Free-to-download Sample HSC Exams with answers

A brand-new format that makes even better use of your study time

Lyn Baker
Section I (10 marks)

1 \[ \frac{2\pi}{3} = \]
A \( \frac{1}{\sqrt{2}} \)  B \( \frac{\sqrt{3}}{2} \)  C \( \frac{1}{2} \)  D 0

2 \[ \frac{4x^3 \times 6x^8}{12x^{12} + 3x^{-7}} \]
A \( \frac{6}{x^8} \)  B \( 6x^6 \)  C \( \frac{x^6}{6} \)  D \( \frac{1}{6x^8} \)

3 The limiting sum of the series
\[ 32 + 16 + 8 + 4 + \ldots \]
A 66  B 72  C 69  D 64

4 The value of \( \log_b 7 \), correct to three decimal places, is
A 1.209  B 1.404  C 0.336  D 2.783

5 \[ \frac{\sin \theta \cos \theta + \sin^2 \theta}{\sin \theta \cos \theta + \cos^2 \theta} = \]
A cosec^2 \( \theta \)  B tan \( \theta \)  C cot \( \theta \)  D sec^2 \( \theta \)

6 Which is a primitive function of \( \frac{1}{x^2} \)?
A \( \frac{3}{x^3} \)  B \( \frac{1}{3x^3} \)  C \( \frac{1}{x} \)  D \( \frac{-1}{x} \)

7 The equation of the circle, centre \((-2, 3)\), radius 4 is
A \( (x - 2)^2 + (y + 3)^2 = 16 \)  B \( (x - 2)^2 - (y + 3)^2 = 16 \)  C \( (x + 2)^2 + (y - 3)^2 = 16 \)  D \( (x + 2)^2 - (y - 3)^2 = 16 \)

8 The roots of the equation \( 2x^2 + 8x + 5 = 0 \) are
A real and rational and equal  B real and rational but not equal  C real but not rational  D not real

9 \[ \frac{1}{\sqrt{3} + 2} = a + b\sqrt{3} \]
Which is correct?
A \( a = 2 \) and \( b = 1 \)  B \( a = -2 \) and \( b = 1 \)
C \( a = 2 \) and \( b = -1 \)  D \( a = -2 \) and \( b = -1 \)

Section II

Question 11 (15 marks)

a P and Q are the points \((4, 2)\) and \((9, -2)\), respectively.

i Find the gradient of PQ.  (1 mark)

ii Find the equation of the line through P perpendicular to PQ.  (2 marks)

iii Find the coordinates of R, the point where this line meets the y-axis.  (1 mark)

iv Find the length of PR.  (1 mark)

v Show that triangle PQR is isosceles.  (2 marks)

vi Find the area of triangle PQR.  (1 mark)
b Differentiate with respect to \( x \):
   
   i  \( x^2e^x \) (2 marks)
   
   ii  \( (1 – \cos x)^3 \) (2 marks)
   
   c 100 tickets are sold in a raffle for which there are two prizes. A ticket is drawn at random for first prize, this ticket is then discarded and another drawn for second prize. Emily buys 5 tickets in the raffle. What is the probability that Emily:
   
   i  wins first prize (1 mark)
   
   ii  does not win a prize? (2 marks)

Question 12 (15 marks)

a OAB is a sector of a circle, centre O. \( \angle AOB = 50^\circ \). The area of the sector is 20\( \pi \) m\(^2\).

b i Find \( \int \frac{x}{x^2 + 1} \) dx (2 marks)

ii Evaluate \( \int_0^\pi \sin x \) dx (2 marks)

c Solve for \( 0 \leq \theta \leq 2\pi \), \( 3\tan^2 \theta = 1 \) (2 marks)

d Consider the curve \( y = 6x^2 – x^3 \)
   
   i  Where does the curve meet the x-axis? (1 mark)
   
   ii Find the stationary points and determine their nature. (4 marks)
   
   iii Find the coordinates of the point of inflection. (1 mark)
   
   iv Where will the curve be concave up? (1 mark)

Question 13 (15 marks)

a The diagram shows the parabola \( x^2 = 4y \) and the tangent to the parabola at the point P(6, 9).

b i Prove that triangles ABX and ADX are congruent. (2 marks)

ii Prove that BD bisects \( \angle ADC \). (2 marks)

c Jenny begins an exercise program. She exercises for 10 minutes the first day and intends to increase that by 3 minutes every day. If Jenny maintains that program:
   
   i  for how long will she exercise on the 14th day (1 mark)
   
   ii on what day will she be exercising for 1 hour and 40 minutes? (1 mark)
   
   iii for how many hours in total will she have exercised at the end of three weeks? (2 marks)

d A spokesperson made the following comment about immigration: ‘Over that period of time the number of immigrants was increasing at a decreasing rate.’
   
   i  If \( I \) is the level of immigration, what does the statement mean about \( \frac{dI}{dt} \) and \( \frac{d^2I}{dt^2} \)? (1 mark)
   
   ii Sketch a graph that fits the above information. (1 mark)

Question 14 (15 marks)

a The velocity \( v \) m s\(^{-1} \) of a moving particle is given by \( v = t^2 + t – 12 \) \( (t \geq 0) \).

b The population, \( P \), of a town is increasing at a rate proportional to the population, so that \( P = P_0e^{kt} \) where \( k \) is a positive constant and \( t \) is the time in years.

i  If the population was 4000 in 1991 and 5400 in 2006, find the percentage growth rate per year. (2 marks)

ii Find the expected population of the town in 2030. (1 mark)

iii At what rate will the population be increasing in 2030? (1 mark)

iv In what year would you expect the population of the town to reach 10 000? (2 marks)
c  
   i  Without the use of calculus, sketch the 
   graph of \( y = 3\sin 2x \)  (2 marks)
   
   ii  On the same set of axes, sketch \( y = x \)  (1 mark)
   
   iii  How many solutions has the equation 
   \( 3\sin 2x = x \)?  (1 mark)

Question 15  (15 marks)

a  Find the volume of the solid generated when the section 
   of the curve \( y = \sec x \) between \( x = 0 \) and \( x = \frac{\pi}{3} \) is 
   rotated about the \( x \)-axis.  (3 marks)

b  The rate \( R \), in tonnes per hour, at which a factory is using 
   its raw resources is given by \( R = 360 - \frac{360}{(1 + t)^2} \).

   i  What value does \( R \) approach as the factory 
   operates for a long period of time?  (1 mark)
   
   ii  Draw a sketch of \( R \) as a function of \( t \).  (1 mark)
   
   iii  Find the total amount of resources used 
   in the first eight hours.  (2 marks)

Point 1  

Lachlan borrows \$170,000\) and agrees to repay 
the principal plus interest over 25 years. The rate 
of reducible interest charged is 0.6% per month. 
Let \( A_n \) be the amount owing after \( n \) months and 
\( M \) be the amount of each monthly repayment.

   i  Show that \( A_n = 170,000(1.006)^n - M(1 + 1.006 + 1.006^2) \)  (3 marks)
   
   ii  Write down a similar expression for \( A_n \) 
   and hence show that 
   \[ A_n = 170,000(1.006)^n - M \left( \frac{1.006^n - 1}{0.006} \right) \]  (2 marks)

   iii  If the loan is to be repaid over 25 years, 
   find the amount of each monthly 
   instalment.  (2 marks)
   
   iv  Find the total amount of interest that 
   Lachlan pays.  (1 mark)

Question 16  (15 marks)

a  The gradient function of a curve is given by 
   \( \frac{dy}{dx} = e^x - e^{-x} \). The curve passes through the 
   origin. Find the equation of the curve.  (2 marks)

b  A cylinder is to be placed in a cone of height 15 cm and 
   radius 9 cm as shown in the diagram.

   If the radius of the cylinder is \( r \) cm and the height is \( h \) cm:

   i  show that \( h = 15 - \frac{5}{3}r \)  (2 marks)
   
   ii  show that the volume \( V \text{ cm}^3 \) of the cylinder is given 
   by \( V = 15\pi r^2 - \frac{5\pi r^3}{3} \)  (1 mark)
   
   iii  show that the maximum volume of the 
   cylinder occurs when \( r = 6 \)  (2 marks)
   
   iv  show that the largest possible cylinder has 
   \( \frac{4}{9} \) of the volume of the cone.  (2 marks)

   c  
   i  Find the exact area bounded by the curve 
   \( y = e^x \), the axes and the line \( x = 2 \).  (2 marks)
   
   ii  Use Simpson’s rule with three function 
   values to find an approximation for the 
   integral \( \int_0^2 e^x \, dx \).

   (Leave the answer in terms of \( e \)).  (2 marks)
   
   iii  Hence show that \( e = 1 + \sqrt{3} \)  (2 marks)
Sample HSC Examination 2

Section I  (10 marks)

1 The value of \( \sin \frac{2\pi}{5} \), to three decimal places, is
   \[ A \ 0.022 \quad B \ 0.309 \quad C \ 0.951 \quad D \ 1.257 \]

2 \( \log_8 3 - \log_2 2 = \)
   \[ A \ 0 \quad B \ 1 \quad C \ 2 \quad D \ 4 \]

3 The first term of a geometric series is 3 and the common ratio is \(-2\). The tenth term of the series is
   \[ A \ -1536 \quad B \ 1536 \quad C \ -3072 \quad D \ 3072 \]

4 The probability that Jo will win a particular game is 0.3. If three games are played, what is the probability that Jo will win all three games?
   \[ A \ 0.9 \quad B \ 0.27 \quad C \ 0.09 \quad D \ 0.027 \]

5 To three significant figures, the value of \( \frac{2}{e} \) is
   \[ A \ 0.736 \quad B \ 0.718 \quad C \ 0.289 \quad D \ 0.271 \]

6 The domain of \( y = \sqrt{9-x^2} \) is
   \[ A \ x \geq 3 \quad B \ x \leq 3 \quad C \ x \leq -3 \quad D \ -3 \leq x \leq 3 \]

7 \( 8x^2 - 27y^3 = \)
   \[ A \ (2x-3y)(4x^2+12xy+9y^2) \quad B \ (2x-3y)(4x^2+6xy+y^2) \quad C \ (2x-3y)(4x^2-12xy+9y^2) \quad D \ (2x-3y)(4x^2-6xy+y^2) \]

