sin (-x) \)  (f) \( 3 \cos x \)  
   (g) \(-20 \sin (5x - 3) \)  (h) \(-6x^2 \sin (x^3) \)  
   (i) \( 14x \sec^2 (x^2 + 5) \)  (j) \( 3 \cos 3x - 8 \sin 8x \)
(k) \( \sec^2(\pi + x) + 2x \)  (l) \( x \sec^2 x + \tan x \)

(m) \( 3 \sin 2x \sec^2 3x + 2 \tan 3x \cos 2x \)

(n) \( \frac{2x \cos x - 2 \sin x}{4x^2} = \frac{\cos x - \sin x}{2x} \)

(o) \( \frac{3 \sin 5x - 5(3x + 4) \cos 5x}{\sin^2 5x} \)

(p) \( 9(2 + 7 \sec^2 7x)(2x + \tan 7x)^9 \)

(q) \( 2 \sin x \cos x \)  (t) \( 45 \sin 5x \cos^2 5x \)

(r) \( e^t + 2 \sin 2x \)  (t) \( \frac{1}{x} \cos (1 - \log x) \)

(s) \( (e^x + 1) \cos (e^x + x) \)

(v) \( \frac{\cos x}{\sin x} = \cot x \)

(w) \(-2e^{ix} \sin 2x + 3e^{ix} \cos 2x = e^{ix}(3 \cos 2x - 2 \sin 2x) \)

(x) \( \frac{2e^{ix} \tan 7x - 7e^{ix} \sec^2 7x}{\tan^2 7x} = \frac{e^{ix}(2 \tan 7x - 7 \sec^2 7x)}{\tan^2 7x} \)

1. \( 4 \cos^2 x \sin^3 x - \sin^5 x \)
   \( = \sin^3 x(4 \cos^2 x - \sin^2 x) \)

2. \( 12 \)

3. \( 6 \sqrt{3} x - 12y + 6 - \pi \sqrt{3} = 0 \)

5. \( -\sin x = -\tan x \quad 6. \quad -\frac{2}{3} = -\frac{2 \sqrt{3}}{9} \)

7. \( \sec^2 x e^{\cos x} \)

8. \( 8 \sqrt{2} x + 48y - 72 \sqrt{2} - \pi \sqrt{2} = 0 \)

9. \( y = 2 \cos 5x \)

\( \frac{dy}{dx} = -10 \sin 5x \)

\( \frac{d^2 y}{dx^2} = -50 \cos 5x \)

\( = -25(2 \cos 5x) \)

\( = -25y \)

10. \( f(x) = -2 \sin x \)

\( f'(x) = -2 \cos x \)

\( f''(x) = 2 \sin 2x \)

\( = -f(x) \)

11. \( \text{LHS} = \frac{d}{dx} [\log(\tan x)] \)

\( = \sec^2 x \tan x \)

\( = \tan^2 x + 1 \)

\( = \tan x \tan x + \frac{1}{\tan x} \)

\( = \tan x + \cot x \)

\( = \text{RHS} \)

\[ \therefore \frac{d}{dx} [\log(\tan x)] = \tan x + \cot x \]

12. \( \left( \frac{\pi}{3}, \sqrt{3} - \frac{\pi}{3} \right) \text{ maximum,} \)

\( \left( \frac{5\pi}{3}, -\sqrt{3} - \frac{5\pi}{3} \right) \text{ minimum} \)

13. \( \frac{\pi}{180} \sec^2 x \)

\( \frac{-\pi}{60} \sin x \)

\( \frac{\pi}{900} \cos x \)

14. \( y = 2 \sin 3x - 5 \cos 3x \)

\( \frac{dy}{dx} = 6 \cos 3x + 15 \sin 3x \)

\( \frac{d^2 y}{dx^2} = -18 \sin 3x + 45 \cos 3x \)

\( = -9(2 \sin 3x - 5 \cos 3x) \)

\( = -9y \)

15. \( a = -7, \quad b = -24 \)

Exercises 5.10

1. \( (a) \sin x \quad (b) \cos x + C \quad (c) \tan x + C \)

\( (d) -\frac{45}{\pi} \cos x^2 + C \quad (e) \frac{1}{3} \cos 3x + C \quad (f) \frac{1}{7} \cos 7x + C \)

\( (g) \frac{1}{5} \tan 5x + C \quad (h) \sin (x + 1) + C \)

\( (i) \frac{1}{2} \cos (2x - 3) + C \quad (j) \frac{1}{2} \sin (2x - 1) + C \)

\( (k) \cos (\pi - x) + C \quad (l) \sin (x + \pi) + C \)

\( (m) \frac{7}{2} \tan 7x + C \quad (n) -8 \cos \frac{x}{2} + C \)

\( (o) 9 \tan \frac{x}{3} + C \quad (p) -\cos (3 - x) + C \)

2. \( (a) 1 \quad (b) \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2 \sqrt{3}}{3} \quad (c) \frac{2}{\sqrt{2}} = \sqrt{2} \quad (d) \frac{1}{3} \)

\( (e) \frac{1}{\pi} \quad (f) \frac{1}{2} \quad (g) \frac{3}{4} \quad (h) -\frac{1}{5} \)

3. \( 4 \quad 4, \quad \frac{1}{3} \quad \frac{1}{6} \text{ units}^2 \quad 5, \quad 0.86 \text{ units}^2 \)

4. \( 0.51 \text{ units}^3 \quad 7, \quad \frac{\pi}{4} \text{ units}^3 \)

5. \( \sqrt{3} - \frac{\pi}{3} = \frac{3 \sqrt{3} - \pi}{3} \text{ units}^2 \quad 9, \quad 2\sqrt{2} \text{ units}^2 \)

10. \( a) \quad V = \int_{0}^{\pi} y^2 \quad dx \)

\( = \pi \int_{0}^{\pi} \cos x \quad dx \)

\( = [\sin x]_{0}^{\pi} \)

\( = \pi(\sin \pi - \sin 0) \)

\( = \pi(1 - 0) \)

\( = \pi \text{ units}^3 \)

(b) 3.1 units$^3$

11. \( y = -2 \sin 3x \)
Test yourself 5

1. (a) \(\frac{5\pi}{6}\) cm  \hspace{1em} (b) \(\frac{25\pi}{12}\) cm\(^2\)  \hspace{1em} (c) 0.295 cm\(^3\)

2. (a) \(\sqrt{3}\)  \hspace{1em} (b) \(\frac{\sqrt{3}}{2}\)  \hspace{1em} (c) \(\frac{\sqrt{3}}{2}\)  \hspace{1em} (d) -\(\frac{1}{\sqrt{2}}\)

3. (a) \(x = \frac{3\pi}{4}\) \hspace{1em} (b) \(x = \frac{\pi}{6}\)

4. (a)

5. (a) \(-\sin x\)  \hspace{1em} (b) \(2 \cos x\)  \hspace{1em} (c) \(\sec^2 x\)  \hspace{1em} (d) \(x \cos x + \sin x\)

(e) \(\frac{x \sec^2 x - \tan x}{x^2}\)  \hspace{1em} (f) -3 \(3\sin 3x\)  \hspace{1em} (g) 5 \(\sec^2 5x\)

6. (a) \(\frac{1}{2} \cos 2x + C\)  \hspace{1em} (b) \(3 \sin x + C\)  \hspace{1em} (c) \(\frac{1}{5} \tan 5x + C\)

(d) \(x - \cos x + C\)

7. (a) \(\frac{1}{\sqrt{2}}\)  \hspace{1em} (b) \(\frac{2\sqrt{3}}{3}\)

8. \(3x + \sqrt{2}y - 1 - \frac{3\pi}{4} = 0\)

9. \(x = \cos 2t\)

\[\frac{dx}{dt} = -2 \sin 2t\]

\[\frac{d^2 x}{dt^2} = -4 \cos 2t\]

\[= -4x\]

10. \(\frac{1}{\sqrt{2}}\) units\(^2\)  \hspace{1em} 11. \(\frac{\pi}{\sqrt{3}}\) units\(^3\) 12. (a) 5  \hspace{1em} (b) 2

13. \(-3\sqrt{3}\)  \hspace{1em} 14. (a) \(\frac{8\pi}{7}\) cm\(^2\)  \hspace{1em} (b) 0.12 cm\(^2\)

15. (a)

16. \(\frac{\sqrt{3} - \sqrt{2}}{2}\) units\(^2\)

17. 2 units\(^2\)  \hspace{1em} 18. \(4x + 8y - 8 = 0\)  \hspace{1em} 19. \(y = -3 \cos 2x\)

20. (a)

Challenge exercise 5

1. 0.27  \hspace{1em} 2. \(\frac{1}{2}(1 - \frac{1}{\sqrt{3}}) = \frac{3 - \sqrt{3}}{6}\)  \hspace{1em} 3. \(r = 64\) units, \(\theta = \frac{\pi}{512}\)

4. (a) Period = 2, amplitude = 3  \hspace{1em} (b)

5. (a) \(y = -\sin 3x\)

(b) \(LHS = \frac{d^2 y}{dx^2} + 9y\)

\[= 9 \sin 3x + 9 (-\sin 3x)\]

\[= 9 \sin 3x - 9 \sin 3x\]

\[= 0\]

\[= RHS\]
6. $y = 5 \sin(x + \pi)$

7. $\frac{\pi}{180}$

8. (a) RHS = $\frac{\sec^2 x}{\tan x}$

9. (2x cos 2x + sin 2x) $\cos e^{2 \tan x}$

10. (a) $\left(\frac{\pi}{4}, 4\right)$ and $\left(\frac{3\pi}{4}, 2\right)$

11. $8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ cm$^2$ = $\frac{4(2\pi - 3\sqrt{3})}{3}$ cm$^2$

12. $-\frac{180}{\pi}$ cos $x + C$

13. $-\frac{1}{2}$

14. 0.204 units$^3$

15. $\cos x - \sin x$ sin $x + \cos x$

16. $\frac{9}{2}\left(\frac{\pi}{2} - 1\right)$ = $\frac{9(\pi - 2)}{4}$ cm$^2$

17. $y = e^{kx}$

$\frac{dy}{dx} = ke^{kx}$

$= 4y$

18. $\left(\frac{\pi}{5}, 0\right), \left(\frac{5\pi}{8}, 0\right), \left(\frac{9\pi}{8}, 0\right), \left(\frac{13\pi}{8}, 0\right)$

19. $\frac{1}{\sqrt{2}} - \frac{1}{2} = \sqrt{2} - 1$ units$^2$

20. $f(x) = 2 \cos 3x$

$a) -6 \sin 3x$

$b) -18 \cos 3x$

$c) -9(2 \cos 3x)$

$= -9f(x)$

Chapter 6: Applications of calculus to the physical world

Exercises 6.1

1. (a) $R = 20 - 8t$ (b) $R = 15t^2 + 4t$ (c) $R = 16 - 4x$

(d) $R = 15t^4 - 4t^2 + 2$ (e) $R = e^t$ (f) $R = -15 \sin 5\theta$

(g) $R = 2\pi - \frac{100}{t}$ (h) $R = \frac{x}{\sqrt{x^2 - 4}}$ (i) $R = 800 - \frac{400}{t^2}$

(j) $R = 4\pi^2$

2. (a) $h = 2t^2 - 4t + C$ (b) $A = 2x^3 + x + C$

(c) $V = \frac{4}{3}\pi x^3 + C$ (d) $d = -7 \cos t + C$

(e) $s = 4e^{3t} - 3t + C$

3. 20 4. 1 5. 6e$^{12}$ 6. 13 7. 900 8. 2e$^3$ + 5

9. $y = x^3 - x^2 + x + 6$ 10. $R = \frac{dM}{dt} = 1 - 4t; R = -19$

[i.e. melting at the rate of 19 g per minute (g min$^{-1}$)]

