COMPLEX NUMBERS

EXERCISE 1

1. Solve for \( z \), expressing answers in the form \( a + ib \).
   (a) \( (1 + i)z = 2 - i \)
   (b) \( \frac{2z}{2 + i} + 3 - 2i = (1 - i)z \)
   (c) \( \frac{z}{z - 1} = 1 + i + \frac{3}{1 - i} \)
   (d) \( \frac{z + 3}{z - 1} = 2 - 3i \)

2. Find the quadratic equation with roots
   (a) \( i, -i \)  (b) \( 1 + i, 1 - i \)  (c) \( 2 + 3i, 2 - 3i \)  (d) \( 3 + i, 1 + 3i \)
   (e) \( 2 + 2i \)

3. Solve for \( z \) and \( w \)
   (a) \( z + iw = 2 + 3i \)
   (b) \( 2x + w = 1 + i \)
   (c) \( (2 + i)z + (2 - i)w = 1 \)
   (d) \( z + (1 - i)w = 2i \)
   (e) \( w + (1 - i)z = 1 \)

4. Show that \( x = i \) is a root of the equation \( x^3 + (1 - i)x^2 + (1 - 2i)x = 1 + i \)

5. If \( x = 1 + i \) is a root of \( x^3 + ax + 4 = 0 \), show that \( a = -2 \)

6. If \( \sqrt{x} + iy = a + ib \) where \( x, y, a, b \) are real and \( a > 0 \), prove that \( a^2 - b^2 = x \) and \( 2ab = y \).
   Hence express the square root of the following in the form \( a + ib \)
   (a) \( 5 + 2i \)  (b) \( 21 - 20i \)  (c) \( i \)  (d) \( -11 - 60i \)
EXERCISE 2

1. If \( \omega \) is a complex cube root of unity (ie a root of \( z^3 = 1 \)), prove that \( \omega^2 \) is also a complex cube root of unity. Further prove that:
   (a) \( 1 + \omega + \omega^2 = 0 \)
   (b) \( \frac{1}{1+\omega} + \frac{1}{1+\omega^2} = 1 \)
   (c) \( (1 + \omega)^3 = -1 \)
   (d) \( (1 + \omega^2)^5 = -\omega^2 \)

2. \( \omega \) is a complex root of the equation \( z^3 - 1 = 0 \). Form a quadratic equation whose roots are given by \( \alpha = 2 + \omega \) and \( \beta = 2 + \omega^2 \).

3. If \( \omega \) is the complex cube root of unity, show that
   (a) \( (1 + \omega - \omega^2) - (1 - \omega + \omega^2)^2 = 0 \)
   (b) \( \frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} = \omega \)
   (c) \( \frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = \omega \)

4. If \( x = a + b, y = a\omega + b\omega^2, z = a\omega^2 + b\omega \), where 1, \( \omega \), \( \omega^2 \) are the other roots of unity, prove that
   (a) \( x + y + z = 0 \)
   (b) \( (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^2 + b^2 + c^2 - ab - bc - ca \)

5. If \( 1, \omega, \omega^2 \) are the three cube roots of unity, prove that
   \( (a + b + c)(a - b\omega + c\omega^2)(a - b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc \)

6. If \( \omega \) is complex root of \( z^5 - 1 = 0 \), show that \( \omega^2, \omega^3, \omega^4 \) are the other complex roots.
   (a) Prove that \( 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0 \)
   (b) Find the quadratic equation whose roots are \( \alpha = \omega + \omega^4 \) and \( \beta = \omega^2 + \omega^3 \)
   (c) Show the roots of \( z^5 - 1 = 0 \) on an Argand diagram
   (d) Find the area of the pentagon formed by the roots (to 2 dec. Pl)

7. If \( \omega \) is a complex root of \( z^6 - 1 = 0 \) then show that the other roots are \( \omega^2, \omega^3, \omega^4, \omega^5 \). Prove that
   (a) \( 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0 \)
   (b) Find all the roots in the form \( a + ib \) and indicate these roots in an Argand diagram.
   Find the area of the hexagon formed by the roots.
   (c) Find the quadratic equation whose roots are
      (i) \( \omega \) and \( \omega^5 \)  (ii) \( \omega^2 \) and \( \omega^4 \)
   (d) Show that
      (i) \( z^6 - 1 = (z^2 - 1)(z^2 + z + 1)(z^2 - z + 1) \)
      \( = (z - 1)(z + 1)(z - \omega)(z - \omega^2)(z - \omega^4)(z - \omega^5)(z - \omega^5) \)
      (ii) The roots of \( z^4 + z^2 + 1 = 0 \) are \( \omega, \omega^2, \omega^4 \) and \( \omega^5 \).
EXERCISE 3