8 A radioactive substance decomposes over time. The mass, in grams, present after \( t \) years is given by \( M = 400e^{-0.06t} \).
   Which is the best approximation to the amount of the mass remaining after 8 years?
   \[ A \ 12 \quad B \ 15 \quad C \ 25 \quad D \ 35 \]

9 The exact value of \( \int_0^\frac{\pi}{2} \sec^2 x \, dx \) is
   \[ A \ \frac{\sqrt{2}}{2} \quad B \ \frac{1}{\sqrt{3}} \quad C \ \frac{\sqrt{3}}{2} \quad D \ \sqrt{3} \]

10 Which is the graph of \( f(x) = \log_4 x \)?
   \[ A \quad B \quad C \quad D \]

Section II

Question 11  (15 marks)

a \( $20000 \) is invested in an account earning 0.75\% per month interest, compounding monthly. Find the value of the investment (to the nearest dollar) at the end of eight years. (1 mark)

b \( P \) is the point \((2, 3)\). The line \( l_1 \) passes through \( P \) and meets the \( x \)-axis at \( Q \) at an angle of 45\( ^\circ \) as shown in the diagram.

\[ P(2, 3) \]

1 Show that the equation of \( l_1 \) is \( x + y - 5 = 0 \). (2 marks)

ii \( l_2 \) is the line \( x + 5y - 1 = 0 \)
   Verify that \( R(-4, 1) \) lies on the line \( l_2 \) (1 mark)

iii Find the coordinates of \( S \), the point of intersection of \( l_1 \) and \( l_2 \) (2 marks)

iv Draw a diagram, showing the points \( P \), \( R \) and \( S \) and the lines \( l_1 \) and \( l_2 \) (1 mark)
v Find the length of the interval RS. Give the answer in simplest surd form. (2 marks)
vii Find the area of triangle PRS. (1 mark)

Of 48 students enrolled in a particular mathematics course at university, all but 6 also study either physics or chemistry or both. 26 study physics and 30 study chemistry. What is the probability that a student chosen at random from those studying that mathematics course:
i studies chemistry (1 mark)
ii studies both physics and chemistry (1 mark)
iii studies physics or chemistry (1 mark)
iv studies physics but not chemistry? (1 mark)

Question 12 (15 marks)
a Differentiate, with respect to $x$

i $e^{x^2}$ (2 marks)

ii $3 \cos 2x$ (2 marks)

b Find $\int \frac{dx}{(2x + 1)^3}$ (2 marks)

c Find the exact value of $\int_1^2 x^{-1} \, dx$ (2 marks)

d Find the value of $b$ if $\int_0^b x \, dx = 12$ $(b > 0)$ (2 marks)

e The rate $(R \text{ mm s}^{-1})$ at which the height of liquid in a container is increasing is given by $R = \frac{12}{\sqrt{t}}$ $(t > 0)$.

i At what rate is the height increasing after 8 seconds? (1 mark)

ii If the container is filled to a height of 180 mm after 8 seconds, find an expression for the height, $h$, of liquid in the container at time $t$. (2 marks)

iii Find the time, in minutes and seconds, for the container to be filled if the depth of liquid is 432 mm when full. (2 marks)

Question 13 (15 marks)
a From a town centre, two roads diverge at an angle of 60°. Every 100 m, each road is intersected by an avenue, numbered 1st Avenue, 2nd Avenue, and so on.

The local fun run is to begin at the town centre, travel up one road to 1st Avenue, along 1st Avenue to the other road, up this road to 2nd Avenue and so on, weaving along each avenue. The finish point is at the end of 12th Avenue.

i What is the total length of the 12 avenues? (2 marks)

ii How long is the fun run? (1 mark)

b The sum of the infinite geometric series $1 + 3^2 + 3^3 + 3^4 + \ldots$ is 1.5

Find the value of $x$. (2 marks)

c If $2 \ln x = \ln (3x + 4)$ find the value of $x$. (2 marks)

d ABDE is a square. △BCD is an equilateral triangle.

vi Find the size of $\angle BAC$. Give reasons. (2 marks)

ii If $F$ is the midpoint of $AC$, prove that triangles $ABF$ and $CBF$ are congruent. (2 marks)

iii Prove that $\angle ABF = \angle EAC$. (2 marks)

e Box A holds 12 red apples and 9 green apples. Box B holds 8 red and 7 green apples.

Kerry chooses a box at random and then takes two apples from the box. By drawing a tree diagram, or otherwise, find the probability that the apples are the same colour. (2 marks)

Question 14 (15 marks)
a For what values of $x$ over the domain $0 \leq x \leq 2\pi$ is the curve $y = x + \cos x$ concave down? (3 marks)

b Find the equation of the normal to the curve $y = e^{2x-1}$ at the point where $x = \frac{1}{2}$. (3 marks)

c Find the exact volume when the area bounded by the curve $y = 5 - x^2$ and the line $y = 3$ is rotated about the $y$-axis. (2 marks)

d $f(x) = x^3 + ax^2 + bx - 5$ where $a$ and $b$ are constants. The function has stationary values at $x = -1$ and $x = 3$.

i Find $a$ and $b$. (3 marks)

ii Determine the nature of the stationary values. (2 marks)

iii If the function is defined for $-2 \leq x \leq 6$ find the absolute maximum and absolute minimum values of the function. (2 marks)
**Question 15 (15 marks)**

a  
- i. Use Simpson's rule with three function values to estimate \( \int_{1}^{3} \log_{e} x \, dx \) (2 marks)
- ii. Differentiate \( x \log_{e} x - x \) (2 marks)
- iii. Hence find the value of \( \int_{1}^{3} \log_{e} x \, dx \) to three decimal places. (2 marks)
- iv. Briefly comment on the accuracy of the approximation of the integral by Simpson's rule. (1 mark)

b  
Two particles, A and B, move along a straight line so that their displacements from the origin at time \( t \) are given by:  
\[
\begin{align*}
x_{A} &= 15t^2 - t^3 \\
x_{B} &= 48t - 9
\end{align*}
\]
- i. Which is moving faster when \( t = 3 \)? (2 marks)
- ii. When do the particles travel with the same velocity? (2 marks)
- iii. Find an expression for the acceleration of particle A. (1 mark)
- iv. What is the maximum displacement of particle A? (2 marks)
- v. What is the acceleration of particle B? Briefly explain any significance of your answer. (1 mark)

**Question 16 (15 marks)**

a  
The diagram shows the curves \( y = x^3 \) and \( y = 3x^2 - 2x \).

\[
\begin{align*}
y &= x^3 \\
y &= 3x^2 - 2x
\end{align*}
\]
- i. The two curves intersect at the origin and at the points where \( x = a \) and \( x = b \) \((0 < a < b)\). Find the value of \( a \). (2 marks)
- ii. Show that the area shaded in the diagram can be found by evaluating the integral \( \int_{0}^{a} (x^3 - 3x^2 + 2x) \, dx \) (2 marks)
- iii. Find the shaded area. (2 marks)

b  
The diagram shows a sector of a circle, centre O. The radius is \( r \) and the angle at the centre \( \theta \).

\[
\begin{align*}
\text{If } P \text{ is the perimeter of the sector, show that } \theta &= \frac{P}{r} - 2 \\
\text{ii. Show that the area, } A, \text{ of the sector is given by } A &= \frac{Pr}{2} - r^2 \\
\text{iii. If } r \text{ and } \theta \text{ are allowed to vary, but } P \text{ remains fixed (that is a constant), find the maximum area of the sector in terms of } r \\
\text{iv. Show that the maximum area occurs when } \theta &= 2. \\
\text{Instead of } P \text{ remaining fixed, } A \text{ will remain fixed.} \\
\text{v. Show that } P &= 2r + \frac{2A}{r} \\
\text{vi. Find the minimum perimeter, in terms of } r, \text{ and the value of } \theta \text{ for which this occurs.} (3 \text{ marks})
Sample HSC Examination 3

All necessary working should be shown.
Attempt all questions.

Time allowed: 3 hours  Total marks: 100

Section I  (10 marks)

1  \( \log_4 16 \), to four decimal places =
   A  0.6021  B  1.3863  C  1.6094  D  54.5982

2  Simplify \( \frac{x - 1}{x^2 - 1} = \)
   A \( \frac{1}{x} \)  B \( \frac{1}{x - 1} \)
   C \( \frac{1}{x + 1} \)  D none of these

3  \( 40^\circ \), expressed in radians in terms of \( \pi \), is
   A \( \frac{2\pi}{9} \)  B \( \frac{3\pi}{8} \)  C \( \frac{4\pi}{7} \)  D \( \frac{5\pi}{8} \)

4  A bag holds 5 red, 4 blue and 3 yellow marbles. What is the probability that a randomly chosen marble is yellow?
   A \( \frac{1}{2} \)  B \( \frac{1}{3} \)  C \( \frac{1}{4} \)  D \( \frac{1}{6} \)

5  If \( f(x) = x^2 - 7x + 4 \), then \( f''(3) = ? \)
   A 2  B 3  C 4  D -7

6  Which is not a primitive function of \( x^2 + 5x? \)
   A \( \frac{x^3}{3} + \frac{5x^2}{2} + 4 \)  B \( x^3 + 5x^2 \)
   C \( \frac{1}{6}(2x^3 + 15x^2 - 1) \)  D \( \frac{1}{3}x^3 + \frac{5}{2}x^2 - 7 \)

7  If \( \alpha \) and \( \beta \) are the roots of the equation \( 3x^2 + 6x - 5 = 0 \) then \( \alpha + \beta = \)
   A \( \frac{1}{2} \)  B \( -\frac{1}{2} \)  C 2  D -2

8

The area of this triangle is closest to
   A 4 cm\(^2\)  B 15 cm\(^2\)  C 20 cm\(^2\)  D 23 cm\(^2\)

9  If \( \log_2 2 = 0.270 \) and \( \log_3 3 = 0.428 \) then \( \log_4 4.5 = \)
   A 0.549  B 0.586  C 0.642  D 0.678

10 The function \( f(x) = x^2 + 2x \) is
    A even  B odd  C neither even nor odd  D There is not enough information to determine whether the function is even or odd or neither.