11. $-11079.25$ cm per second (cms$^{-1}$) 12. 21 000 L

13. 165 cm$^2$ per second (cm$^2$s$^{-1}$) 14. $-0.25$

15. 41 cm$^3$ per minute (cm$^3$ min$^{-1}$) 16. 31 cm$^3$

17. 108 731 people per year 18. (a) 27 g

(b) $-2.7$ (i.e. decaying at a rate of 2.7 g per year)

19. $y = e^{kt}$

$\frac{dy}{dx} = ke^{kt}$

$= 4y$

20. $S = 2e^{kt} + 3$

$\frac{dS}{dt} = 2(2e^{kt})$

$= 2(2e^{2t} + 3 - 3)$

$= 2(S - 3)$
Exercises 6.2

1. (a) 80 (b) 146 (c) 92 days (d) 

[Graph showing exponential growth with points (10, 50), (20, 100), (30, 150)]

2. (a) 99,061 (b) 7 hours

3. (a) When \( t = 0, M = 100 \)
\[ M = 100e^{-kt} \]
When \( t = 5, M = 95 \)
\[ 95 = 100e^{-5k} \]
\[ 0.95 = e^{-5k} \]
\[ \ln 0.95 = -5k \]
\[ 0.01 = k \]
So \( M = 100e^{-0.01t} \)
(b) 90.25 kg
(c) 67.6 years

4. (a) 35.6 L (b) 26.7 minutes

5. (a) \( P = 5000 \) (b) \( k = 0.157 \) (c) 12,800 units (d) 8.8 years

6. 2.3 million m²
7. (a) \( P = 50,000e^{0.06t} \) (b) 70,599 (c) 4,871 people per year (d) 2,040

8. (a) 65.61°C (b) 1 hour 44 minutes

9. (a) 92 kg (b) Reducing at the rate of 5.6 kg per hour (c) 18 hours

10. (a) \( M_0 = 200; k = 0.00253 \) (b) 192.5 g (c) Reducing by 0.49 g per year (d) 273.8 years

11. (a) \( B = 15,000e^{0.07m} \) (b) 36,008 (c) 79.6 hours

12. 11.4 years

13. (a) 19% (b) 3200 years

14. (a) \( P(t) = P(t_0)e^{-kt} \)
\[ \frac{dP(t)}{dt} = -kP(t)e^{-kt} \]
(b) 23% (c) 2% decline per year (d) 8.5 years

15. 12.6 minutes

16. 12.8 years

17. (a) 76.8 mg/dL (b) 9 hours

Exercises 6.3

1. (a) \( v \) (b) \( a \)

2. (a) \( Q = Ae^{kt} \)
\[ \frac{dQ}{dt} = kAe^{kt} = kQ \]
(b) \[ \frac{dQ}{dt} = kQ \]
So \[ \frac{dt}{dQ} = \frac{1}{kQ} \]
\[ t = \int \frac{1}{kQ} dQ \]
\[ = \frac{1}{k} \ln Q + C \]
\[ kt = \ln Q + C_i \]
\[ kt - C_i = ln Q \]
\[ e^{kt} - C_i = Q \]
\[ e^{kt} \times e^{-C_i} = Q \]
\[ Ae^{-C} = Q \]

18. 15.8 s

19. 8.5 years

20. (a) \( Q = Ae^{kt} \)
\[ \frac{dQ}{dt} = kAe^{kt} \]
(b) \[ \frac{dQ}{dt} = kQ \]
So \[ \frac{dt}{dQ} = \frac{1}{kQ} \]
\[ t = \int \frac{1}{kQ} dQ \]
\[ = \frac{1}{k} \ln Q + C \]
\[ kt = \ln Q + C_i \]
\[ kt - C_i = ln Q \]
\[ e^{kt} - C_i = Q \]
\[ e^{kt} \times e^{-C_i} = Q \]
\[ Ae^{-C} = Q \]
Exercises 6.4

1. (a) $18 \text{ cm/s}$  (b) $12 \text{ cm/s}^2$  (c) When $t = 0$, $x = 0$; after 3 s
   (d) After 5 s

2. (a) $-8 \text{ m/s}^2$  (b) $a = 4$; constant acceleration of $4 \text{ m/s}^2$
   (c) $13 \text{ m}$  (d) after 2 s  (e) $-5 \text{ m}$

3. (a) $4 \text{ m}$  (b) $40 \text{ m/s}$  (c) $39 \text{ m}$  (d) $84 \text{ m}$

4. (a) $2 \text{ cm}$  (b) After 1 s  (c) $-4 \text{ cm}$  (d) $6 \text{ cm}$
   (e) $-7 \text{ cm/s}^2$

5. (a) $2 \text{ m/s}^2$  (b) $4e^2 \text{ m/s}^2$  (c) $a = 4e^{2t} = 2(2e^{2t}) = 2v$
   (d) $1 \text{ cm}$

6. (a) $v = -2\sin 2t$  (b) $a = -4\cos 2t$  (c) $1 \text{ cm}$
   (d) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$  (e) $1 \text{ cm}$
   (f) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$
   (g) $a = -4\cos 2t = -4x$

7. (a) $v = 3t^2 + 12t - 2; a = 6t + 12$  (b) $266 \text{ m}$
   (c) $133 \text{ m/s}^2$  (d) $42 \text{ m/s}^2$
8. (a) $\dot{x} = 20(4t - 3)^2$, $\ddot{x} = 320(4t - 3)^2$
(b) $x = 1$ cm, $\dot{x} = 20$ cm s$^{-1}$, $\ddot{x} = 320$ cm s$^{-2}$
(c) The particle is on the RHS of the origin, travelling to the right and accelerating.

9. (a) $v = 5 - 10t$ (b) $-95$ ms$^{-1}$ (c) $a = -10 = g$

10. $v = \frac{17}{3t + 1}$, $a = \frac{-102}{(3t + 1)^2}$

11. (a) At the origin (b) $\frac{1}{6}$ cm s$^{-1}$ (c) $-\frac{1}{36}$ cm s$^{-2}$
(d) The particle is moving to the right but decelerating
(e) $(e^t - 1)$ s

12. (a) $3$ ms$^{-1}$ (b) When $t = 0$, $1$, $3$ s (c) $10$ ms$^{-2}$

13. (a) $x$
(b) $\dot{x} = 6\cos 2t$, $\ddot{x} = -12\sin 2t$
(c) $-6\sqrt{3}$ cm s$^{-2}$
(d) $\ddot{x} = -12\sin 2t = -4(3\sin 2t) = -4x$

14. (a) $7$ m (b) $16$ m (c) After $7$ s
(d) $h$
(e) $10$ m

15. (a) $18.75$ m (b) $-15$ ms$^{-1}$ (c) $5$ s

16. (a) At the origin: $x = 0$
   $2t^3 - 3t^2 + 42t = 0$
   $t(2t^2 - 3t + 42) = 0$
   $t = 0, 2t^2 - 3t + 42 = 0$
Since $t = 0$, the particle is initially at the origin.
   $2t^2 - 3t + 42 = 0$
   $b^2 - 4ac = (-3)^2 - 4(2)(42)$
   $= -327 < 0$
So the quadratic equation has no real roots.
So the particle is never again at the origin.

(b) $\frac{dx}{dt} = 6t^2 - 6t + 42$
At rest: $\frac{dx}{dt} = 0$
$6t^2 - 6t + 42 = 0$
$t^2 - t + 7 = 0$
$b^2 - 4ac = (-1)^2 - 4(1)(7)$
$= -27 < 0$
So the quadratic equation has no real roots.
So the particle is never at rest.

17. (a) $0$ cm (at the origin) (b) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$ s (c) $\pm 12$ cm

18. (a) $8e^{x^6}$ cm$^{-1}$ (b) $0$ s (initially) (c) $1$ cm

19. (a) $7$ s (b) $\frac{7}{\sqrt{2}}$ or $\frac{7\sqrt{2}}{2}$ s (c) $49$ cm

Exercises 6.5

1. $12$ cm 2. $28$ m 3. $-42.5$ cm
4. $570$ cm$^{-2}$ (b) $135$ cm (c) After $0.5$ s
5. $(e^t + 1)$ cm 6. $163$ m 7. $(a) 95$ cm$^{-1}$ (b) $175$ cm
8. $h = -4.9t^2 + 4t + 2$ 9. $262$ m 10. $(e^t - 3)$ m
11. $-744$ cm 12. $(2\pi - 3)$ cm 13. $1.77$ m 14. $893$ m
15. $(\sqrt{3} + 3)$ m (b) $-4\sqrt{3}$ ms$^{-2}$
16. $(\frac{4}{15})$ ms$^{-1}$ (b) $x = \frac{2n}{3} - 2\ln (n + 3)$ m

(c) $v = \frac{2}{3} - \frac{2}{3t + 3}$
$= \frac{2(t + 3) - 6}{3(t + 3)}$
$= \frac{2t + 6 - 6}{3(t + 3)}$
$= \frac{2t}{3t + 9}$
$t \geq 0$
When $t = 0$: $v = 0$
When $t > 0$: $v > 0$
Also $2t > 0$ and $3t + 9 > 0$ when $t > 0$
So $2t < 3t + 9$
$
\frac{2t}{3t + 9} < 1$
\therefore $0 \leq v < 1$

17. $(a) 5e^{45}$ ms$^{-1}$ (b) $e^{10}$ m (c) $\ddot{x} = 25e^{6t}$ (d) $50$ ms$^{-2}$

Test yourself 6

1. (a) $0$ m, $0$ ms$^{-1}$, $8$ ms$^{-2}$ (b) $0, 0.8$ s (c) $0.38$ m
2. $-76$ m, $-66$ ms$^{-2}$ 3. $39.6$ years
4. (a) 6 cms\(^{-1}\)  (b) 145 855.5 cms\(^{-2}\)
   (c) \(x = 2e^{3t}\)
   \(\dot{x} = 6e^{3t}\)
   \(\ddot{x} = 18e^{3t}\)
   \(= 9(2e^{3t})\)
   \(= 9x\)

5. 1 m

6. \(x = 2 \sin 3t\)
   \(\dot{x} = 6 \cos 3t\)
   \(\ddot{x} = -18 \sin 3t\)
   \(= -9(2 \sin 3t)\)
   \(= -9x\)

7. (a) 2, 6 s
   (b) (i) 16 cm
       (ii) 15 cms\(^{-1}\)
       (iii) –18 cms\(^{-2}\)
   (c) Particle is 16 cm to the right of the origin, travelling at 16 cms\(^{-1}\) to the right. Acceleration is –18 cms\(^{-2}\) (to the left), so the particle is slowing down.

8. (a) (i) 18 ms\(^{-2}\)
       (ii) 15 ms\(^{-1}\)
       (iii) –28 m
   (b) Particle is 28 m to the left of the origin, travelling at 15 ms\(^{-1}\) to the right, with 18 ms\(^{-2}\) acceleration (to the right), so the particle is speeding up.