Section II

Question 11  (15 marks)

a  P, Q and R are the points \((-2, 1)\), \((1, 6)\) and \((6, 3)\) respectively.
   i  Find the midpoint, M, of PR.  (1 mark)
   ii  Find the gradient of PR.  (1 mark)
   iii  Find the length of PR.  (1 mark)
   iv  If M is the midpoint of QS, find the coordinates of S.  (1 mark)
   v  Show that QS is perpendicular to PR.  (2 marks)
   vi  Using the information from the previous parts, what type of quadrilateral is PQRS? Justify your answer.  (2 marks)

b  Use Simpson's rule with five function values to approximate \( \int_0^4 3^x \, dx \).  (2 marks)
c PQRS is a square of side 2 m. M is the midpoint of SR. PM and QS intersect at X.

\[ \text{P} \quad 2 \text{ m} \quad \text{Q} \\
\text{2 m} \quad \text{X} \\
\text{S} \quad \text{M} \quad \text{R} \]

i Show that triangles PQX and MSX are similar.  
(2 marks)

ii Show that the length of PM is \( \sqrt{5} \) m.  
(1 mark)

iii Hence find the exact length of PX.  
(2 marks)

**Question 12**  (15 marks)

a Differentiate:

i \( e^x \sin x \)  
(2 marks)

ii \( \log(x^2 + 4) \)  
(2 marks)

b Evaluate:

i \( \int_1^4 \frac{dx}{\sqrt{x}} \)  
(2 marks)

ii \( \int_0^\pi \sin 2x \, dx \)  
(2 marks)

c Solve \( 2 \cos \alpha - 1 = 0 \) for \( 0 \leq \alpha \leq 2\pi \).  
(2 marks)

d The diagram shows a sector of a circle, centre O, radius 10 cm.

\[ \text{A} \quad 10 \text{ cm} \quad \text{B} \\
\frac{3\pi}{5} \quad \text{O} \\
\text{10 cm} \quad \text{C} \]

i Find the exact length of the arc ABC.  
(1 mark)

ii Find the area of the sector, correct to one decimal place.  
(2 marks)

e If \( \sin x \neq \pm 1 \) show that  
\[ 1 + \sin^2 x + \sin^4 x + \sin^6 x + \ldots = \sec^2 x \]  
(2 marks)

**Question 13**  (15 marks)

a Find the gradient of the tangent to the curve  
\( y = \tan x \) at the point where \( x = \frac{\pi}{4} \).  
(1 mark)

b Four girls and five boys play a game where all have an equal chance of winning. Three games are played and the winner of any game is not permitted to play in the following games. What is the probability that:

i all games are won by girls  
(1 mark)

ii at least one game is won by a boy  
(1 mark)

iii exactly two games are won by boys?  
(2 marks)

c The diagram shows the parabola \( x^2 = 16y \).

\[ x^2 = 16y \]

i Show that the equation of the normal to the curve \( x^2 = 16y \) at the point \( P(4, 1) \) is \( y = -2x + 9 \).  
(2 marks)

ii Find the coordinates of the point \( Q \) where the normal cuts the curve again.  
(2 marks)

iii Hence find the area shaded in the diagram.  
(2 marks)

d An orchard is trapezoidal in shape. There are 20 trees in the first row and 95 trees in the last row. The difference between the number of trees between successive rows is constant. There is a total of 1495 trees in the orchard.

i Find the number of rows of trees.  
(2 marks)

ii In any row, how many more trees are there than in the previous row?  
(2 marks)

**Question 14**  (15 marks)

a The acceleration (in m s\(^{-2}\)) of a moving particle is  
\( \frac{d^2x}{dt^2} = -2 \). When \( t = 0 \), \( \frac{dx}{dt} = 6 \) and \( x = 7 \).

i Find an expression for the velocity of the particle.  
(1 mark)

ii Find an expression for the displacement.  
(1 mark)

iii Find the time when the particle is at rest.  
(1 mark)

iv Find the distance travelled in the first five seconds.  
(2 marks)
b The section of the curve \( y = \ln x \) between \( y = 0 \) and \( y = 2 \) is rotated about the \( y \)-axis.

\[
\begin{align*}
y(x) &= \ln x \\
0 &\leq x \leq 2
\end{align*}
\]

i Show that the volume of the solid generated is given by \( V = \pi \int_0^2 e^{2y} \, dy \) (2 marks)

ii Hence show that the exact volume is \( \frac{\pi}{2} (e^4 - 1) \) units\(^3\) (2 marks)

c A child’s swimming pool, which at first had 900 litres of water in it, is being filled. The volume flow rate \( R \) of water in litres per minute is given by \( R = 80 - 2t \)

i For how long will water flow into the pool? (1 mark)

ii At what rate is the water flowing into the pool after 15 minutes? (1 mark)

iii Find an expression for \( V \), the volume of water in the pool at time \( t \) minutes. (1 mark)

iv How long will it take for the pool to hold 2400 litres? (2 marks)

Question 15 (15 marks)

a A rectangular box is made from cardboard. The box has a square base of side length \( l \) cm and height \( h \) cm. The box has an extra flap at the top, effectively making two lids.

\[
\text{If the volume of the box is } 768 \text{ cm}^3:
\]

i show that the area, \( A \) m\(^2\), of cardboard needed to make the box is given by \( A = 3l^2 + \frac{3072}{l} \) (2 marks)

ii find the dimensions of the box that uses the minimum amount of cardboard. (3 marks)

b The population of two towns, Whenever and Whynot, are given by the following formulas

\[
\begin{align*}
\text{Whenever:} & \quad P = 3000e^{0.02t} \\
\text{Whynot:} & \quad Q = 5000e^{-0.03t}
\end{align*}
\]

where \( t \) is the time in years since 2001.

i Briefly compare the two populations over time, with reference to initial populations and growth rates. (2 marks)

ii During what year will the population of Whenever first exceed that of Whynot? (3 marks)

iii The population of which town will be larger in 2020? Justify your answer. (2 marks)

b Consider the curve \( y = 2xe^{-x} \)

i Where does the curve cross the \( y \)-axis? (1 mark)

ii Show that \( \frac{dy}{dx} = 2e^{-x}(1 - x) \) (2 marks)

iii Show that \( \frac{d^2y}{dx^2} = -2e^{-x}(2 - x) \) (2 marks)

iv Find the turning point and determine its nature. (2 marks)

v Find the point where the curve changes concavity. (1 mark)

vi Determine the behaviour of \( y \) as \( x \to \infty \) and as \( x \to -\infty \) (1 mark)

vii Sketch the graph of \( y = 2xe^{-x} \) (1 mark)

Go to p 234 for Quick Answers
or pp 327–31 for Worked Solutions
Quick Answers to Sample HSC Examinations

Sample HSC Examination 1

Section I

1 B 2 A 3 D 4 A 5 B 6 D 7 C 8 C 9 C 10 B

Section II

11a i $\frac{4}{5}$ ii $5x - 4y - 12 = 0$ iii $(0, -3)$ iv $\sqrt{41}$ units v $20.5$ units ii $i$ $xe^{x+2}$ ii $3\sin x(1 - \cos x)$ c i $\frac{1}{20}$ ii $\frac{893}{990}$

12a $12$ m b i $\frac{1}{2}\log_{e}(x^2 + 1) + C$ ii $\frac{1}{2}$ c $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$ d i $(0, 0)$ and $(6, 0)$ ii minimum $(0, 0)$, maximum at $(4, 32)$ iii $(2, 16)$ iv $x < 2$ 13a i $y = 3x - 9$ ii $4.5$ units ii $49$ minutes i $31$ ii $14$ hours d i $\frac{dy}{dt} > 0, \frac{d^2I}{dt^2} < 0$ ii See diagram below

14a i $t = 3$ ii $16$ m to the right of O iii $1$ m $s^{-2}$ iv It is initially $2$ m left of O moving left and slowing down. It comes to rest after $3$ seconds and then moves right at increasing speed reaching a position $16$ m to the right of O after the $6$ seconds. b i $2\%$ ii $8700$

iii $175$ people per year iv $2037$ c i and ii See diagram below 15a $\pi \sqrt{3}$ units iii $360$ ii See diagram below iii $2560$ t c i $\int_{A} = 170000 \times 1.006^{n} - M(1 + 1.006 + \ldots + 1.006^{n-1})$ iii $1223.30$ iv $196990$ 16a $y = e^t + e^{-t} - 2$ c i $(e^t - 1)$ units ii $\frac{1}{3}[1 + 4e + e^2]$

Diagrams: 13 d ii

Sample HSC Examination 2

Section I

1 C 2 B 3 A 4 D 5 A 6 D 7 B 8 C 9 D 10 C

Section II

11a $\frac{40978}{5}$ b iii $(6, -1)$ iv See the diagram on the right v $2\sqrt{26}$ units vi $\frac{16}{26}$ units

vii $16$ units ii $i$ $\frac{5}{8}$ ii $\frac{7}{24}$ iii $\frac{7}{8}$ iv $\frac{1}{4}$ 12a i $2xe^x$ ii $-6\sin 2x$ b $-\frac{1}{4(2x+1)^2} + C$ c $\log_2$

d i $5$ ii $3$ mm $s^{-1}$ iii $h = 36r^2 + 108 \left[ \text{or } 36 \sqrt{r} + 108 \right]$ iv $12$ min v $9$ sec 13a i $7800$ m ii $9$ km

b i $-1$ c iv $10$ d $15^\circ$ e $\frac{10}{21}$ 14a $0 \leq x < \frac{\pi}{2}$ ii $\frac{3\pi}{2} < x \leq 2\pi$ b $2x + 4y - 5 = 0$ c $2\pi$ units iii $a = -3, b = -9$ ii maximum when $x = -1$, minimum when $x = 3$ iii absolute maximum $49$, absolute minimum $-32$ 15a i $0.386$ ii $\log_2 x$ iii $0.386$ iv Simpson’s rule is accurate to $3$ decimal places. b i $A$ ii $t = 2$ and $t = 8$

iii $30 - 6t$ iv $500$ v $0$, Particle B is undergoing no acceleration. It is travelling at a constant speed. 16a i $a = 1$ ii $\frac{1}{4}$ units iii 2 c iii $\frac{2\sqrt{2}}{3}$ vi $4r$ when $\theta = 2$

Sample HSC Examination 3

Section I

1 B 2 C 3 A 4 C 5 A 6 B 7 D 8 D 9 B 10 C

Section II

11a i $(2, 2)$ ii $\frac{1}{4}$ iii $2\sqrt{17}$ units iv $(3, -2)$ vi Rhombus, the diagonals bisect each other at right angles. b $73 \frac{1}{3}$ c iii $\frac{2\sqrt{3}}{3}$ units

12a i $e^{(\cos x + \sin x)}$ ii $\frac{2x^2 + 4}{x^2}$ b i $2$ ii $\frac{1}{4}$ c $\alpha = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ d $\frac{1}{6}$ c $\text{cm}$ ii $94.2$ cm $^2$ 13a 2 b $\frac{1}{21}$ i $20 \frac{1}{21}$ ii $10 \frac{1}{21}$ iii $(-36, 81)$

iii $666 \frac{2}{3}$ units ii $26$ rows iii $13$ 14a i $v = -2t + 6$ ii $x = -t^2 + 6t + 7$ iii $t = 3$ iv $13$ m c $i$ $40$ minutes ii $50$ litres per minute iii $V = 80t - t^2 + 900$ iv $2500$ litres v $30$ minutes

15a i $8$ cm long and $12$ cm high b i Whenever had a population of $3000$ in 2001 and is growing at $2\%$ per year. Why not had a population of $5000$ in 2001, but its population is decreasing at $3\%$ per year. ii Whenever, $4400$ compared to $2800$ iii $2012$ c $459946$ 16b i $(0, 0)$ iv maximum $(1, 2e^{-t})$ v $(2, 4e^{-t})$ vi As $x \to \infty$, $y \to 0^+$. As $x \to -\infty$, $y \to -\infty$ vii See diagram on the right

y = $2xe^{-x}$
Sample HSC Examinations
Worked Solutions to Sample HSC Examinations

Section I (10 marks)

1. \[ \sin \frac{2\pi}{3} = \sin (\pi - \frac{\pi}{3}) \]
   \[ = \frac{\sqrt{3}}{2} \]
   \[ \text{B} \]

2. \[ 4x^3 \times 6x^8 = \frac{24x^{11}}{4x^9} = \frac{6}{x^2} \]
   \[ \text{A} \]

3. \[ 32 + 16 + 8 + 4 + \ldots \]
   Geometric series: \(a = 32, \ r = \frac{1}{2} \)
   \[ S = \frac{a}{1-r} \]
   \[ = \frac{32}{1-\frac{1}{2}} \]
   \[ = 64 \]
   \[ \text{D} \]

4. \[ \log_y 7 = \frac{\log_5 7}{\log_5 y} \]
   \[ = \frac{1.20906195\ldots}{1.209} \approx 1.000 \]
   \[ \text{A} \]

5. \[ \frac{\sin \theta \cos \theta + \sin^2 \theta}{\sin \theta \cos \theta + \cos^2 \theta} = \frac{\sin \theta (\cos \theta + \sin \theta)}{\cos \theta (\sin \theta + \cos \theta)} = \frac{\sin \theta}{\cos \theta} = \tan \theta \]
   \[ \text{B} \]

6. \[ f(x) = \frac{1}{x^2} \]
   \[ F(x) = -\frac{1}{x} + C \]
   \[ \text{D} \]

7. Centre \((-2,3)\), radius 4
   The equation of the circle is
   \[ (x + 2)^2 + (y - 3)^2 = 4^2 \]
   \[ (x + 2)^2 + (y - 3)^2 = 16 \]
   \[ \text{C} \]

8. \[ 2x^2 + 8x + 5 = 0 \]
   \[ \Delta = b^2 - 4ac \]
   \[ = 8^2 - 4 \times 2 \times 5 \]
   \[ = 24 \]
   \[ \therefore \text{the roots are real} \quad [\Delta > 0] \]
   \[ \text{but not rational.} \quad [\Delta \text{is not a perfect square}] \]
   \[ \text{C} \]

Section II (90 marks)

11a. P(4, 2), Q(9, -2)

1. \[ \frac{1}{\sqrt{3}+2} = \frac{a+b\sqrt{3}}{1} \]
   \[ = \frac{1}{\sqrt{3}+2} \times \frac{\sqrt{3}-2}{\sqrt{3}-2} \]
   \[ = \frac{\sqrt{3}-2}{(\sqrt{3})^2 - 2^2} \]
   \[ = \frac{\sqrt{3}-2}{3-4} \]
   \[ = -\frac{1}{1} = -1 \]
   \[ \text{So } a = 2 \text{ and } b = -1. \]
   \[ \text{C} \]

2. \[ xy - y = x - x \]
   \[ = \frac{9}{2} \]
   \[ \text{B} \]

3. \[ 10 \]
   \[ \text{C} \]

4. \[ \text{vi} \]
   Perpendicular line has gradient
   \[ m = \frac{5}{4} \]
   \[ m = \frac{5}{4} (4, 2) \]
   \[ y - y_1 = m(x - x_1) \]
   \[ y - 2 = \frac{5}{4} (x - 4) \]
   \[ 4y - 8 = 5x - 20 \]
   \[ 5x - 4y = 12 \]
   \[ \text{C} \]

5. \[ \text{v} \]
   \[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
   \[ = \sqrt{(0 - 4)^2 + (-3 - 2)^2} \]
   \[ = \sqrt{16 + 25} \]
   \[ = \sqrt{41} \]
   \[ \therefore \text{length of } PR = \sqrt{41} \text{ units} \]
   \[ \text{B} \]

6. \[ \text{vi} \]
   \[ A = \frac{1}{2} bh \]
   \[ = \frac{1}{2} \times \sqrt{41} \times \sqrt{41} \]
   \[ = 20.5 \]
   \[ \therefore \text{area of } \Delta PQR = 20.5 \text{ units}^2 \]
   \[ \text{B} \]

7. \[ \text{b i} \]
   \[ y = x^2e^x \]
   \[ \frac{dy}{dx} = x^2 \times \frac{d}{dx}(e^x) + e^x \times \frac{d}{dx}(x^2) \]
   \[ = x^2e^x + e^x \times 2x \]
   \[ = xe^{x+2} \]
   \[ \text{(2 marks)} \]

8. \[ \text{b ii} \]
   \[ y = (1 - \cos x)^3 \]
   \[ \frac{dy}{dx} = 3(1 - \cos x)^2 \times \sin x \]
   \[ = 3(1 - \cos x)^2 \text{sin } x \]
   \[ \text{(2 marks)} \]
ii \[ P(\text{no prize}) = \frac{95}{100} \times \frac{94}{99} \] (2 marks)

Total for Q12: 15 marks

ii \[ \int_0^1 \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int_0^1 \frac{2x}{x^2 + 1} \, dx \] (2 marks)

\[ = \frac{1}{2} \log_2 \left( x^2 + 1 \right) + C \] (2 marks)

\[ = \frac{1}{2} \log_2 \left( \frac{1}{2} \right) + 1 \] (1 mark)

Total for Q12: 15 marks

13a i \[ x^2 = 4y \]

\[ y = \frac{x^2}{4} \]

\[ \frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2} \] (2 marks)

When \( x = 6 \), \[ \frac{dy}{dx} = \frac{6}{2} = 3 \] (3 marks)

\[ m = 3 \] (1 mark)

\[ (6, 9), m = 3 \]

\[ y - y_1 = m(x - x_1) \]

\[ y - 9 = 3(6 - x) \]

\[ y = 9 - 3x - 18 \]

\[ y = 3x - 9 \] (2 marks)

i The tangent cuts the x-axis when \( y = 0 \)

i.e. \[ 3x - 9 = 0 \]

\[ 3x = 9 \]

\[ x = 3 \] (3 marks)

Area of shaded triangle:

\[ A = \frac{1}{2} \times 3 \times 9 \]

\[ = 13.5 \] (2 marks)

Area under parabola:

\[ A = \int_0^4 \frac{x}{4} \, dx \]

\[ = \frac{1}{12} \times 6 \]

\[ = \frac{1}{2} \times 3 \frac{1}{2} \]

\[ = 3 \] (2 marks)

Required area:

\[ = (18 - 13.5) \text{ units}^2 \]

\[ = 4.5 \text{ units}^2 \] (3 marks)

b i In \( \triangle ABD \), \( \angle ABD = \angle ADB \) (both 90°, \( AB \perp BD \))

\( AB = AD (sides \ of \ rhombus) \)

\( AX = AX \ (common \ side) \)

\( \therefore \triangle ABD \equiv \triangle ADB \) (RHS) (2 marks)

ii \( \angle ABD = \angle ADB \) (corresp. \( \angle s \) of congruent \( \triangle s \))

But \( \angle ABD = \angle CDB \) (alternate \( \angle s \), parallel lines) (2 marks)

\( \therefore \triangle ADB \equiv \triangle CDB \)

\( \therefore BD \) bisects \( \triangle ADC \) (2 marks)

c 10 + 13 + 16 + 19 + ... Arithmetic series: \( a = 10, d = 3 \)

i \( T_n = a + (n - 1)d \)

\( T_{14} = 10 + (14 - 1) \times 3 \)

\[ = 49 \] (1 mark)

\( \therefore \) Jenny will exercise for 49 minutes on the 14th day

ii 1 hour 40 min = 100 min

\[ 100 = 10 + (n - 1) \times 3 \]

\[ 100 = 10 + 3n - 3 \]

\[ 93 = 3n \]

\[ n = 31 \]

On the 31st day, Jenny will exercise for 1 hour and 40 minutes. (1 mark)
iii Three weeks = 21 days
\[ S_n = \frac{n}{2} [2a + (n-1)d] \]
\[ S_{21} = \frac{21}{2} [2(10) + (21-1)\times3] \]
\[ = 840 \] 
840 minutes = 14 hours
Jenny will have exercised for a total of 14 hours at the end of three weeks. ✓ (2 marks)

d i Immigration was increasing.
\[ \therefore \frac{dI}{dt} > 0 \]
It was increasing at a decreasing rate.
\[ \therefore \frac{d^2I}{dt^2} < 0 \] ✓ (1 mark)