9. (a) \(t_1, t_2, t_3\)
    (b) \(t_4, t_5\)
    (c) \(t_6, t_7\) and after \(t_8\)
    (d) (i)

10. (a) 48.2%  (b) 1052.6 years

11. (a) \(\frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots\)
    (b) \(\frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots\)
    (c) \(-\frac{1}{\sqrt{2}}\) ms\(^{-2}\)

12. (a) 15 m  (b) 20 m  (c) 4 s

13. (a) 16 941  (b) 1168 birds/year
    (c) 18.3 years

14. (a) (i)

15. 55 033 m

Challenge exercise 6

1. (a) \(x = -\frac{2}{3} \cos 3t + \frac{8}{3} = \frac{2(4 - \cos 3t)}{3}\)
   (b) \(\ddot{x} = -9\left(x - \frac{8}{3}\right)\)

2. (a) 1 m, 0 ms\(^{-1}\)  (b) 3.26 × 10\(^7\) ms\(^{-2}\)
   (c) Show \((t^3 + 1)^6 = 0\) has no solution for \(t \geq 0\)

3. (a) \(x = \cos 4t\)  (b) ±2√3 cms\(^{-1}\)

4. (a) \(v = 5 \cos 5t\)
    \(x = \int (5 \cos 5t) dt\)
    = \(\sin 5t + C\)
   When \(t = 0, x = 0\)
   \(0 = \sin 0 + C\)
   = \(C\)
   \(\therefore x = \sin 5t\)
   \(a = \frac{d}{dt}(5 \cos 5t)\)
   = \(-25 \sin 5t\)
   = \(-25x\)
   (b) 25 ms\(^{-2}\)  (c) –7.5 ms\(^{-2}\)

5. (a) 19.9 years  (b) 16%
6. \[ N = \frac{bN_0}{bN_0 + (k - bN_e)e^{kt}} \]
\[ \frac{dN}{dt} = -\frac{kN}{bN_0 + (k - bN_e)e^{kt}} \]
\[ = \frac{bN_0 + (k - bN_e)e^{kt}}{kN_0} \]
\[ = \frac{[bN_0 + (k - bN_e)e^{kt}]}{kN_0} \]
\[ = \frac{bN_0 + (k - bN_e)e^{kt}}{kN_0} \]
\[ = \frac{bN_0 + (k - bN_e)e^{kt}}{kN_0} \]
\[ = \frac{bN_0 + (k - bN_e)e^{kt}}{kN_0} \]
\[ = \frac{bN_0 + (k - bN_e)e^{kt}}{kN_0} \]
\[ = \frac{bN_0 + (k - bN_e)e^{kt}}{kN_0} \]
\[ = \frac{bN_0 + (k - bN_e)e^{kt}}{kN_0} \]
\[ = kN - bN^2 \]

7. \( 3e \text{ cm}^2 \)

Practice assessment task set 2

1. 3.2 years 2. 1.099 3. 27 m 4. -2
5. \( x = e^t, x = 3.42 \) 6. \( 3x^2 + 2e^{2t} \) 7. \( x = \pm \frac{1}{2} \)
8. (a) 47.5 g (b) 3.5 g/year (c) After 9.3 years
9. (a) 0 cm/s (b) \( a = -18 \sin 3t = -9x \)
10. (a) 7750 L (b) 28 minutes
11. 622.1 units³ 12. \( \frac{12}{4x + 3} \)
13. \( \frac{1}{3} \log (3x^2 + 3x - 2) + C \)
14. (a) 100 L (b) 40 L (c) -16 L per minute, i.e. leaking at the rate of 16 L per minute (d) 12.2 minutes
15. \( 2x^3 - x^2 + 4 \log x + C \) 16. \( 3 \log 2 \)
17. (a) 7.8 cm (b) -0.06 cm/s²
18. \( \frac{1}{4}e^t + x + C \) 19. \( \frac{1 - 2x}{e^{2x}} \)
20. (a) \( k = 0.101 \) (b) 2801 (c) 20 days (d) (i) 11 people per day (ii) 283 people per day
21. (a) 1.77 (b) \( \frac{1}{x \log 3} \)
22. \( e^{2\pi} (e^t - 1) \) units³
23. (a) \( v = 6 \text{ ms}^{-1}, a = 0 \text{ ms}^{-2} \) (b) 3 m (c) \( \frac{3\pi}{4}, \frac{5\pi}{4} \ldots \) seconds (d) \( a = -12 \sin 2t \)
\[ = -4 \left( 3 \sin 2t \right) = -4x \]
24. 4.67 units² 25. -27 m 26. \( x = 0.28 \)
27. \( x - y + 2 = 0 \) 28. \( \left( -\frac{1}{2} - \frac{1}{2e} \right) \) minimum
29. (a) 1.60 cm² (b) 0.17 cm²
30. 15 months
31. (a) \( \frac{5\pi}{6} \) cm (b) \( \frac{25\pi}{12} \) cm²
32. (a) \( x = \cos \frac{\pi}{2} \)
33. 1 34. \( \cot x \) 35. \( 5e^{xt} \) sec² \( (e^{xt} + 1) \) 36. \( \sqrt{3} \)
37. (a) \( y = \sin 2x \)
(b) 2 units²
38. 0.348
39. (a) \( e^t \) (sin \( x + \cos x \)) (b) \( 3 \tan^2 x \) sec² \( x \)
\[ = -6 \sin \left( 3x - \frac{\pi}{2} \right) \]
40. (a) 546 ms⁻¹ (b) \( a = 20e^{2t} \) (c) 20 ms⁻² (d) \( x = 4(5e^{2t}) \)
\[ = 4x \]
41. \( \ln 8 - \ln 3 = \ln \frac{8}{3} \) 42. 6 m \times 12 m
43. \( \frac{\sqrt{3} \pi}{2} \) units³
Chapter 7: Series

Exercises 7.1

1. 14, 17, 20  2. 23, 28, 33  3. 44, 55, 66  4. 85, 80, 75  5. 1, −1, −3  6. 87, 83, 79

Exercises 7.2

1. (a) $T_1 = 3$, $T_2 = 11$, $T_3 = 19$  (b) $T_1 = 5$, $T_2 = 7$, $T_3 = 9$  (c) $u_1 = 5$, $u_2 = 11$, $u_3 = 17$  (d) $T_1 = 3$, $T_2 = 0$, $T_3 = 7$  (e) $t_1 = 19$, $t_2 = 18$, $T_3 = 71$  (f) $u_1 = 3$, $u_2 = 9$, $u_3 = 27$  (g) $Q_1 = 9$, $Q_2 = 11$, $Q_3 = 15$  (h) $t_1 = 2$, $t_2 = 12$, $t_3 = 58$  (i) $T_1 = 8$, $T_2 = 31$, $T_3 = 70$  (j) $T_1 = 2$, $T_2 = 10$, $T_3 = 30$

2. (a) $T_1 = 1$, $T_2 = 4$, $T_3 = 7$  (b) $t_1 = 4$, $t_2 = 16$, $t_3 = 64$  (c) $T_1 = 2$, $T_2 = 6$, $T_3 = 12$

3. (a) 349  (b) 105  (c) 248  (d) −110  (e) −342

4. (a) 1029  (b) −59 039  (c) 1014  (d) 53  (e) 1002

5. (a), (b), (d)  6. (b), (d)  7. 16th term  8. Yes

9. 7th term  10. 23rd term  11. (a) 1728  (b) 25th term

12. (a) −572  (b) 17th term

13. $n = 33$  14. $n = 9$  15. $n = 88$, 89, 90,

16. $n = 41, 42, 43$  17. $n = 501$  18. $n = 151$

19. (a) $n = 14$  (b) −4  20. −6

Exercises 7.3

1. (a) 128  (b) 54  (c) 70  (d) 175  (e) 220  (f) $\frac{47}{60}$  (g) 40  (h) 21  (i) 126  (j) 1024

2. (a) 65  (b) 99  (c) 76  (d) 200  (e) 11  (f) 39  (g) 97  (h) 66  (i) 75  (j) 45

3. (a) $\sum_{n=1}^{k} 2n - 1$  (b) $\sum_{n=1}^{k} 7n$  (c) $\sum_{n=1}^{k} n!$  (d) $\sum_{n=1}^{k} 6n - 4$

(e) $\sum_{k=1}^{n} k^2$  (f) $\sum_{n=1}^{k} (-n)$  (g) $\sum_{n=0}^{k} 32^n$  (h) $\sum_{n=0}^{k} \frac{1}{2}$

(i) $\sum_{k=1}^{n} a + (k - 1)d$  (j) $\sum_{k=1}^{n} \arctan k$

Exercises 7.4

1. (a) $y = 13$  (b) $x = -4$  (c) $x = 72$  (d) $b = 11$  (e) $x = 7$

(f) $x = 42\frac{1}{2}$  (g) $k = 4\frac{1}{2}$  (h) $x = 1$  (i) $t = -2$  (j) $t = 3$
2. (a) 46 (b) 78 (c) 94 (d) –6 (e) 67
3. (a) 590 (b) –850 (c) 414 (d) 1610 (e) –397
4. (a) –110 (b) 12.4 (c) –8.3 (d) 37 (e) $15 \frac{4}{5}$
5. $T_n = 2n + 1$
6. (a) $T_n = 8n + 1$ (b) $T_n = 2n + 98$ (c) $T_n = 3n + 3$
   (d) $T_n = 6n + 74$ (e) $T_n = 4n - 25$ (f) $T_n = 20 - 5n$
   (g) $T_n = \frac{n + 6}{8}$ (h) $T_n = -2n - 28$ (i) $T_n = 1.2n + 2$
   (j) $T_n = \frac{3n - 1}{4}$
7. 28th term 8. 54th term 9. 30th term
10. 15th term 11. Yes 12. No
13. Yes 14. $n = 13$ 15. $n = 30, 31, 32 \ldots$
16. –2 17. 103 18. 785
19. (a) 98 (b) 87
20. $d = 9$ 21. $a = 12, d = 7$
22. 173 23. $a = 5$
24. 280 25. 1133
26. (a) $T_2 - T_1 = \log_2 x^2 - \log_2 x$
   $= 2 \log_2 x - \log_2 x$
   $= \log_2 x$
   $T_3 - T_2 = \log_2 x^3 - \log_2 x^2$
   $= 3 \log_2 x - 2 \log_2 x$
   $= \log_2 x$
Since $T_1 - T_0 = T_0 - T_2$ it is an arithmetic series with $d = \log_2 x$.
   (b) $80 \log_3 x$ or $\log_3 x^{80}$ (c) 8.6
27. (a) $T_2 - T_1 = \sqrt{12} - \sqrt{3}$
   $= \sqrt{4 \times 3} - \sqrt{3}$
   $= 2\sqrt{3} - \sqrt{3}$
   $= \sqrt{3}$
   $T_3 - T_2 = \sqrt{27} - \sqrt{12}$
   $= \sqrt{9 \times 3} - \sqrt{4 \times \sqrt{3}}$
   $= 3\sqrt{3} - 2\sqrt{3}$
   $= \sqrt{3}$
Since $T_1 - T_0 = T_0 - T_2$ it is an arithmetic series with $d = \sqrt{3}$.
   (b) $50\sqrt{3}$
28. 26 29. 122b 30. 38th term

Exercises 7.5
1. (a) 375 (b) 555 (c) 480
2. (a) 2640 (b) 4365 (c) 240
3. (a) 2050 (b) –2575
4. (a) –4850 (b) 4225
5. (a) 28 875 (b) 3276 (c) –1419 (d) 6426
   (e) 6604 (f) 598 (g) –2700
   (h) 11 704 (i) –290 (j) 1284
6. (a) 700 (b) –285 (c) –1170 (18 terms)
   (d) 6525 (e) –2286
7. 21. 8. 8. 9. 11. 10. $a = 14, d = 4$
8. (a) 1944 (b) 9216 (c) –8192
   (d) 3125 (e) $\frac{64}{729}$
9. (a) 256 (b) 26 244 (c) 1.369
   (d) –768 (e) $\frac{3}{1024}$
10. (a) 234 375 (b) 268.8 (c) –81 920
    (d) $\frac{2187}{156250}$ (e) 27
11. (a) $3 \times 2^{19}$ (b) $7^{19}$ (c) $1.04^{20}$
(d) \( \frac{1}{4} \left( \frac{1}{2} \right)^5 = \frac{1}{2^5} \) \( \text{c) } \left( \frac{3}{4} \right)^{20} \)

8. \( 11^{th} \) 9. 6th term
10. 5th term 11. No
12. 7th term 13. 11th term
14. 9th term 15. \( n = 5 \) 16. \( r = 3 \)
17. (a) \( r = -6 \)
(b) \(-18 \)
18. \( a = \frac{1}{10} \) \( r = \pm 2 \) 19. \( n = 7 \) 20. \( 208 \frac{2}{7} \)

Exercises 7.7
1. (a) 2 097 150  (b) 7 324 218
2. (a) 720 600  (b) 26 240
3. (a) 131 068  (b) 32 769 65 536
4. (a) 7 812  (b) 35 \( \frac{55}{64} \)
(c) 8 403  (d) 273  (e) 255
5. (a) 255  (b) 364 \( \frac{729}{729} \)
(d) 1 127 \( \frac{128}{128} \)  (e) 87 376
6. (a) 1 792  (b) 3 577
7. 148.58  8. 133.33
9. \( n = 9 \) 10. 10 terms
11. \( a = 9 \) 12. 10 terms
13. (a) 333 502.39  (b) $178 550.21
14. (a) \( \sum_{k=1}^{n} 2(-5)^{k-1} \)  (b) \( S_n = \frac{(-5)^n - 1}{3} = \frac{1 - (-5)^n}{3} \)
15. 2 146

Puzzles
1. Choice 1 gives $465.00. Choice 2 gives $10 737 418.23!
2. 382 apples

Exercises 7.8
1. (a) Yes LS = 13 \( \frac{1}{2} \)  (b) No  (c) Yes LS = 12 \( \frac{4}{5} \)  (d) No
(e) Yes LS = 3  (f) Yes LS = \( \frac{25}{32} \)  (g) No
(h) Yes LS = \( -\frac{5}{22} \)  (i) No  (j) Yes LS = \( \frac{7}{2} \)
2. (a) 80  (b) 426 \( \frac{2}{3} \)  (c) 66 \( \frac{2}{3} \)  (d) 12  (e) \( \frac{7}{10} \)  (f) 54
(g) \(-10 \frac{2}{7} \)  (h) \( \frac{9}{20} \)  (i) 48  (j) \(-\frac{16}{39} \)
3. (a) \( \frac{7}{12} \)  (b) \( \frac{4}{27} \)  (c) \( \frac{1}{12500} \)  (d) \( \frac{1}{64} \)  (e) 3 645 \( \frac{4096}{25} \)
4. (a) \( 1\frac{1}{4} \)  (b) \( \frac{2}{5} \)  (c) \( \frac{1}{48} \)  (d) \( \frac{1}{2} \)  (e) 3  (f) 5  (g) \( \frac{2}{5} \)

(b) \(-5 \frac{1}{3} \)  (i) \( \frac{4}{5} \)  (j) \( \frac{5}{6} \)
5. \( a = 4 \)  6. \( r = \frac{2}{5} \)  7. \( a = 5 \frac{3}{5} \)  8. \( r = \frac{7}{8} \)  9. \( r = -\frac{1}{4} \)
10. \( r = -\frac{2}{3} \) 11. \( a = 3, r = \frac{2}{3} \) and \( a = 6, r = \frac{1}{3} \)
12. \( a = 192, r = -\frac{1}{4} \)  LS = 153 \( \frac{2}{5} \)
13. \( a = 1, r = \frac{2}{3} \)  LS = 3, \( a = -1, r = -\frac{2}{3} \)  LS = \( -\frac{3}{4} \)
14. \( a = 150, r = \frac{3}{5} \)  LS = 375
15. \( a = \frac{2}{5}, r = \frac{2}{5} \)  LS = \( \frac{1}{5} \) 16. \( a = 3, r = \frac{2}{5} \) and \( a = 2, r = \frac{3}{5} \)
17. \( x = \frac{21}{32} \) 18. (a) \(-1 < k < 1 \)  (b) \(-\frac{2}{5} \)  (c) \( k = \frac{3}{4} \)
19. (a) \(-\frac{1}{2} < p < \frac{1}{2} \)  (b) \( \frac{5}{7} \)  (c) \( p = \frac{1}{14} \)

20. \( LS - S_n = a \left( \frac{1 - r^n}{1 - r} \right) = \frac{a - a(1 - r^n)}{1 - r} = \frac{a - a + ar^n}{1 - r} \)
= \( \frac{ar^n}{1 - r} \)

Exercises 7.9
1. (a) 210  (b) 13th  (c) 57
2. (a) 39  (b) 29th  (c) 32
3. (a) 3n + 3
(b) \( S_n = \frac{n}{2}[2a + (n - 1)d] \)
= \( \frac{n}{2}[2 \times 6 + (n - 1) \times 3] \)
= \( \frac{n}{2}(12 + 3n - 3) \)
= \( \frac{n}{2}(3n + 9) \)
= \( \frac{3}{2}n(n + 3) \)
4. (a) (i) $23 200  (ii) $26 912  (iii) $31 217.92
(b) $102 345.29  (c) 6.2 years
5. (a) (i) 93%  (ii) 86.49%  (iii) 80.44%
(b) 33.67%  (c) 19 weeks
6. (a) 0.01 m  (b) 91.5 m
7. (a) 49  (b) 4 mm
8. (a) 3k m  (b) k(3k + 3) m  (c) 9
9. (a) 96.04%  (b) 34  (c) 68.6
10. (a) 77.4%  (b) 13.5  (c) 31.4
11. (a) \( \frac{4}{9} \)  (b) \( \frac{7}{9} \)  (c) \( \frac{2}{9} \)
(d) \( \frac{25}{99} \)  (e) \( \frac{2 - \frac{9}{11}}{7 - \frac{7}{30}} \)
12. (a) 74.7 cm (b) 75 m
17. (a) 74.7 cm (b) 75 m
18. 300 cm 19. 3.5 m 20. 32 m
21. (a) 1, 8, 64... (b) 16 777 216 people (c) 19 173 961 people

Exercises 7.10
1. (a) $740.12 (b) $14 753.64 (c) $17 271.40 (d) $9385.69 (e) $5298.19
2. (a) $2007.34 (b) $2015.87 (c) $2020.28
3. (a) $4930.86 (b) $4941.03
4. $408.24
5. $971.40
6. $1733.99
7. $3097.06
8. $22 800.81
9. $691.41
10. $1776.58
11. $14 549.76
12. $1 301 694.62

Exercises 7.11
1. $27 882.27 2. $83 712.95
3. $50 402.00 4. $163 907.81
5. $40 728.17 6. $29 439.16
7. $67 596.72 8. $62 873.34
9. $164 155.56 (28 years) 10. $106 379.70
11. $3383.22 12. $65 903.97
13. $2846.82 14. $13 601.02
15. $6181.13 16. $4646.71 17. $20 405.74
18. (a) $26 361.59 (b) $46 551.94
19. $45 599.17
20. (a) $7335.93 (b) $1467.18
21. $500 for 30 years 22. Yes, $259.80 over

Exercises 7.12
1. $1047.62 2. $394.46 3. $139.15
4. (a) $966.45 (b) $1265.79
5. $2519.59
6. (a) $592.00 (b) $39 319.89
7. (a) $77.81 (b) $2645.42
8. $78 700
9. (a) Get Rich $949.61, Capital Bank $491.27 (b) $33 427.80 more through Capital Bank
10. $43 778.80 11. $61 292.20
12. NSW Bank $175.49 a month ($5791.25 altogether) Sydney Bank $154.39 a month ($5557.88 altogether) Sydney Bank is better
13. (a) $249.69 (b) $13 485.12
14. (a) $13 251.13 (b) $374.07 (c) $20 199.77
15. (a) $1835.68 (b) $9178.41

Test yourself 7
1. (a) $T_n = 4n + 5 (b) $T_n = 14 - 7n (c) $T_n = 2 \cdot 3^{n-1} (d) $T_n = 200 \left( \frac{1}{4} \right)^{n-1} (e) $T_n = -2^n
2. (a) 2 (b) 1185 (c) 1183 (d) $T_{15} = S_{14} - S_{14}$ (e) $n = 16$
3. (a) 11 125 (b) 1 $\frac{13}{140}$ (c) 3985 785 (d) $34 750 (e) $\frac{1}{2}$
4. (a) Each slat rises 3 mm so the bottom one rises up 30 \times 3 mm or 90 mm. (b) $87\, \text{mm}$ (c) $90, 87, 84, \ldots$ which is an arithmetic sequence with $a = 90, d = -3$ (d) $42\, \text{mm}$ (e) $1395\, \text{mm}$
5. $3400.01
6. (a) (i) (b) (ii) (c) (i) (d) (iii) (e) (i) (f) (ii) (g) (ii) (h) (i) (i) (j) (i)
7. \( n = 108 \)
8. (a) $24,050 (b) $220,250
9. \( a = -33, d = 13 \)
10. (a) 59 (b) 80 (c) 18th term
11. (a) \( x = 25 \) (b) \( x = \pm 15 \)
12. (a) \( \frac{4}{9} \) (b) \( \frac{13}{18} \) (c) \( \frac{19}{33} \)
13. \( x = 3 \)
14. (a) 136 (b) 44 (c) 6
15. 121 \( \frac{1}{2} \)
16. $8066.42
17. (a) \( T_n = 4n + 1 \) (b) \( T_n = 1.07n - 1 \)
18. (a) \(-1 < x < 1\) (b) \( \frac{1}{2} \) (c) \( x = \frac{1}{3} \)
19. \( d = 5 \)
20. (a) 39 words/min (b) 15 weeks
21. (a) $59,000 (b) $15,988.89
22. 4.8 \( \text{m} \)
23. \( x = -\frac{2}{17} \)
24. (a) $2385.04 (b) $2392.03
25. 1300
26. (a) 735 (b) 4315
27. (a) $1432.86 (b) $343,886.91
28. \( n = 20 \)
29. \( n = 11 \)

Challenge exercise 7
1. (a) 8.1 (b) 19th term
2. (a) \( \frac{\pi}{4} \) (b) \( \frac{9\pi}{4} \) (c) \( \frac{33\pi}{4} \)
3. (a) 2097 170 (b) -698 775
4. (a) $40 (b) $2880
5. 6th term 6. 17 823
7. 5 terms 8. \( n = 1, 2, 3 \)
9. \( n = 56 \)
10. $1799.79
11. \( x = \frac{3}{8} \)
12. $8522.53
13. \( k = 20 \)
14. (a) $10 100 (b) $11 268.25 (c) $4212.41 (d) $2637.23
15. (a) \( \csc^2 x \)
(b) \( -1 \leq \cos x \leq 1 \)
So \( 0 \leq \cos^2 x \leq 1 \)
So the limiting sum exists.
16. $240,652.62

Chapter 8: Probability

Exercises 8.1
1. \( \frac{1}{30} \)
2. \( \frac{1}{52} \)
3. \( \frac{1}{6} \)
4. \( \frac{1}{40} \)
5. \( \frac{1}{20000} \)
6. (a) \( \frac{4}{7} \) (b) \( \frac{3}{7} \)
7. \( \frac{3}{37} \)
8. \( \frac{1}{12} \)
9. (a) \( \frac{11}{20} \) (b) \( \frac{3}{4} \)
10. (a) \( \frac{1}{6} \) (b) \( \frac{1}{2} \) (c) \( \frac{1}{3} \)
11. (a) \( \frac{1}{62} \) (b) \( \frac{3}{31} \) (c) \( \frac{1}{2} \) (d) \( \frac{99}{124} \)
12. (a) \( \frac{8}{15} \) (b) \( \frac{7}{15} \) (c) \( \frac{12}{31} \)
13. \( \frac{1}{50} \)
14. \( \frac{1}{2} \)
15. \( \frac{23}{44} \)
16. (a) \( \frac{7}{31} \) (b) \( \frac{7}{31} \) (c) \( \frac{12}{31} \)
17. \( \frac{1}{175} \)
18. 8
19. \( \frac{25}{43} \)
20. 34
21. \( \frac{1}{3} \)
22. (a) \( \frac{1}{6} \) (b) \( \frac{1}{3} \) (c) \( \frac{5}{6} \)

23. (a) False: outcomes are not equally likely. Each horse and rider has different skills.
(b) False: outcomes are not equally likely. Each golfer has different skills.
(c) False: outcomes are not dependent on the one before. Each time the coin is tossed, the probability is the same.
(d) False: outcomes are not dependent on the one before. Each birth has the same probability of producing a girl or boy.
(e) False: outcomes are not equally likely. Each car and driver has different skills.