14 a \[ v = t^2 + t - 12 \]
i Particle is at rest when \( v = 0 \)
\[ i.e. \ t^2 + t - 12 = 0 \]
\[ (t+4)(t-3) = 0 \]
\[ t = -4 \text{ or } t = 3 \]
But \( t \geq 0 \) \( \therefore t = 3 \)
The particle is at rest after 3 seconds. ✓ (1 mark)

ii \[ v = t^2 + t - 12 \]
\[ x = \frac{t^3}{3} + \frac{t^2}{2} - 12t + C \]
When \( t = 0 \), \( x = -2 \)
\[ -2 = \frac{0^3}{3} + \frac{0^2}{2} - 12(0) + C \]
\[ C = -2 \]
\[ \therefore x = \frac{t^3}{3} + \frac{t^2}{2} - 12t - 2 \] ✓
When \( t = 6 \),
\[ x = \frac{6^3}{3} + \frac{6^2}{2} - 12(6) - 2 = 16 \]
After 6 seconds the particle is 16 m to the right of \( O \). ✓ (2 marks)

iii \[ a = 2t + 1 \]
When \( t = 0 \), \( a = 2(0) + 1 = 1 \)
\[ \therefore \text{initial acceleration is } 1 \text{ m s}^{-2} \] ✓ (1 mark)

iv The particle is initially 2 m to the left of the origin moving left \( (\text{the velocity is negative}) \) and slowing down \( (\text{the acceleration is positive, opposite the velocity}) \). It comes to rest after 3 s and then moves right at increasing speed reaching a position 16 m to the right of the origin after 6 s. ✓ (1 mark)

b \[ P = P_0 e^{kt} \]
i Let \( t = 0 \) in 1991
\[ \therefore P_0 = 4000 \]
In the year 2006, \( t = 2006 - 1991 = 15 \)
When \( t = 15 \), \( P = 5400 \)
\[ 5400 = 4000e^{15k} \]
\[ 1.35 = e^{15k} \]
\[ \ln 1.35 = 15k \]
\[ k = \frac{\ln 1.35}{15} \] ✓ (2 marks)

ii In the year 2030,
\[ t = 2030 - 1991 = 39 \]
\[ P = 4000e^{39k} \]
\[ = 8728,262,306... \]
\[ = 8700 \text{ [nearest hundred]} \]
In 2030, the population will be approximately 8700. ✓ (1 mark)

iii \[ \frac{dP}{dt} = kP \]
\[ = k \times 8728,262,306... \]
\[ = 174,626,106... \]
\[ = 175 \text{ [nearest unit]} \]
In 2030, the population will be increasing at the rate of 175 people per year. ✓ (1 mark)

iv \[ P = 10000 \]
\[ 10000 = 4000e^{kt} \]
\[ 2.5 = e^{kt} \]
\[ \ln 2.5 = t.k \]
\[ t = \frac{\ln 2.5}{k} \]
\[ = 45,798,569.31... \]
During the 46th year, the population should reach 10 000
\[ \text{i.e. in the year } 2037. \] ✓ (2 marks)

c i and ii
\[ y = 3 \sin 2x \]
\[ y = x \]
\[ y = 3 \sin 2x \text{ have three points of intersection.} \]
\[ \therefore 3 \sin 2x = x \text{ has three solutions.} \] ✓ (1 mark)

Total for Q14: 15 marks

15 a

\[ V = \int_0^e \pi y^2 \, dx \]
\[ = \pi \left[ \tan x \right]_0^e \]
\[ = \pi \left( \tan \frac{\pi}{3} - \tan 0 \right) \]
\[ = \pi \sqrt{3} \]
The volume is \( \pi \sqrt{3} \text{ units}^3 \). ✓ (3 marks)

b \[ R = 360 - \frac{360}{(1+t)^2} \]
\[ \text{As } t \to \infty, \frac{1}{(1+t)^2} \to 0 \]
\[ R \to 360 \] ✓ (1 mark)

ii \[ R = \frac{360}{t} \]
\[ \text{When } t = 20, \]
\[ R = \frac{360}{20} = 18 \]
\[ \therefore 2560 \text{ tonnes of resources will be used in the first } 8 \text{ hours} \] ✓ (2 marks)

Total for Q15: 15 marks

\[ A_1 = 170000(1.006) - M \]
\[ A_2 = A_1(1.006) - M \]
\[ = [170000(1.006) - M] \times (1.006) - M \]
\[ = 170000(1.006)^2 - M(1.006)^2 - M \]
\[ = 170000(1.006)^2 - M(1+1.006) \] ✓ (cont.)
\[
A_3 = A_2 (1.006) - M \\
= [100000(1.006)]^2 - M + 1 + 1.006^2) (1.006)^3 - M \\
= 170000(1.006)^3 - M \\
- M (1.006 + 1.006^2) - M \\
= 170000(1.006)^3 - M \\
- M (1 + 1.006 + 1.006^2) (1.006)^n \\
= 170000(1.006)^3 - M \\
- M (1 + 1.006 + 1.006^2) \\
\]

\[\text{(3 marks)}\]

ii \(A_n = 170000(1.006)^n\)

\[\text{Geometric series:}\]

\[a = 1, r = 1.006, n = n\]

\[S_n = \frac{a(r^n - 1)}{r - 1} \]

\[= \frac{1.006^n - 1}{0.006} \]

\[\therefore A_n = 170000(1.006)^n\]

\[\text{(2 marks)}\]

iii \(25\) years = 300 months

\[A_{300} = 170000(1.006)^{300}\]

\[-M \left(\frac{1.006^{300} - 1}{0.006}\right)\]

\[\text{If the loan is repaid after}\]

\[25\] years, \(A_{300} = 0\)

\[0 = 170000(1.006)^{300}\]

\[-M \left(\frac{1.006^{300} - 1}{0.006}\right)\]

\[M \left(\frac{1.006^{300} - 1}{0.006}\right)\]

\[= 1223.300 775...\]

\[= 1223.30 (2\text{ d.p.})\]

\[\text{The amount of each monthly}\]

\[\text{installment would be } \$1223.30\]

\[\text{(2 marks)}\]

iv \(\text{Total repaid:}\)

\[\$1223.30 \times 300\]

\[= \$366990\]

\[\text{Total interest:}\]

\[\$366990 - \$170000\]

\[= \$196990\]

\[\text{(1 mark)}\]

\[\text{Total for Q15: 15 marks}\]

\[\text{Sample HSC Examinations Exam 1 Q15–16 and Exam 2 Q1}\]

16a \(\frac{dy}{dx} = e^r - e^{-y}\)

\[y = e^r - e^{-y} + C\]

\[= e^r + e^{-y} + C\]

\[\text{The curve passes through the origin.}\]

\[\therefore: \text{when } x = 0, y = 0\]

\[0 = e^0 + e^{-0} + C\]

\[0 = 1 + C\]

\[C = -1\]

\[\text{The equation of the curve}\]

\[y = e^r + e^{-y} - 2.\]

\[\text{(2 marks)}\]

b i \(\triangle AOB = \triangle CDDB\) (both 90\°)

\(\triangle ABO = \triangle CDB\) (common \(\angle\))

\[\therefore \triangle AOB \text{ and } \triangle CDB \text{ are similar}\]

\[\therefore h = 15 - \frac{3r}{9}\]

\[\text{(corresponding sides of similar } \triangle \text{s)}\]

\[h = 15 - \frac{r}{3}\]

\[h = 15 - \frac{15r}{9}\]

\[= 15 - \frac{5r}{3}\]

\[\text{(1 mark)}\]

ii \(V = \pi r^2h\)

\[= \pi r^2 \left(15 - \frac{5r}{3}\right)\]

\[= 15\pi r^2 - \frac{5\pi r^3}{3}\]

\[\text{(2 marks)}\]

iii \(\frac{dV}{dr} = 30\pi r - 5\pi r^2\)

\[\text{When } \frac{dV}{dr} = 0\]

\[0 = 30\pi r - 5\pi r^2\]

\[5\pi r(6 - r) = 0\]

\[r = 0\text{ or } r = 6\]

\[\frac{d^2V}{dr^2} = 30\pi - 10\pi r\]

\[\frac{d^2V}{dr^2} < 0 \text{ when } r = 6\]

\[\text{The maximum volume occurs when } r = 6.\]

\[\text{(2 marks)}\]

iv \(V = 15\pi r^2 - \frac{5\pi r^3}{3}\)

\[\text{When } r = 6,\]

\[V = 15\pi (6)^2 - \frac{5\pi (6)^3}{3}\]

\[= 180\pi\]

\[\text{The maximum possible volume of the cylinder is } 180\pi \text{ cm}^3.\]

\[\text{(2 marks)}\]
2 \[ \log_a 8 - \log_a 2 = \log_a \frac{8}{2} = \log_a 4 = 1 \]

3 Geometric series:
   \[ a = 3, r = -2, \ n = 10 \]
   \[ T_n = ar^{n-1} \]
   \[ T_{10} = 3 \times (-2)^{10-1} = -1536 \]
The tenth term is -1536.

4 \( P(\text{Jo wins all 3 games}) = 0.3 \times 0.3 \times 0.3 = 0.027 \)

5 \( 2 \pi e = 0.7357588... \]
   \( e = 0.736 \) [3 sig. figs.]

6 \( y = \sqrt{9 - x^2} \)
   Domain: \( 9 - x^2 \geq 0 \)
   \( x^2 \leq 9 \)
   \( -3 \leq x \leq 3 \)

7 \[ 8x^3 - 27y^3 = (2x)^3 - (3y)^3 \]
   \[ = (2x - 3y)(4x^2 + 2xy + 9y^2) \]

8 \( M = 40e^{-0.06t} \)
When \( t = 8 \),
\[ M = 40e^{-0.06 \times 8} = 25.75133... \]
\[ = 25 \] (nearest whole number)
\( \therefore \) the mass remaining is about 25 g.