Exercises 8.2
1. \( \frac{5}{11} \)
2. \( \frac{2}{9} \)
3. 99.8%
4. 0.73
5. 38%
6. 98.5%
7. \( \frac{22}{23} \)
8. \( \frac{5}{18} \)
9. 0.21
10. 91.9%
11. \( \frac{7}{8} \)
12. \( \frac{46}{49} \)
13. (a) \( \frac{2}{15} \) (b) \( \frac{13}{15} \)
14. \( \frac{7}{11} \)
15. \( \frac{15}{16} \)

Exercises 8.3
1. (a) \( \frac{3}{10} \) (b) \( \frac{3}{5} \) (c) \( \frac{11}{20} \) (d) \( \frac{7}{10} \)
2. (a) \( \frac{1}{5} \) (b) \( \frac{1}{2} \) (c) \( \frac{3}{5} \) (d) \( \frac{3}{5} \) (e) \( \frac{19}{50} \)
5. (a) $\frac{27}{45}$  (b) $\frac{4}{9}$  (c) $\frac{2}{3}$  
6. (a) $\frac{3}{14}$  (b) $\frac{13}{28}$  (c) $\frac{9}{28}$

7. (a) $\frac{21}{80}$  (b) $\frac{17}{80}$  (c) $\frac{21}{40}$  
8. (a) $\frac{1}{10}$  (b) $\frac{11}{20}$  (c) $\frac{7}{20}$

9. (a) $\frac{7}{25}$  (b) $\frac{2}{15}$  (c) $\frac{44}{75}$  
10. (a) $\frac{3}{10}$  (b) $\frac{2}{5}$  (c) $\frac{3}{10}$

Exercises 8.4

1. $\frac{1}{36}$  2. $\frac{1}{4}$  3. $\frac{1}{8}$  4. $\frac{1}{4}$  5. $\frac{25}{121}$

6. (a) 0.0441  (b) 0.6241  7. 80.4%  8. 32.9%

9. (a) $\frac{9}{49}$  (b) $\frac{15}{91}$  10. $\frac{3}{2075}$  11. $\frac{19}{99}$  12. $\frac{1}{16170}$

13. (a) $\frac{29791}{35937}$  (b) $\frac{8}{35937}$  (c) $\frac{35929}{35937}$

14. (a) $\frac{1}{2400}$  (b) $\frac{1}{5760000}$  (c) $\frac{5755201}{5760000}$

15. (a) $\frac{1}{7776}$  (b) $\frac{3125}{7776}$  (c) $\frac{4651}{7776}$

16. (a) $\frac{9}{25000000}$  (b) $\frac{24970009}{25000000}$  (c) $\frac{29991}{25000000}$

17. (a) $\frac{1}{4}$  (b) $\frac{9}{100}$  (c) $\frac{9}{100}$

18. (a) $\frac{1}{22}$  (b) $\frac{1}{11}$  (c) $\frac{7}{22}$  (d) $\frac{15}{22}$

19. (a) 61.41%  (b) 0.34%  (c) 99.66%

20. (a) $\frac{1}{2^n}$  (b) $\frac{1}{2^{n^2}}$  (c) $1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

Exercises 8.5

1. (a) $\frac{1}{4}$  (b) $\frac{1}{4}$  (c) $\frac{1}{2}$  
2. (a) $\frac{1}{8}$  (b) $\frac{3}{8}$  (c) $\frac{7}{8}$

3. (a) $\frac{1}{900}$  (b) $\frac{1}{900}$  (c) $\frac{1}{450}$  
4. (a) $\frac{1}{25}$  (b) $\frac{2}{25}$

5. (a) $\frac{25}{169}$  (b) $\frac{80}{169}$

6. (a) 27.5%  (b) 23.9%  (c) 72.5%

7. (a) 0.42  (b) 0.09  (c) 0.49  
8. (a) $\frac{189}{1000}$  (b) $\frac{441}{1000}$

(c) $\frac{657}{1000}$  9. (a) 0.325  (b) 0.0034  (c) 0.997

10. (a) $\frac{60}{121}$  (b) $\frac{6}{11}$

Test yourself 8

1. (a) 80.4%  (b) 1.4%  (c) 99.97%

2. (a)

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(b) (i) $\frac{1}{6}$  (ii) $\frac{1}{6}$  (iii) $\frac{1}{2}$  
3. (a) (i) $\frac{1}{40}$  (ii) $\frac{39}{40}$

(b) $\frac{39}{796}$

4. (a) $\frac{4}{15}$  (b) $\frac{1}{10}$

5. False: the events are independent and there is the same chance next time $\left(\frac{1}{4}\right)$

6. (a) $\frac{1}{2}$  (b) $\frac{29}{100}$  (c) $\frac{1}{5}$  (d) $\frac{11}{25}$  (e) $\frac{16}{25}$

7. (a) $\frac{2}{5}$  (b) $\frac{7}{15}$  (c) $\frac{2}{15}$  
8. (a) $\frac{35}{72}$  (b) $\frac{35}{66}$

9. $\frac{1}{56}$

10. (a) 0.009%  (b) 12.9%  (c) $\frac{1}{13}$  (b) $\frac{3}{13}$  (c) $\frac{5}{26}$

11. (a) $\frac{1}{13}$  (b) $\frac{3}{13}$  (c) $\frac{5}{26}$
12. (a) $\frac{5}{12}$  (b) $\frac{1}{3}$
13. (a) $\frac{9}{40}$  (b) $\frac{3}{10}$  (c) $\frac{27}{160}$  (d) $\frac{4}{25}$
14. (a) $\frac{1}{200}$  (b) $\frac{81}{200}$  (c) $\frac{11}{100}$  (d) $\frac{3}{25}$
15. (a) $\frac{1}{15}$  (b) $\frac{4}{5}$
16. (a) $\frac{1}{50}$  (b) $\frac{147}{7450}$  (c) $\frac{377}{3725}$
17. (a) $\frac{80}{361}$  (b) $\frac{40}{171}$  (c) $\frac{2}{9}$  (d) $\frac{1}{3}$
18. (a) $\frac{21}{50}$  (b) $\frac{3}{25}$  (c) $\frac{23}{50}$

Challenge exercise 8
1. (a) $\frac{1}{7}$  (b) $\frac{4}{7}$
2. (a) 0.04  (b) 0.75  (c) 0.25
3. (a) $\frac{1}{54,145}$  (b) $\frac{33}{173,264}$
4. (a) $\frac{4}{13}$  (b) $\frac{25}{52}$  (c) $\frac{4}{13}$
5. No—any combination of numbers is equally likely to win.
6. (a) 0  (b) $\frac{1}{10}$  (c) $\frac{3}{10}$
7. (a) $\frac{1}{7776}$  (b) $\frac{1}{1296}$
8. (a) $\frac{3}{10}$  (b) $\frac{12}{145}$  (c) $\frac{4}{9}$
9. (a) $\frac{1}{144}$  (b) $\frac{5}{144}$  (c) $\frac{7}{144}$  (d) $\frac{3}{144}$

Practice assessment task set 3
1. $\frac{324}{625}$  2. (a) $\frac{1296}{2401}$  (b) $\frac{864}{2401}$  (c) $\frac{1105}{2401}$
3. $\frac{4}{9}$  4. $\frac{3}{4}$  5. $\$2929.08$  6. $\pm 104 + 52 \pm 26 + \ldots$
7. 44th term  8. $\$945$
9. (a) $\frac{1}{36}$  (b) $\frac{1}{6}$  (c) $\frac{36}{11}$  (d) $\frac{5}{36}$  (e) $\frac{5}{12}$
10. $\frac{9841}{6561}$  11. 2.4 m
12. (a) 3 000 000  (b) 3 000 336  (c) 146 insects per day
13. (2, 0), inflexion

14. (a) $\frac{7}{50}$  (b) $\frac{11}{20}$
15. (a)

\[ d^2 = (230 - 65t)^2 + (125 - 80t)^2 \]
\[ = 52900 - 29900t + 4225t^2 + 15625 - 20000t + 6400t^2 \]
\[ = 10625t^2 - 49900t + 68525 \]
(b) 2.3 h  (c) 109.7 km
16. 199; 5050  17. $\frac{1}{110}$
18. (a) $T_1 = 4$, $T_2 = 11$, $T_3 = 18$, $T_4 = 81$
(b) 1410  (c) 29th term
19. $-\frac{1}{\sqrt{2}}$
20. (a)

\[ y = x^3 - 5x + 6 \]
21. $-15 - 4 + 7 + \ldots$
22. (a) $\frac{5}{33}$  (b) $\frac{35}{66}$
23. $\$2851.52$
24. (a) $v = -12 \sin 4t$  (b) $a = -48 \cos 4t$  (c) 3 cm
(d) $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \ldots$  (e) $\pm 3$ cm  (f) $t = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \ldots$
(g) $a = -48 \cos 4t$
\[ = -16(3 \cos 4t) \]
\[ = -16x \]
25. \( \frac{26}{45} \)  26. (a) \( \frac{1}{12} \) (b) \( \frac{1}{4} \)  27. $180.76

28. \[ AC^2 = 16^2 = 256 \]
   \[ AB^2 + BC^2 = 9.6^2 + 12.8^2 = 256 \]
   Since \( AC^2 = AB^2 + BC^2 \), 
   \( \triangle ABC \) is right angled at \( \angle B \).

29. \( x = \frac{5}{6} \)

30. 

Let \( ABCD \) be a rhombus with \( AB = BC \).
   \( AB = DC \) and \( AD = BC \)
   (opposite sides in a parallelogram)
   \( \therefore \) \( AB = BC = DC = AD \)
   \( \therefore \) all sides are equal

31. \( \frac{1}{40000} \)

32. (a) \( k \neq 0.025 \)  (b) after 42.4 years  (c) 20.6 years

33. 76473  34. 450 cm²

35. (a) 12 ms⁻¹  (b) 48e⁴ ms⁻²  
   (c) \( x = 3e^{4t} + 2 \)
   \[ \dot{x} = 12e^{4t} \]
   \[ \ddot{x} = 48e^{4t} \]
   \[ \dddot{x} = 4(12e^{4t}) \]
   \[ \dddot{x} = 48 \]
   (d)

36. \( n = 4 \)

37. \( y = \sin 7x \)
   \[ \frac{dy}{dx} = 7 \cos 7x \]
   \[ \frac{d^2y}{dx^2} = 7(-7 \sin 7x) \]
   \[ = -49 \sin 7x \]
   \[ = -49y \]
   \( \frac{1}{10} \)

38. \( \frac{1}{10} \)

39. (a) Square 46.3 m × 46.3 m, rectangle 30.9 m × 92.7 m
   (b) $8262.38

40. (a) \( \frac{5}{15} \)  (b) \( \frac{5}{12} \)  (c) \( \frac{1}{6} \)  (d) \( \frac{7}{36} \)  (e) \( \frac{1}{2} \)

41. 

42. (a) \( \theta = 76°52' \)  (b) 0.92 cm²  43. $18399.86

44. (a) \[ \log 3 + \log 9 + \log 27 + \ldots \]
   \[ = \log 3 + \log 3^2 + \log 3^3 + \ldots \]
   \[ = \log 3 + 2 \log 3 + 3 \log 3 + \ldots \]
   Arithmetic series, since
   \[ 2 \log 3 - \log 3 = 3 \log 3 - 2 \log 3 \]
   \[ = \log 3 \]
   (b) 210 log 3

45. 

46. (a) 12.6 mL  (b) 30 minutes  47. (a)

48. $277.33  49. (d)  50. (b)  51. (c)  52. (c), (d)

53. (d)  54. (a)  55. (c)  56. (a)  57. (d)

Sample examination papers

Mathematics—Paper 1

1. (a) 0.75  
   (b) \( (3x - 2)(x - 3) \)
   (c) \[ \frac{6}{x} + \frac{6}{x-1} \]
   \[ = 6 \left( \frac{x-1}{x} \right) \]
   \[ = 6(5) \]
   \[ 3x - 2(x - 1) = 30 \]
   \[ 3x - 2x + 2 = 30 \]
   \[ x + 2 = 30 \]
   \[ x = 28 \]
   (d) \[ 12 = \frac{1}{3} \pi r^2 \]
   \[ 36 = \pi r^2 \]
   \[ \frac{36}{\pi} = r^2 \]
   \[ \sqrt{\frac{36}{\pi}} = r \]
   \[ 3.39 = r \]
(e) 3300
(f) \( x + 3 < 7 \)
\[ x < 4 \]
or \(-x + 3 < 7\)
\[ x > -10 \]
\[ \therefore -10 < x < 4 \]

2. (a) (ii) Since \( AB = BD, AB:AD = 1:2 \)
\[ \angle ABC = \angle ADE \quad \text{(corresponding \( s, BC \mid DE \))} \]
\[ \angle ACB = \angle AED \quad \text{(similarly)} \]
\[ \angle A \text{ is common} \]
\[ \therefore \Delta ABC \parallel \Delta ADE \]
\[ \because \quad \frac{AB}{AD} = \frac{AE}{AF} = \frac{1}{2} \]
\[ \therefore \quad AE = 2AC \]
\[ AC + CE = 2AC \]
\[ \therefore \quad CE = AC \]

(iii) \[ \frac{AB}{BC} = \frac{1}{3} \]
\[ \frac{AD}{DE} = \frac{2}{3} \]
\[ \therefore \quad 3.4 = \frac{1}{2} \]
\[ DE = 2 \times 3.4 \]
\[ = 6.8 \text{ cm} \]

(b) (ii) \( \angle NOM = 105^\circ \) (straight angle)
\[ \angle NMO = 32^\circ \] (angle sum of \( \Delta \))
\[ \frac{a}{\sin A} = \frac{b}{\sin B} \]
\[ \frac{MO}{\sin 43^\circ} = \frac{5}{\sin 32^\circ} \]
\[ MO = \frac{5 \sin 43^\circ}{\sin 32^\circ} \]
\[ = 6.4 \text{ m} \]

(iii) \[ \frac{a}{\sin A} = \frac{b}{\sin B} \]
\[ \frac{MP}{\sin 75^\circ} = \frac{6.4}{\sin 53^\circ} \]
\[ MP = \frac{6.4 \sin 75^\circ}{\sin 53^\circ} \]
\[ = 8 \text{ m} \]

3. (a) (i) \( \sqrt{x} + 5x^4 + 1 = x^\frac{1}{2} + 5x^3 + 1 \)
\[ \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + 15x^3 = \frac{1}{2\sqrt{x}} + 15x^3 \]

(ii) \( 3 \ln x + \frac{1}{x} = 3 \ln x + x^{-1} \)
\[ \frac{dy}{dx} = 3 \cdot \frac{1}{x} - x^{-2} \]
\[ = \frac{3}{x} - \frac{1}{x^2} \]
\[ = 3x - 1 \]
\[ = \frac{2x + 3}{x^2} \]

(iii) \[ \frac{dy}{dx} = 5(2x + 3)^3 \times 2 \]
\[ = 10(2x + 3)^3 \]

(b) (i) \[ \frac{x^2}{2} - \frac{1}{-1}e^{-x^2} + C = \frac{x^2}{2} + e^{-x^2} + C \]

(ii) \[ (-\cos \theta + \theta)^{\prime} = (-\cos \pi + \pi) - (-\cos 0 + 0) \]
\[ = (-1) + \pi + 1 \]
\[ = 2 + \pi \]

(c) (i) \[ 5 \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \]
\[ = 5\sqrt{2} + 5 \]
\[ \frac{\sqrt{2} + 1}{\sqrt{2} - 1} \]
\[ = 5 \sqrt{2} + 5 \]
\[ = 5 \sqrt{50} \]
\[ \therefore \quad a = 5, b = 50 \]

4. (a) (i) \( y = 2 \)
\[ B (-1, 5) \]
\[ A (3, 2) \]
\[ 6 \]
\[ 4 \]
\[ 2 \]
\[ 1 \]
\[ 0 \]
\[ -3 -2 -1 \]
\[ 1 \]
\[ 2 \]
\[ 3 \]
\[ x \]
\[ y \]

(ii) Substitute \( A(3, 2) \) into \( 3x + 4y - 17 = 0 \)
\[ 3(3) + 4(2) - 17 = 0 \]
\[ 9 + 8 - 17 = 0 \]
\[ 0 = 0 \quad \text{(true)} \]

\[ \therefore \quad A \text{ lies on the line} \]
Substitute \( B(-1, 5) \) into \( 3x + 4y - 17 = 0 \)
\[ 3(-1) + 4(5) - 17 = 0 \]
\[ -3 + 20 - 17 = 0 \]
\[ 0 = 0 \quad \text{(true)} \]

\[ \therefore \quad B \text{ lies on the line} \]
Since both \( A \) and \( B \) lie on the line \( 3x + 4y - 17 = 0 \),
this is the equation of \( AB \)

(iii) \[ d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \]
\[ = \frac{|3(0) + 4(0) - 17|}{\sqrt{3^2 + 4^2}} \]
\[ = \frac{|-17|}{\sqrt{9 + 16}} \]
\[ = \frac{17}{\sqrt{25}} \]
\[ = \frac{17}{5} \]
\[ = 3.4 \text{ units} \]

(iv) Length \( AB: d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
\[ = \sqrt{[3(-1)]^2 + (2 - 5)^2} \]
\[ = \sqrt{16 + 9} \]
\[ = \sqrt{25} \]
\[ = 5 \]
Maths In Focus Mathematics HSC Course

5. (a) (i) $\frac{800}{24600} \times 100\% = 3.25\%$

(ii) Arithmetic series with $a = 24600$, $d = 800$ and $n = 12$

$T_n = a + (n-1)d$

$T_{12} = 24600 + (12 - 1)800$

$= 33400$

So Kate earns $33400 in the 12th year.

(iii) $S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{12}{2}[2 \times 24600 + (12 - 1)800]$

$= 348000$

So Kate earns $348000 over the 12 years.

(b) (i) $y = x^3 - 3x^2 - 9x + 2$

$y' = 3x^2 - 6x - 9$

$y'' = 6x - 6$

For stationary points, $y' = 0$

i.e. $3x^2 - 6x - 9 = 0$

$3(x - 3)(x + 1) = 0$

$\therefore x = 3, -1$

When $x = 3$, $y = 3^3 - 3(3)^2 - 9(3) + 2$

$= -25$

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 2$

$= 7$

So $(3, -25)$ and $(-1, 7)$ are stationary points.

At $(3, -25)$, $y'' = 6(3) - 6$

$> 0$ (minimum point)

At $(-1, 7)$, $y'' = 6(-1) - 6$

$< 0$ (maximum point)

$\therefore (-1, 7)$ is a maximum, $(3, -25)$ minimum stationary point

(ii) For inflexions, $y'' = 0$

i.e. $6x - 6 = 0$

$6x = 6$

$x = 1$

When $x = 1$, $y = 1^3 - 3(1)^2 - 9(1) + 2$

$= -9$

$\therefore (1, -9)$ is a point of inflexion

6. (a) $\Delta OAB: A = \frac{1}{2}bh$

$= \frac{1}{2} \times 5 \times 3.4$

$= 8.5 \text{ units}^2$

(b) $c^2 = a^2 + b^2 - 2ab \cos C$

$BC^2 = 6^2 + 4^2 - 2(6)(4) \cos 87^\circ$

$\div 49.49$

$BC = 49.49$

$\div 7 \text{ cm}$

(i) $P(2W, 1N) = P(WNW) + P(WNW)$

$= \frac{2}{7} \times \frac{5}{7} \times \frac{2}{7} \times \frac{5}{7} \times \frac{2}{7}$

$= \frac{5}{7} \times \frac{5}{7} \times \frac{2}{7}$

$= 218 \div 343$

(ii) $P(\text{at least one } W) = 1 - P(\text{NNN})$

$= 1 - \frac{5}{7} \times \frac{5}{7} \times \frac{5}{7}$

$= 218 \div 343$

(b) $x = 60^\circ, 180^\circ - 60^\circ$ (first, second quadrants)

$= 60^\circ, 120^\circ$

(c) (i) $v = \int (6t + 4) \, dt$

$= 3t^2 + 4t + C$

When $t = 0$, $v = 0$

$\therefore 0 = 3(0)^2 + 4(0) + C$

$= C$

$\therefore v = 3t^2 + 4t$

When $t = 5$,

$v = 3(5)^2 + 4(5)$

$= 95 \text{ cm/s}$

(ii) $x = \int (3t^2 + 4) \, dt$

$= t^3 + 2t^2 + C$

When $t = 0$, $x = 0$

$\therefore 0 = (0)^3 + 2(0)^2 + C$

$= C$

$\therefore x = t^3 + 2t^2$

When $t = 2$,

$\therefore x = (2)^3 + 2(2)^2$

$= 16 \text{ cm}$

7. (a) (ii) $DC = AB = 2$ (opposite sides of a gram)

$\therefore DX = AD = 1 \left( DX = \frac{1}{2} \text{ DC} \text{ -- given} \right)$

$\therefore \Delta DXA$ is isosceles

$\therefore \angle DXA = \angle DXA = (180^\circ - 60^\circ) \div 2$

$= 60^\circ$

$\therefore \Delta DXA$ is equilateral
(iii) \( \angle XCB = 180^\circ - 60^\circ \)  
\( \angle ADC, \angle XCB \) cotointer \( \angle s, AD \parallel BC \)  
\[ \angle ADC = 120^\circ \]

\( \Delta CxB \) is isosceles  
\( \angle XCB = \angle CBX = (180^\circ - 120^\circ) = 30^\circ \)

\( \angle AXB = 180^\circ - (60^\circ + 30^\circ) \)  
\( \angle DXC \) straight angle  
\[ \angle DXC = 90^\circ \]

\( \therefore \angle AXB = 90^\circ \)

(iv) \( AX = 1 \)  
\( \Delta ADX \) equilateral  
\[ \therefore \frac{c^2}{4} + 1 = BX^2 \]

\[ 2^2 = 1^2 + BX^2 \]

\[ 4 = 1 + BX^2 \]

\[ 3 = BX^2 \]

\[ \sqrt{3} = BX \]

(b) Sketch \( y = x^2 \) and \( x + y = 3 \) as unbroken lines.