9 \[ \int_0^2 \sec^2 x \, dx = \left[ \tan x \right]_0^2 = \tan \frac{\pi}{3} - \tan 0 = \sqrt{3} - 0 = \sqrt{3} \]

10 \( y = \log_x x \)
   \( y = \frac{1}{x} \)

\[ \text{Section II (90 marks)} \]

11 \( a \)
   \[ P = \$20000, r = 0.75, n = 96 \]
   \[ A_1 = P \left( 1 + \frac{r}{100} \right)^n \]
   \[ A_{96} = 20000 \left( 1.075 \right)^{96} \]
   \[ = 40978.42456... \]
   \[ \approx 40978 \] [nearest dollar] \( \checkmark \) (1 mark)

\[ b \]
   \[ P(2, 3), x + 5y - 1 = 0 \]
   \[ d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \]
   \[ = \frac{16}{\sqrt{26}} \] units \( \checkmark \) (1 mark)

12 \( a \)
   \[ y = e^{2x} \]
   \[ \frac{dy}{dx} = e^{2x} \times \frac{d}{dx}(2x) \]
   \[ = 2e^{2x} \] \( \checkmark \) (2 marks)

\[ \text{Total for Q11: 15 marks} \]

13 \( a \)
   \[ y = 3\cos 2x \]
   \[ \frac{dy}{dx} = 3\cos 2x \times (-2\sin 2x) \]
   \[ = -6\sin 2x \] \( \checkmark \) (2 marks)

\[ b \]
   \[ \int \frac{dx}{(2x + 1)^2} = \int (2x + 1)^{-2} \, dx \]
   \[ = \frac{1}{4(2x + 1)^2} + C \] \( \checkmark \) (2 marks)
c \[ \int_1^2 x^{-1} \, dx = \int_1^2 \frac{1}{x} \, dx \]
= \left[ \log_2 x \right]_1^2 \quad \checkmark
= \log_2 2 - \log_2 1
= \log_2 2
= \log_2 \sqrt{2} \quad \checkmark \quad (2 \text{ marks})

d \[ \int_1^2 x \, dx = 12 \]
= \left[ \frac{x^2}{2} \right]_1^2
= \frac{2^2}{2} - \frac{1^2}{2} \quad \checkmark
\therefore \quad \frac{b^2 - 1}{2} = 12
b^2 - 1 = 24
b^2 = 25
b = 5 \quad (b > 0) \quad \checkmark \quad (2 \text{ marks})

e \quad R = \frac{12}{3} \sqrt{2^2}

i When \( t = 8 \), \[ R = \frac{12}{\sqrt{8^2}} = 3 \]
After 8 seconds the height is increasing at 3 mm s\(^{-1}\) \quad (1 \text{ mark})

ii \[ \frac{dh}{dt} = \frac{12}{\sqrt{r^2}} \]
= \frac{12r^2 - \sqrt{r^2} + C}{3}
= \frac{36r^2 + C}{3}
When \( t = 8, h = 180 \)
180 = 36(8) \quad + C
180 = 72 + C
C = 108
\therefore \quad h = 36t^2 + 108 \quad \checkmark \quad (2 \text{ marks})

iii When \( h = 432 \)
432 = 36t^2 + 108
324 = 36t^2
t^2 = 9
\therefore \quad t = 3 \quad \checkmark
729 seconds = 12 minutes 9 seconds
It takes 12 minutes and 9 seconds for the container to be filled. \quad \checkmark \quad (2 \text{ marks})

Total for Q12: 15 marks

b \[ 1 + 3x + 3x^2 + 3x^3 + \ldots \]
Infinite geometric series:
\( a = 1, r = 3, S = 1.5 \)
\[ S = \frac{a}{1-r} \]
\[ 1.5 = \frac{1}{1-3} \quad \checkmark \]
\[ 1 - 3^r = \frac{1}{1-3} \]
\[ 1 - 3^t = \frac{1}{1.5} \]
\[ = \frac{2}{3} \]
\[ 3^t = \frac{3}{2} \]
\[ = \frac{3}{2} \quad \checkmark \]
\[ x = -1 \quad \checkmark \quad (2 \text{ marks})

\[ 2 \ln x = \ln (3x + 4) \]
\[ \ln x^2 = \ln (3x + 4) \quad \checkmark \]
\[ x^2 = 3x + 4 \]
\[ x^2 - 3x - 4 = 0 \]
\[ (x+1)(x-4) = 0 \]
\[ x = -1 \text{ or } x = 4 \]
But \( x > 0 \) \quad \therefore \quad x = 4 \quad \checkmark \quad (2 \text{ marks})

c \[ P(\text{apples same colour}) = P(A; RR) + P(A; GG) \]
\[ + P(B; RR) + P(B; GG) \]
\[ = \frac{1}{2} \times \frac{12}{21} + \frac{1}{2} \times \frac{11}{20} + \frac{1}{2} \times \frac{9}{20} + \frac{1}{2} \times \frac{8}{20} \]
\[ = \frac{1}{2} \times \frac{12}{21} + \frac{1}{2} \times \frac{11}{20} + \frac{1}{2} \times \frac{9}{20} + \frac{1}{2} \times \frac{8}{20} \]
\[ = \frac{10}{21} \quad \checkmark \quad (2 \text{ marks}) \]

Total for Q13: 15 marks
\[ y = x + \cos x \]
\[ \frac{dy}{dx} = 1 - \sin x \]
\[ \frac{d^2y}{dx^2} = -\cos x \]

The curve is concave down when \( \frac{d^2y}{dx^2} < 0 \), i.e. \( \cos x < 0 \) or \( \cos x > 0 \). The curve is concave down when \( 0 < x < \frac{\pi}{2} \) and when \( \frac{3\pi}{2} < x \leq 2\pi \). (3 marks)

\[ y = e^{2x-1} \]
\[ \frac{dy}{dx} = 2e^{2x-1} \]

When \( x = \frac{1}{2} \):
\[ \frac{dy}{dx} = 2e^{2\left(\frac{1}{2}\right)-1} = 2e^0 = 2 \]

\( m = 2 \)
\[ y - y_1 = m(x-x_1) \]
\[ y - 1 = 2\left(x - \frac{1}{2}\right) \]
\[ y = 2x - 1 + 1 = 2x \]

\( 4y - 4 = -2x + 1 \)
\[ 2x + dy - 5 = 0 \] (3 marks)

The absolute maximum of the function is 49 (when \( x = 6 \)) and the absolute minimum is 32 (when \( x = 3 \)). (2 marks)

\[ f(x) = x^3 + ax^2 + bx - 5 \]
\[ \frac{dy}{dx} = x \times \frac{d}{dx}(\log_e x) \]
\[ + (\log_e x) \times \frac{d}{dx}(x) - 1 \]
\[ = x \times \frac{1}{x} + (\log_e x) \times 1 - 1 \]
\[ = 1 + \log_e x - 1 \]
\[ = \log_e x \] (2 marks)

\[ \int_0^1 \log_e x \, dx \]
\[ = \left[ (\log_e x - x)^2 \right] \]
\[ = [2 \log_2 x - 2 - 2(\log_2 1 - 1)] \]
\[ = 0.386 \times 294 \times 361 \]
\[ = 0.386 \] (3 d.p.) (2 marks)

iv Simpson’s rule gives a good approximation of the integral. It is accurate to 3 decimal places. (1 mark)

b \[ x_A = 15t^2 - t^3 \]
\[ x_B = 48t - 9 \]

i \( v_B = 30t - 3t^2 \)
When \( t = \), \( v_B = 30(3) - 3(3)^2 = 63 \)
\[ v_B = 48 \]
:. particle A is moving faster when \( t = 3 \) (2 marks)

ii The particles have the same velocity when \( 30t - 3t^2 = 48 \)
\[ 3t^2 - 30t + 48 = 0 \]
\[ t^2 - 10t + 16 = 0 \]
\[ (t - 2)(t - 8) = 0 \]
\[ t = 2 \] or \( t = 8 \)
:. the particles travel at the same velocity when \( t = 2 \) and \( t = 8 \) (2 marks)

iii \( a_A = 30 - 6t \) (1 mark)

iv Maximum displacement will occur when \( v_B = 0 \)
\[ \text{i.e. } 30t - 3t^2 = 0 \]
\[ 3(10 - t) = 0 \]
\[ t = 0 \text{ or } t = 10 \]
When \( t = 0 \), \( a_A = 30 \) (0) \:. minimum
When \( t = 10 \), \( a_A = 30 \) (0) \:. maximum
When \( t = 10 \), \( x_A = 15(10)^2 - 10^3 \)
\[ = 500 \]
\[ a_A = 0 \]
Particle \( B \) is undergoing no acceleration. It is travelling at a constant speed. (1 mark)

Total for Q14: 15 marks

Total for Q15: 15 marks
At points of intersection:
\[ x^3 - 3x^2 + 2x = 0 \]
\[ x(x^2 - 3x + 2) = 0 \]
\[ x = 0 \text{ or } x = 1 \text{ or } x = 2 \]
\[ 0 < a < b \implies a = 1 \]  \((2 \text{ marks})\)

Let \( x = c \) be the point where the curve cuts the \( x \)-axis \((0 < c < a)\).