![Graph of y = x^2 and x + y = 3](image)

Substitute \( (1, 0) \) into \( y \geq x^2 \)  
\[ 0 \geq 1^2 \]  
(false)

Substitute \( (1, 0) \) into \( x + y \leq 3 \)  
\[ 1 + 0 \leq 3 \]  
(true)

(c) (i)  
\[ P = 20e^{0.06t} \]

When \( t = 10 \),  
\[ P = 20e^{0.06(10)} \]

\[ P = 20e^{0.6} \]

\[ P = 292 \] mice

(iii) When \( P = 500 \),  
\[ 500 = 20e^{0.268t} \]

\[ 25 = e^{0.268t} \]

\[ \ln 25 = \ln e^{0.268t} \]

\[ \ln 25 = 0.268t \ln e \]

\[ \ln 25 = 0.268t \]

\[ \frac{0.268}{25} = t \]

\[ 12 = t \]  
(i.e. after 12 weeks)

(ii) The curve is increasing so \( \frac{dp}{dt} > 0 \).  
The curve is concave downwards so \( \frac{d^2p}{dt^2} < 0 \).
\[
\text{(ii) } \quad V' = 80 - 3\pi^2
\]

For max./min. volume, \( V' = 0 \)
i.e. 
\[
\begin{align*}
80 - 3\pi^2 &= 0 \\
3\pi &= r
\end{align*}
\]
\[
\pm \frac{\sqrt{80}}{3\pi} = r \quad \pm 2.91 \div r
\]
\[
\therefore \quad V'' = -6\pi
\]

When \( r = 2.91, V'' = -6\pi(2.91) \)
\[
< 0 \quad \text{(maximum)}
\]
\[
\therefore \quad r = 2.91 \text{ cm}
\]

(iii) When \( r = 2.91, V = 80(2.91) - \pi r^2 \)
\[
= 206.4 \text{ cm}^3
\]

(b) 1996 to 2025 inclusive is 30 years.
\[
A = 500(1.12^{30} + 500(1.12^{29}) + 500(1.12^{28}) + \ldots + 500(1.12))
\]
\[
= 500(1.12^{30} + 1.12^{29} + \ldots + 1.12)
\]
\[
= 500(1.12^{30} + 1.12^{29} + \ldots + 1.12^{0})
\]
\[
1.12 + 1.12^2 + \ldots + 1.12^{30} \text{ is a geometric series}
\]
\[
a = 1.12, \quad r = 1.12
\]
\[
S_n = \frac{a(r^n - 1)}{r - 1}
\]
\[
S_{30} = \frac{1.12(1.12^{30} - 1)}{1.12 - 1} \div 270.29
\]
\[
\therefore \quad A = 500(270.29)
\]
\[
= 135146.30
\]

10. (a) (i) Substitute \( y = 4 \) into \( y = e^x \)
\[
4 = e^x
\]
\[
\therefore \quad \ln 4 = x \quad \text{(by definition of log)}
\]
\[
\therefore \quad \text{point of intersection is (ln 4, 4)}
\]

(ii)
\[
A = \text{Area of rectangle OABC} - \int_{\ln 4}^{\ln 4} e^x \, dx
\]
\[
= 4\ln 4 - \int_{\ln 4}^{\ln 4} e^x \, dx
\]
\[
= 4\ln 4 - (e^{\ln 4} - e^0)
\]
\[
= 4\ln 4 + 4 + 1
\]
\[
= (4\ln 4 - 3) \text{ units}^2
\]

(b) (i) For real, equal roots, \( \Delta = 0 \)
i.e. 
\[
b^2 - 4ac = 0
\]
\[
(k - 1)^2 - 4(1)(k) = 0
\]
\[
k^2 - 2k + 1 - 4k = 0
\]
\[
k^2 - 6k + 1 = 0
\]
\[
k = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)}
\]
\[
= \frac{6 \pm \sqrt{36}}{2}
\]
\[
= \frac{6 \pm 6\sqrt{2}}{2}
\]
\[
= 3 \pm 3\sqrt{2}
\]

(ii) When \( k = 5 \),
\[
x^2 + (k - 1)x + k = x^2 + 4x + 5
\]
\[
a > 0
\]
\[
\Delta = b^2 - 4ac
\]
\[
= 4^2 - 4(1)(5)
\]
\[
= -4
\]
\[
\therefore \quad \Delta < 0
\]

Since \( a > 0 \) and \( \Delta < 0 \), \( x^2 + 4x + 5 > 0 \) for all \( x \)

(c) (i)
\[
y = x^2 + 2px + q
\]
\[
y - q = x^2 + 2px
\]
\[
y - q + p^2 = x^2 + 2px + p^2
\]
\[
y - (q - p^2) = (x + p)^2
\]

This is in the form \( (x - h)^2 = 4a(y - k) \), where \( a \) is the focal length and \( (h, k) \) is the vertex. \( h = -p \) and \( k = q - p^2 \)
\[
\therefore \quad \text{vertex is } (-p, q - p^2)
\]

(ii)
\[
4a = 1
\]
\[
\therefore \quad a = \frac{1}{4}
\]

(iii) For \( P: \quad x = m \) since it is vertically below \( (m, 3m^2 + q) \)
When \( x = m \)
\[
m^2 = 8y
\]
\[
\frac{m^2}{8} = y
\]
So \( P = \left( m, \frac{m^2}{8} \right) \)
Distance = \[ \sqrt{3m^2 + q - \frac{m^2}{8}} \]
= \[ \frac{23m^2}{8} + q \]
= \[ \frac{23m^2}{8} + q \] since \( m^2 \geq 0 \) and \( q > 0 \)

(iv) \( m + q = 5 \)
\[ \therefore q = 5 - m \]
\[ D = \frac{23m^2}{8} + q \]
= \[ \frac{23m^2}{8} + 5 - m \]
\[ \frac{dD}{dm} = 46m - 1 \]

For stationary points \( \frac{dD}{dm} = 0 \)
\[ \frac{46m}{8} - 1 = 0 \]
\[ 46m = 8 \]
\[ m = 8 \]
\[ m = \frac{8}{46} \]
\[ m = \frac{4}{23} \]

So there is a stationary point at \( m = \frac{4}{23} \).

To determine its nature
\[ \frac{d^2D}{dm^2} = \frac{46}{8} \]
\[ > 0 \]

So concave upwards.
\[ \therefore \text{minimum turning point} \]

When \( m = \frac{4}{23} \)
\[ D = \frac{23m^2}{8} + 5 - m \]
= \[ \frac{23}{8} \left( \frac{4}{23} \right)^2 + 5 - \frac{4}{23} \]
= \[ \frac{21}{23} \]

So minimum distance is \( \frac{4}{23} \) units.

**Mathematics—Paper 2**

1. (a) (i) \( x - 3 = 5 \)
   \( x = 8 \)

(ii) \( x - 3 = -5 \)
   \( x = -2 \)

(b) \( 5 - x^2 = -4 \)
   \( 9 - x^2 = 0 \)
   \( 9 = x^2 \)
   \( \pm 3 = x \)

(c) \( \sin \frac{5\pi}{6} \)
= \( \sin \left( \frac{\pi - \frac{\pi}{6}}{6} \right) \)
= \( \sin \frac{\pi}{6} \) (2nd quadrant)
= \( \frac{1}{2} \)

(d) \( a^2(a - 2) - 4(a - 2) \)
= \( (a - 2)(a^2 - 4) \)
= \( (a - 2)(a + 2)(a - 2) \)
= \( (a + 2)(a - 2)^2 \)

(e) \( 2 \times \sqrt{4} \times \sqrt{6} - \sqrt{25} \times \sqrt{6} \)
   = \( 2 \times 2 \times \sqrt{6} - 5 \times \sqrt{6} \)
   = \( 4\sqrt{6} - 5\sqrt{6} \)
   = \( -\sqrt{6} \)

(f) \( \log_a 20 = \log_a (5^2 \times 2) \)
= \( \log_a 5^2 + \log_a 2 \)
= \( 2 \log_a 5 + \log_a 2 \)
= \( 2 \times 1.3 + 0.43 \)
= \( 3.03 \)

(g) \( P = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \)
= \( \left( \frac{-3 + 0}{2}, \frac{4 + 2}{2} \right) \)
= \( \left( -1, \frac{1}{2} \right) \)

2.

(a) Substitute \( A \left( 5, -\frac{1}{2} \right) \) into \( 3x + 4y - 9 = 0 \)
LHS = \( 3 \times 5 + 4 \times -\frac{1}{2} - 9 \)
= \( 0 \)
= RHS

\[ \therefore A \text{ lies on the line} \]

Substitute \( B \left( 1, 1 \frac{1}{2} \right) \) into \( 3x + 4y - 9 = 0 \)
LHS = \( 3 \times 1 + 4 \times 1 \frac{1}{2} - 9 \)
= \( 0 \)
= RHS

\[ \therefore B \text{ lies on the line} \]

\[ \therefore AB \text{ has equation } 3x + 4y - 9 = 0 \]

(b) \( 3x + 4y - 9 = 0 \)
\( 4y = -3x + 9 \)
\( y = \frac{3}{4} x + \frac{9}{4} \)

\[ \therefore m_1 = \frac{3}{4} \]

\( l \) is perpendicular to \( AB \), so \( m_1 m_2 = -1 \)
\[
\frac{3}{4}m_2 = -1
\]
\[
\therefore \quad m_2 = \frac{4}{3}
\]

Equation of \( l \):
\[
y - y_1 = m(x - x_1)
\]
\[
y - 1 = \frac{4}{3}(x - 4)
\]
\[
3y + 3 = 4(x + 4)
\]
\[
= 4x + 16
\]
\[
0 = 4x - 3y + 13
\]

(c)
\[
4x - 3y + 13 = 0
\]
\[
3x + 4y - 9 = 0
\]
\[
(1) \times 4: \quad 16x - 12y + 52 = 0
\]
\[
(2) \times 3: \quad 9x + 12y - 27 = 0
\]
\[
25x + 25 = 0
\]
\[
25x = -25
\]
\[
x = -1
\]

Substitute \( x = -1 \) in (1):
\[
4 \times (-1) - 3y + 13 = 0
\]
\[
9 - 3y = 0
\]
\[
9 = 3y
\]
\[
3 = y
\]

So point of intersection is \((-1, 3)\).

(d) \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \)

\[
AB:
\]
\[
d = \sqrt{(5 - 1)^2 + \left(-\frac{1}{2} - \frac{1}{2}\right)^2}
\]
\[
= \sqrt{4^2 + (-3)^2}
\]
\[
= \sqrt{16 + 9}
\]
\[
= \sqrt{25}
\]
\[
= 5
\]

\( CP: \)
\[
d = \sqrt{(-1 - 4)^2 + (3 - 1)^2}
\]
\[
= \sqrt{3^2 + 4^2}
\]
\[
= \sqrt{9 + 16}
\]
\[
= \sqrt{25}
\]
\[
= 5
\]

\( A = \frac{1}{2}bh \)
\[
= \frac{1}{2} \times 5 \times 5
\]
\[
= 12.5 \text{ units}^2
\]

(e) Midpoint \( AC = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \)
\[
= \left( -4 + 5, -1 + \frac{1}{2} \right)
\]
\[
= \left( \frac{1}{2}, -\frac{1}{2} \right)
\]

Midpoint \( AC = \) midpoint \( BD \)

where
\[
D = (x, y)
\]
\[
x = \frac{x_1 + x_2}{2}
\]
\[
\frac{1}{2} = \frac{x + 1}{2}
\]
\[
1 = x + 1
\]
\[
x = 0
\]
\[
y = \frac{y_1 + y_2}{2}
\]
\[
= \frac{-1 + \frac{1}{2}}{2}
\]
\[
= -\frac{1}{4}
\]
\[
-\frac{1}{2} = y + \frac{1}{2}
\]
\[
-4 = y
\]

(f) So \( D = (0, -4) \)

3. (a) \( (i) \) \( \frac{dy}{dx} = u'v + v'u \)
\[
= 1 \cdot \cos x + (-\sin x)x
\]
\[
= \cos x - x \sin x
\]
\[
(ii) \quad \frac{dy}{dx} = 5e^x
\]
\[
(iii) \quad \frac{dy}{dx} = \frac{4x}{2x^2 - 1}
\]

(b) \( (i) \) \( \frac{(3x - 2)^2}{3} + C \)
\[
= \frac{3 \times 5}{15} + C
\]
\[
= \frac{(3x - 2)^2}{15} + C
\]
\[
(ii) \quad -3 \times \frac{1}{2} \cos 2x + C
\]
\[
= -\frac{3}{2} \cos 2x + C
\]

(c) \( \left[ e^x - \frac{1}{1 - e^{-x}} \right] \)
\[
\left[ e^x + e^{-x} \right] \]
\[
\left[ e^x - e^{-x} \right] - \left[ e^x + e^{-x} \right]
\]
\[
e^x + e^{-x} - 1 - 1
\]
\[
e^x + e^{-x} - 2
\]

(d) \( \frac{dy}{dx} = \int (18x - 6) \, dx \)
\[
= 9x^2 - 6x + C
\]

At \((2, -1)\), \( \frac{dy}{dx} = 0 \)
\[
0 = 9(2)^2 - 6(2) + C
\]
\[
= 24 + C
\]
\[
-24 = C
\]
\[
\therefore \frac{dy}{dx} = 9x^2 - 6x - 24
\]
\[
y = \int (9x^2 - 6x - 24) \, dx
\]
\[
= 3x^3 - 3x^2 - 24x + C
\]