Required area
\[ = \int_{0}^{c} x^3 \, dx + \int_{c}^{1} (3x^2 - 2x) \, dx \]
\[ = \left[ \frac{x^4}{4} - x^3 \right]_{0}^{c} + \int_{c}^{1} (3x^2 - 2x) \, dx \]
\[ = \int_{0}^{c} x^3 \, dx - \int_{0}^{c} (3x^2 - 2x) \, dx \]
\[ = \left[ \frac{x^4}{4} - x^3 \right]_{0}^{1} - \left[ x^3 - 3x^2 + 2x \right] \]
\[ = \int_{0}^{1} x^3 - 3x^2 + 2x \, dx \]  \((2 \text{ marks})\)

\[ A = \int_{0}^{1} x^3 - 3x^2 + 2x \, dx \]
\[ = \left[ \frac{x^4}{4} - x^3 + x^2 \right]_{0}^{1} \]
\[ = \frac{1}{4} - 1 + 1 \]
\[ = \frac{1}{4} \]

The area is \( \frac{1}{4} \) units\(^2\) \((2 \text{ marks})\)

\[ y = x^3 \]
\[ y = x^3 - 2x \]

\[ i \quad \text{Arc length: } l = r \theta \]
Perimeter: \( P = r + r + r \theta \)
\[ = 2r + r \theta \]
\[ = r(2 + \theta) \]
\[ 2 + \theta = \frac{P}{r} \]
\[ \theta = \frac{P}{r} - 2 \]  \((1 \text{ mark})\)

\[ ii \quad A = \frac{1}{2} r^2 \theta \]
\[ = \frac{1}{2} \left( \frac{P}{r} - 2 \right) \]
\[ = \frac{Pr}{2} - r^2 \]  \((1 \text{ mark})\)

\[ iii \quad A = \frac{Pr}{2} + r^2, \quad \frac{dA}{dr} = \frac{P}{2} - 2r \]
Maximum or minimum occurs when \( \frac{dA}{dr} = 0 \)
i.e. \( \frac{P}{2} - 2r = 0 \)
\[ P = 2r \]
\[ P = 2r \]
\[ P = 4r \]  \((3 \text{ marks})\)

\[ \frac{d^2 A}{dr^2} = -2 \]  \((< 0)\)
\[ \therefore \text{ maximum area occurs when } P = 4r \]
When \( P = 4r \),
\[ A = \frac{4r^2 - r^2}{2} \]
\[ = r^2 \]  \((2 \text{ marks})\)

\[ iv \quad \theta = \frac{P}{r} - 2 \]
When \( P = 4r \),
\[ \theta = \frac{4r}{r} - 2 \]
\[ = 2 \]
\[ \therefore \text{ maximum area, } r^2, \]
occur when \( \theta = 2 \)  \((1 \text{ mark})\)

\[ v \quad A = \frac{Pr}{2} - r^2 \]
\[ \frac{Pr}{2} = A + r^2 \]
\[ Pr = 2A + 2r^2 \]
\[ P = 2\frac{A}{r} + 2r \]
\[ P = 2r + \frac{2A}{r} \]  \((1 \text{ mark})\)

\[ vi \quad P = 2r + \frac{2A}{r} \]
\[ = 2r + 2Ar^{-1} \]
\[ \frac{dP}{dr} = 2 - 2Ar^{-2} \]
\[ = 2 - 2r^{-2} \]  \((2 \text{ marks})\)

A maximum or minimum occurs when \( \frac{dP}{dr} = 0 \)

\[ \alpha + \beta = \frac{2b}{a} \]
\[ = \frac{6}{3} \]
\[ = -2 \]  \((2 \text{ marks})\)

Sample HSC Examination 3

Section I (10 marks)

1. \( \log_{4} 1386 = 13863 \) \((3 \text{ d.p.})\)
2. \( x - 1 \frac{1}{x^2 - 1} = \frac{x - 1}{(x+1)(x-1)} \]
3. \( \frac{40}{90} = \frac{40}{180} \)
4. Number of marbles = \( 5 + 4 + 3 \)
5. \( f(x) = x^2 - 7x + 4 \)
6. \( F(x) = \frac{x^3 + 5x + C}{2} \)

Sample HSC Mathematics Study Guide

Worked Solutions Sample HSC Examinations Exam 2 Q16 and Exam 3 Q1–7
8

\[ A = \frac{1}{2} ab \sin C \]
\[ = \frac{1}{2} \times 5.9 \times 7.9 \times \sin 80^\circ \]
\[ = 22.950 \ 944 \ 68... \]

Area = 23 cm²

\[ \log_2 a = 0.270 \]
\[ \log_3 b = 0.428 \]

\[ \log_2 4.5 = \log_3 b \]
\[ = \log_2 3 \cdot \log_2 2 \]
\[ = \log_2 3 - \log_2 2 \]
\[ = 2 \log_2 3 - \log_2 2 \]
\[ = 2 \times 0.428 - 0.270 \]
\[ = 0.558 \]

\[ f(x) = x^2 + 2x \]
\[ f(-x) = (-x)^2 + 2(-x) \]
\[ = x^2 - 2x \]
\[ f(-x) \neq f(x) \), so the function is not even.
\[ f(-x) \neq -f(x) \), so the function is not odd.
\[ The function is neither even nor odd.

Section II (90 marks)

11a i \[ M(x, y) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = (2, 2) \) (1 mark)

ii \[ \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{6 - (-2)} \]
\[ = \frac{2}{8} \]
\[ = \frac{1}{4} \] (1 mark)

iii \[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
\[ = \sqrt{(6 - (-2))^2 + (3 - 1)^2} \]
\[ = \sqrt{68} \]
\[ = 2\sqrt{17} \] (1 mark)

b 5 function values: \[ n = 4 \]
\[ h = \frac{b - a}{n} \]
\[ = \frac{4 - 0}{4} \]
\[ = 1 \]
\[ \int_a^b f(x) \, dx \]
\[ = \int_0^4 3 \, dx \]
\[ = \frac{1}{3} \left[ 3^0 + 3^4 + 4(3^1 + 3^3) + 2(3^2) \right] \]
\[ = 73 \frac{1}{2} \] (2 marks)

\[ \int_0^5 \sin 2x \, dx \]
\[ = \left[ \frac{1}{2} \cos 2x \right]_0^5 \]
\[ = \frac{1}{2} \cos \left( 2 \times \frac{\pi}{6} \right) \]
\[ - \left( - \frac{1}{2} \cos (2 \times 0) \right) \]
\[ = \frac{1}{2} \] (2 marks)

\[ \int_1^4 \frac{dx}{x} \]
\[ = \left[ \frac{x^4}{4} \right]_1^4 \]
\[ = \frac{24}{4} - \frac{1}{4} \]
\[ = 5 \] (2 marks)

12a i \[ y = e^x \sin x \]
\[ \frac{dy}{dx} = e^x \left[ \sin x + \cos x \frac{dx}{dx} \right] \]
\[ = e^x \cos x \sin x \cos x \]
\[ = e^x \left( \sin x + \sin x \right) \] (2 marks)

ii \[ y = \log_{\frac{1}{2}} (x^2 + 4) \]
\[ \frac{dy}{dx} = \frac{1}{x^2 + 4} \cdot \frac{1}{\log_{\frac{1}{2}} e} \cdot \frac{2x}{x^2 + 4} \]
\[ = \frac{2x}{x^2 + 4} \] (2 marks)

b i \[ \int_1^4 \, dx = \int_1^4 x^{-1} \, dx \]
\[ = \left[ x^{-\frac{1}{2}} \right]_1^4 \]
\[ = \frac{2}{2} - \frac{1}{2} \]
\[ = 2 \] (2 marks)

ii \[ \int_0^\pi \sin 2x \, dx \]
\[ = \left[ -\frac{1}{2} \cos 2x \right]_0^\pi \]
\[ = -\frac{1}{2} \cos (2 \times \pi) \]
\[ - \left( -\frac{1}{2} \cos (2 \times 0) \right) \]
\[ = \frac{1}{2} \] (2 marks)
13 a  \[ y = \tan x \]
\[ \frac{dy}{dx} = \sec^2 x \]
When \( x = \frac{\pi}{4} \)
\[ \frac{dy}{dx} = \sec^2 \frac{\pi}{4} = 2 \]

\( \therefore \) the gradient of the tangent is \( 2 \) \( \sqrt{2} \) \( (2 \text{ marks}) \)

b i  \[ P(\text{GGG}) = \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{21} \] \( \sqrt{2} \) \( (1 \text{ mark}) \)

ii  \[ P(\text{at least one } B) = 1 - P(\text{GGG}) = 1 - \frac{1}{21} = \frac{20}{21} \] \( \sqrt{2} \) \( (1 \text{ mark}) \)

iii  \[ P(\text{2 boys, 1 girl}) = P(\text{BBG}) + P(\text{GBB}) = \frac{5}{9} \times \frac{4}{8} \times \frac{4}{7} + \frac{4}{9} \times \frac{5}{8} \times \frac{4}{7} = \frac{10}{21} \] \( \sqrt{2} \) \( (2 \text{ marks}) \)

c  \[ x^2 = 16y \]
\[ y = \frac{x^2}{16} \]
\[ \frac{dy}{dx} = \frac{2x}{16} = \frac{x}{8} \]
When \( x = 4 \)
\[ \frac{dy}{dx} = \frac{4}{8} = \frac{1}{2} \]
\( \therefore \) gradient of tangent is \( \frac{1}{2} \)
\( \therefore \) gradient of normal is \( m = -2 \)
\( P(4, 1) \)
\[ y - y_1 = m(x - x_1) \]
\[ y - 1 = -2(x - 4) \]
\[ y = -2x + 8 \] \( (2 \text{ marks}) \)

14 a i  \[ \frac{d^2x}{dt^2} = -2 \]
\[ \frac{dx}{dt} = -2t + C_1 \]
When \( t = 0 \)
\[ \frac{dx}{dt} = 6 \]
\( \therefore \) \( 6 = -2(0) + C_1 \)
\( C_1 = 6 \)
\[ \frac{dx}{dt} = -2t + 6 \] \( (1 \text{ mark}) \)

ii  \[ x = -t^2 + 6t + C_2 \]
When \( t = 0 \)
\[ x = 7 \]
\( \therefore \) \( 7 = -0^2 + 6(0) + C_2 \)
\( C_2 = 7 \)
\( \therefore \) \( x = -t^2 + 6t + 7 \) \( (1 \text{ mark}) \)