Substitute \((2, -1)\):
\[
-1 = 3(2)^2 - 3(2)^2 - 24(2) + C
\]
\[
= -36 + C
\]
\[
35 = C
\]
\[
\therefore \ y = 3x^3 - 3x^2 - 24x + 35
\]

4. (a) Game 1
\[
\text{Game 2}
\]
\[
W \subset L \subset W
\]
\[
\frac{3}{5} \subset \frac{3}{5}
\]
\[
L \subset L
\]
(i) \( P(WW) = \frac{2}{5} \times \frac{3}{8} = \frac{3}{20} \)

(ii) \( P(WL) + P(LW) = \frac{2}{5} \times \frac{5}{8} + \frac{3}{5} \times \frac{3}{8} = \frac{19}{40} \)

(iii) \( P(\text{at least 1W}) = 1 - P(\text{LL}) = 1 - \frac{3}{5} \times \frac{5}{8} = \frac{5}{8} \)

(b) (i)

\[ y = x + 2 \]

\[ y^2 = (x + 2)^2 \]

\[ V = \pi \int_{-2}^{2} y^2 \, dx = \pi \int_{-2}^{2} (x + 2)^2 \, dx = \pi \left[ \frac{3}{2} + 2(3) \right] \left\{ \frac{(-2)^3}{2} + 2(-2) \right\} = 12.5 \text{ units}^3 \]

(iii) \( y = x + 2 \)

\[ y^2 = (x + 2)^2 \]

\[ V = \pi \int_{-2}^{2} y^2 \, dx = \pi \int_{-2}^{2} (x + 2)^2 \, dx = \pi \left[ \frac{3}{2} + 2(3) \right] \left\{ \frac{(-2)^3}{2} + 2(-2) \right\} = 12.5 \text{ units}^3 \]

or \[ V = \frac{1}{3} \pi rh = \frac{1}{3} \pi (5)^2 \times 5 = \frac{125\pi}{3} \text{ units}^3 \]

5. (a) \( y = 2x^3 - 9x^2 + 12x - 7 \)

\[ \frac{dy}{dx} = 6x^2 - 18x + 12 \]

\[ \frac{d^2y}{dx^2} = 12x - 18 \]

(i) For stationary points, \( \frac{dy}{dx} = 0 \)

\[ 6x^2 - 18x + 12 = 0 \]

\[ x^2 - 3x + 2 = 0 \]

\[ (x - 2)(x - 1) = 0 \]

\[ x = 1, 2 \]

When \( x = 1, y = 2(1)^3 - 9(1)^2 + 12(1) - 7 = -2 \)

When \( x = 2, y = 2(2)^3 - 9(2)^2 + 12(2) - 7 = -3 \)

So \((1, -2)\) and \((2, -3)\) are stationary points.

At \((1, -2)\), \( \frac{d^2y}{dx^2} = 12(1) - 18 = -6 \)

\( \therefore (1, -2) \) is a maximum turning point

At \((2, -3)\), \( \frac{d^2y}{dx^2} = 12(2) - 18 = 6 \)

\( \therefore (2, -3) \) is a minimum turning point

(ii) For points of inflexion \( \frac{d^2y}{dx^2} = 0 \)

12 \(x - 18 = 0 \)

\[ 12x = 18 \]

\[ x = 1.5 \]

When \( x = 1.5, y = 2(1.5)^3 - 9(1.5)^2 + 12(1.5) - 7 \]

\[ = -2.5 \]

Check concavity:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1.25</th>
<th>1.5</th>
<th>1.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d^2y}{dx^2} )</td>
<td>-3</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

Concavity changes, so \((1.5, -2.5)\) is a point of inflexion.

(iii) When \( x = -3, y = 2(-3)^3 - 9(3)^2 + 12(-3) - 7 \]

\[ = -178 \]

When \( x = 3, y = 2(3)^3 - 9(3)^2 + 12(3) - 7 \]

\[ = 2 \]
(b) (i) The particle is at the origin when \( x = 0 \), i.e. at \( t = t_1 \) and \( t_2 \).

(ii) At rest, \( \frac{dx}{dt} = 0 \) (at the stationary points, i.e. \( t_2 \) and \( t_3 \)).

(c) \( T = T_0 e^{kt} \)

When \( t = 0 \), \( T = 97 \)

\[ T = 97 e^{kt} \]

When \( t = 5 \), \( T = 84 \)

\[ 84 = 97 e^{5k} \]

\[ k = \frac{5 \ln \left( \frac{84}{97} \right)}{e} \]

\[ k = 0.029 \]

So \( T = 97 e^{-0.029t} \)

(i) When \( t = 15 \)

\[ T = 97 e^{-0.029 \times 15} \]

\[ T = 63 \]

So the temperature is 63°C after 15 minutes.
7. (a) (i) \[ \int f(x) \, dx = \int \left( y_0 + y_2 + 4y_1 + 2y_2 \right) \, dx \]

\[ \int_{\tan x}^{\tan x} \, dx = \frac{\pi}{3} \left( \tan 0 + \tan \frac{\pi}{4} \right) + 4 \left( \frac{\pi}{16} + \frac{3\pi}{16} + 2 \frac{\pi}{8} \right) \]

\[ = 0.35 \text{ units}^2 \]

(ii) \[ \frac{dy}{dx} = -\sin x \cos x = -\tan x \]

(iii) \[ \int_{0}^{\tan x} \tan x \, dx = \left[ -\ln (\cos x) \right]_0^\pi \]

\[ = -\pi \ln (\cos 0) = -\pi \]

(b) (i) \[ T = 20 \]

\[ 20 = 97 e^{-0.029 t} \]

\[ \ln 20 = \ln 97 \cdot e^{-0.029 t} \]

\[ \ln 20 = \ln 97 \cdot -0.029 \ln e \]

\[ \ln 20 = \ln 97 \cdot -0.029 \]

\[ \ln 20 \]

\[ = \frac{-0.029 \cdot 97}{-0.029 \cdot 20} = t \]

\[ = 54.9 \text{ minutes} \]

So the temperature is 20°C after 54.9 minutes.

8. (a) (i) & (ii) Amplitude = 3, period = \( \frac{2\pi}{\pi} = \pi \)

(b) \[ 2 \sin x - 1 = 0 \]

\[ 2 \sin x = 1 \]

\[ \sin x = \frac{1}{2} \] (1st, 2nd quadrants)

\[ x = 30^\circ, 180^\circ - 30^\circ = 30^\circ, 150^\circ \]

(c) \( \angle CBE = 50^\circ \) (base \( \angle s \) of isosceles \( \triangle \))

\[ \angle DCE = 50^\circ + 50^\circ \] (exterior \( \angle \) of \( \triangle CBE \))

\[ = 100^\circ \]

\[ \angle ABC = 130^\circ - 50^\circ = 80^\circ \]

\[ \angle DAB = 360^\circ - (100^\circ + 80^\circ + 80^\circ) \]

\[ = 100^\circ \]

\[ \angle DAB = \angle DCE \text{ and } \angle ABC = \angle ADC \]

\[ \angle ABC \text{ is a parallelogram (opposite } \angle \text{s equal) } \]

(ii) \[ \angle DAB = \angle DCE \text{ and } \angle ABC = \angle ADC \]

\[ \angle ABC \text{ is a parallelogram (opposite } \angle \text{s equal) } \]

(d) (i) \[ 1 + 3 + 5 + \ldots \text{ is an arithmetic series with } a = 1, d = 2 \]

\[ T_n = a + (n - 1)d \]

\[ T_{12} = 1 + (12 - 1) \times 2 \]

\[ = 23 \]

So there are 23 oranges in the 12th row.

(ii) Total number of oranges is 289, so \( S_n = 289 \)

\[ S_n = \frac{n}{2} [2a + (n - 1)d] \]

\[ 289 = \frac{n}{2} [2 \times 1 + (n - 1)2] \]

\[ 578 = n(2 + 2n - 2) \]

\[ = n \times 2n \]

\[ = 2n^2 \]

\[ 289 = n^2 \]

\[ \sqrt{289} = n \]

17 = \( n \)

So there are 17 rows of oranges altogether.
9. (a) \[ y = \frac{3}{x} \]

(b) The statement would only be true if there were equal numbers of each colour. It is probably false.

(c) \[ \ln x^2 = \ln (2x + 3) \]
\[ \therefore x^2 = 2x + 3 \]
\[ x^2 - 2x - 3 = 0 \]
\[ (x - 3)(x + 1) = 0 \]
\[ x = 3, -1 \]
But \( x \neq -1 \) (in -1 does not exist)
so the solution is \( x = 3 \)

(d) (i) \[ \frac{dy}{dx} = e^x \]
When \( x = k \)
\[ \frac{dy}{dx} = e^k \]
So gradient \( m = e^k \)

(ii) When \( x = k, y = e^k \)
\[ y - y_1 = m(x - x_1) \]
\[ y - e^k = e^k(x - k) \]
\[ = e^k x - ke^k \]
\[ y = e^k x - ke^k + e^k \]
\[ = e^k(x - k + 1) \]

(iii) Substitute \((2, 0)\) into the equation
\[ 0 = e^k(2 - k + 1) \]
\[ 3 - k = 0 \]
\[ 3 = k \]

(e) \( 180^\circ = \pi \) radians
\[ 1^\circ = \frac{\pi}{180^\circ} \]
\[ \therefore 53^\circ = \frac{\pi}{180^\circ} \times 53^\circ \]
\[ = \frac{53\pi}{180} \]
\[ A = \frac{1}{2} r^2 \theta \]
\[ = \frac{1}{2} \times 7^2 \times \frac{53\pi}{180} \]
\[ = 22.7 \text{ cm}^2 \]

10. (a) (i) \( y = (x + 2)^2 \)
\[ \therefore y^2 = (x + 2)^4 \]
\[ V = \pi \int_{-2}^{1} y^2 dx \]
\[ = \pi \int_{-2}^{1} (x + 2)^4 dx \]
\[ = \pi \left[ \frac{1}{5} \times (x + 2)^5 \right]_{-2}^{1} \]
\[ = \pi \left[ \frac{1}{5} \times (1 + 2)^5 - \frac{1}{5} \times (-2 + 2)^5 \right] \]
\[ = \pi \left[ \frac{243}{5} - 0 \right] \]
\[ = \frac{243\pi}{5} \text{ units}^3 \]

(b) (i) \( s = \frac{d}{t} \)

So \( t = \frac{d}{s} \)

Cost of trip taking \( t \) hours:
\[ C = (s^2 + 7500)t \]
\[ = (s^2 + 7500) \frac{3000}{s} \]
\[ = 3000s + 7500 \times 3000 \frac{1}{s} \]
\[ = 3000 \left( s + \frac{75000}{s} \right) \]

(ii) \[ C = 3000 \left( s + \frac{7500}{s} \right) \]
\[ = 3000(s + 7500s^{-1}) \]
\[ = 3000 \left( 1 + \frac{7500}{s^2} \right) \]

For minimum cost, \( \frac{dC}{ds} = 0 \)
\[ 3000 \left( 1 - \frac{7500}{s^2} \right) = 0 \]
\[ 1 - \frac{7500}{s^2} = 0 \]
\[ s = \sqrt{7500} \text{ (speed is positive)} \]
\[ = 86.6 \text{ km/h} \]

Check:
\[ \frac{d^2C}{ds^2} = 3000 \left( 15000s^{-3} \right) \]
\[ = 3000 \left( \frac{15000}{s^3} \right) \]

When \( s = 86.6 \)
\[ \frac{d^2C}{ds^2} = 3000 \left( \frac{15000}{86.6^3} \right) \]
\[ > 0 \]

Concave upwards
So minimum when \( s = 86.6 \)

(iii) \[ C = 3000 \left( 86.6 + \frac{7500}{86.6} \right) \]
\[ = 519.615 \text{ cents} \]
\[ = $5196.15 \]