Excel HSC Mathematics Study Guide
iii Particle is at rest when
\[ \frac{dx}{dt} = 0 \]
i.e. \(-2t + 6 = 0\)
\[ 2t = 6 \]
\[ t = 3 \]
The particle is at rest after 3 seconds. ✓ (1 mark)

iv When \( t = 0, x = 7 \)
The particle is at rest at \( t = 3 \), so the maximum [\( x < 0 \)] displacement occurs at \( t = 3 \)
When \( t = 3 \),
\[ x = -3t^2 + 6(3) + 7 
\[ = 16 \]
When \( t = 5 \),
\[ x = -5t^2 + 6(5) + 7 
\[ = 12 \]
Total distance
\[ = (16 - 7) + (16 - 12) \text{ m} 
\[ = 13 \text{ m} \]
✓ (2 marks)

b i \[ y = \ln x \]
\[ ∴ e^y = x \]
✓

V = \( \int \pi x^2 \, dy \)
\[ = \int_0^2 \pi \left( e^y \right)^2 \, dy \]
\[ = \pi \int_0^2 e^{2y} \, dy \]
✓ (2 marks)

ii \( V = \pi \int_0^2 e^{2y} \, dy \)
\[ = \frac{\pi}{2} \left( e^4 - e^0 \right) \]
\[ = \frac{\pi}{2} \left( e^4 - 1 \right) \]
The exact volume is
\[ \frac{\pi}{2} \left( e^4 - 1 \right) \text{ units}^3. \]
✓ (2 marks)

c i \( R = 80 - 2t \)
Water stops flowing when \( R = 0 \)
i.e. \( 80 - 2t = 0 \)
\[ 2t = 80 \]
\[ t = 40 \]
The water will flow for 40 minutes. ✓ (1 mark)

ii When \( t = 15, R = 80 - 2(15) \)
\[ = 50 \]
After 15 minutes the flow rate is 50 litres per minute. ✓ (1 mark)

iii \[ V = 80t - t^2 + C \]
When \( t = 0, V = 900 \)
\[ 900 = 80(0) - 0^2 + C \]
\[ C = 900 \]
\[ ∴ V = 80t - t^2 + 900 \] ✓ (1 mark)

iv When \( t = 40 \),
\[ V = 80(40) - 40^2 + 900 
\[ = 2500 \]
The pool holds 2500 litres. ✓ (1 mark)

v When \( V = 2400 \),
\[ 80t - t^2 + 900 = 2400 \]
\[ t^2 - 80t + 1500 = 0 \]
\[ (t - 30)(t - 50) = 0 \]
\[ t = 30 \text{ or } t = 50 \]
But \( 0 ≤ t ≤ 40 \) \( ∴ t = 30 \)
∴ the pool will hold 2400 litres after 30 minutes ✓ (2 marks)

Total for Q14: 15 marks

15 a

i \[ V = AH \]
\[ = l^2h \]
But the volume is 768 cm\(^3\)
\[ ∴ l^2h = 768 \]
\[ 768 \]
\[ h = \frac{768}{l^2} \]
✓

Base and two lids have area \( l^2 \text{ cm}^2 \)
Four sides have area \( lh \text{ cm}^2 \)
\[ A = 3l^2 + 4lh \]
\[ = 3l^2 + 4l \left( \frac{768}{l^2} \right) \]
\[ = 3l^2 + \frac{3072}{l} \] ✓ (2 marks)

ii \[ A = 3l^2 + 3072l^{-1} \]
\[ \frac{dA}{dl} = 6l - 3072l^{-2} \]
\[ = 6l - \frac{3072}{l^2} \]
✓

Maximum or minimum occurs when \( \frac{dA}{dl} = 0 \)
i.e. \( 6l - \frac{3072}{l^2} = 0 \)
\[ 6l = \frac{3072}{l^2} \]
\[ l^3 = 3072 \]
\[ l = 16 \]
✓

\[ \frac{d^2A}{dl^2} = 6 + 6144l^{-3} \]
\[ \frac{d^2A}{dl^2} > 0 \text{ when } l = 8 \]
The minimum area occurs when \( l = 8 \).

When \( l = 8, \quad h = \frac{768}{8^2} \]
\[ = 12 \]
∴ a box that is 8 cm long and 12 cm high will use the minimum amount of cardboard ✓ (3 marks)

b i Whenever had a population of 3000 in 2001 and is growing at 2% per year.
\[ \text{Why not had a population of 5000 in 2001, but its population is decreasing at 3% per year. } \]
✓ (2 marks)

ii In the year 2020, \( t = 19 \)
\[ P = 3000e^{0.02(19)} \]
When \( t = 19 \),
\[ P = 3000e^{0.02(19)} \]
\[ = 4386.853768... \]
\[ = 4400 \text{ (near. hundred)} \]
✓

Q = 5000e^{-0.03(19)}
When \( t = 19 \),
\[ Q = 5000e^{-0.03(19)} \]
\[ = 2827.627194... \]
\[ = 2800 \text{ (near. hundred)} \]
✓

∴ Whenever will have the larger population in 2020 ✓ (2 marks)

iii Populations will be the same when:
\[ 3000e^{0.02t} = 5000e^{-0.03t} \]
\[ 3000 \]
\[ 5000 \]
\[ = \frac{e^{0.02t}}{e^{-0.03t}} \]
\[ = 0.6 = e^{-0.05t} \]
\[ \ln(0.6) = -0.05t \]
\[ t = 10.2165... \]
✓

During the eleventh year the populations will be equal.
In the year 2012 the population of Whenever will first exceed that of Whynot. ✓ (3 marks)

c i Last amount invested earns interest for 1 year:
\[ A_1 = $5000(1.065) \]
First investment earns interest for 30 years:
\[ A_{30} = $5000(1.065)^{30} \]
Total investment
\[ = [5000(1.065) + 5000(1.065)^2 + ... + 5000(1.065)^{30}] \]
✓

Geometric series:
\[ a = 5000(1.065), r = 1.065, n = 30 \]
\[ S_n = \frac{a(r^n - 1)}{r - 1} \]
\[ S_{30} = \frac{5000(1.065)(1.065^{30} - 1)}{1.065 - 1} \]
\[ = 459946.151... \]
The total investment is worth
\[ $459946, \text{ to the nearest dollar. } \]
✓ (3 marks)

Total for Q15: 15 marks
i. In $\triangle ABC$,
\[ b^2 = a^2 + c^2 - 2ac \cos B \]
\[ x^2 = 2^2 + (\sqrt{3})^2 \]
\[ -2 \times 2 \times \sqrt{3} \times \cos 120^\circ \]
\[ x^2 = 4 + 3 - 4\sqrt{3} \times \left( -\frac{1}{2} \right) \]
\[ x^2 = 7 + 2\sqrt{3} \quad \checkmark (1 \text{ mark}) \]

ii. $DC^2 = BD^2 + BC^2$
\[ = 2^2 + 2^2 \]
\[ = 8 \]
$DC = \sqrt{8}$ (DC > 0)
$2\sqrt{2}$ \quad \checkmark \]

In $\triangle ADC$,
\[ d^2 = a^2 + c^2 - 2ac \cos D \]
\[ x^2 = 2(\sqrt{2})^2 + \sqrt{2} \]
\[ -2 \times 2\sqrt{2} \times 1 \times \cos 105^\circ \]
\[ x^2 = 8 + 1 - 4\sqrt{2} \cos 105^\circ \]
But $\cos 105^\circ = \cos (180^\circ - 75^\circ)$
\[ = -\cos 75^\circ \]
\[ \therefore x^2 = 9 + 4\sqrt{2} \cos 75^\circ \quad (2 \text{ marks}) \]

iii. $x^2 = 7 + 2\sqrt{3}$ and
\[ x^2 = 9 + 4\sqrt{2} \cos 75^\circ \]
\[ \therefore 9 + 4\sqrt{2} \cos 75^\circ = 7 + 2\sqrt{3} \]
\[ 4\sqrt{2} \cos 75^\circ = -2 + 2\sqrt{3} \]
\[ \cos 75^\circ = \frac{-2 + 2\sqrt{3}}{4\sqrt{2}} \quad \checkmark \]
\[ = \frac{-1 + \sqrt{3} \times \sqrt{2}}{2\sqrt{2}} \]
\[ = \frac{\sqrt{6} - \sqrt{2}}{4} \]
\[ (2 \text{ marks}) \]

b. i. $y = 2xe^{-x}$
Curve crosses y-axis when $x = 0$
\[ y = 2 \times 0 \times e^{-0} \]
\[ = 0 \]
The curve cuts the y-axis at the origin. \quad \checkmark (1 \text{ mark}) \]

ii. \[ \frac{dy}{dx} = 2x \times \frac{d}{dx} (e^{-x}) \]
\[ + e^{-x} \times \frac{d}{dx} (2x) \]
\[ = 2x \times -e^{-x} + e^{-x} \times 2 \quad \checkmark \]
\[ = 2e^{-x} (x + 1) \]
\[ = 2e^{-x} (1-x) \quad \checkmark (2 \text{ marks}) \]

iii. \[ \frac{d^2y}{dx^2} = 2e^{-x} \times \frac{d}{dx} (1-x) \]
\[ + (1-x) \times \frac{d}{dx} (2e^{-x}) \quad \checkmark \]
\[ = 2e^{-x} \times -1 \]
\[ + (1-x) \times -2e^{-x} \]
\[ = -2e^{-x} (1 + 1 - x) \]
\[ = -2e^{-x} (2 - x) \quad \checkmark (2 \text{ marks}) \]

iv. At turning point $\frac{dy}{dx} = 0$
i.e. $2e^{-x} (1-x) = 0$
$2e^{-x} = 0$ or $1 - x = 0$
no solution or $x = 1$
When $x = 1$,
\[ \frac{d^2y}{dx^2} = -2e^{-1} (2 - 1) < 0 \]
\[ \therefore$ there is a maximum when $x = 1$ \quad \checkmark \]
When $x = 1$,
\[ y = 2 \times 1 \times e^{-1} \]
\[ = 2e^{-1} \]
\[ \therefore$ turning point is a maximum at $(1, 2e^{-1})$ \quad \checkmark (2 \text{ marks}) \]

v. If the curve changes concavity
\[ \frac{d^2y}{dx^2} = 0 \]
i.e. $-2e^{-x} (2 - x) = 0$
$2e^{-x} = 0$ or $2 - x = 0$
no solution or $x = 2$
If $x < 2$, $\frac{d^2y}{dx^2} < 0$
If $x > 2$, $\frac{d^2y}{dx^2} > 0$
\[ \therefore$ the curve changes concavity when $x = 2$ \]
When $x = 2$,
\[ y = 2 \times 2 \times e^{-2} \]
\[ = 4e^{-2} \]
The curve changes concavity at $(2, 4e^{-2})$. \quad \checkmark (1 \text{ mark}) \]

vi. As $x \to \infty$, $y \to 0^+$
As $x \to -\infty$, $y \to -\infty$ \quad \checkmark (1 \text{ mark}) \]

vii. \[ y = 2xe^{-x} \quad \checkmark (1 \text{ mark}) \]