1. If \( z_1 = 2 + 3i \), \( z_2 = -1 + 4i \), show on separate Argand diagrams
   (a) \( z_1 \)
   (b) \( z_2 \)
   (c) \( z_1 + z_2 \)
   (d) \( z_1 - z_2 \)
   (e) \( z_2 - z_1 \)
   (f) \( z_1z_2 \)
   (g) \( iz_1 \)
   (h) \( iz_2 \)

2. Show on separate Argand diagrams the points representing
   (a) \( 2 - i \)
   (b) \( 3 + 4i \)
   (c) \( (2 - i) + (3 + 4i) \)
   (d) \( (2 - i) - (3 + 4i) \)
   (e) \( (2 - i)(3 + 4i) \)
   (f) \( i(2 - i) \)
   (g) \( i(3 + 4i) \)

3. Verify the triangle inequalities, \( |z_1 + z_2| \leq |z_1| + |z_2| \) and \( |z_1 - z_2| \geq |z_1| - |z_2| \) when
   (a) \( z_1 = 2 + 3i \), \( z_2 = -1 + 4i \)
   (b) \( z_1 = 2 - i \), \( z_2 = 3 + 4i \)
EXERCISE 4

1. If P represents the complex number \( z \), sketch the locus of P if
   (a) \( |z| = 4 \)
   (b) \( |z| \leq 4 \)
   (c) \( |z - 3| < 3 \)
   (d) \( |z + 3i| < 1 \)
   (e) \( |2z - 3| = 1 \)
   (f) \( |z - 1 - 2i| = 4 \)
   (g) \( \arg z = \frac{\pi}{2} \)
   (h) \( \arg z = -\frac{\pi}{3} \)
   (i) \( \text{Re}(z) = 2 \)
   (j) \( \text{Im}(z) = -2 \)
   (k) \( 1 < |z| < 2 \)
   (l) \( 3 < |z| \leq 4 \)
   (m) \( 2 \leq |z| \leq 5 \)
   (n) \( 1 \leq |z + 2| \leq 2 \)
   (o) \( 2 \leq \text{Im}(z) < 3 \)
   (p) \( 2 < \text{Re}(z) \leq 3 \)
   (q) \( 0 < \arg z < \frac{\pi}{6} \)
   (r) \( \frac{\pi}{2} < \arg z < \frac{2\pi}{3} \)
   (s) \( \frac{1}{2} < \frac{|z|}{2} \leq \frac{1}{9} \)
   (t) \( 1 < |z^2 - 1 + i| < 2 \)
   (u) \( \text{Re}(z^2) = 0 \)
   (v) \( \text{Im}(z^2) = 2 \)
   (w) \( \text{Re}(z) = |z - 1| \)
   (x) \( 0 < \text{Re}(z) \leq 2 \)
   (y) \( \text{Re}(z - iz) \geq 2 \)

2. Mark clearly on an Argand diagram the regions of the z plane satisfied by
   (a) \( \text{Re}(z) \geq 1 \) and \( 1 \leq \text{Im}(z) \leq 2 \)
   (b) \( 3 < |z| \) and \( \frac{\pi}{4} < \arg z \leq \pi \)
   (c) \( |z| \leq 3 \) and \( \text{Im}(z) > 1 \)
   (d) \( 2 < |z| \leq 3 \) and \( \text{Im}(z) > 1 \)
   (e) \( \text{Im}(z) \geq 1 \) and \( 0 \leq \arg z \leq \frac{\pi}{4} \)
   (f) \( 1 \leq \text{Re}(z) \leq 2 \) and \( 2 \leq \text{Im}(z) \leq 3 \)
   (g) \( 1 < |z + i| < 2 \) and \( \pi < \arg z < \frac{5\pi}{2} \)
   (h) \( 4 \leq \text{Im}(z) \leq 5 \) and \( |z| \geq 5 \)
   (i) \( |2z - 3| < 2 \) and \( \frac{\pi}{6} < \arg z < \frac{\pi}{2} \)
EXERCISE 5

1. Find the Cartesian equation of the following curves, and sketch and describe them
   (a) \(|z - 2| = |z + i|
   (b) \(|z + 2 - 3i| = |z + 2 + i|
   (c) \(|z - 2i| = 2|z + 1|
   (d) \(|z + 2 - 3i| = 2|z + 2 + i|

2. For the following, describe the locus of the complex number \(w\), where \(z\) is restricted as indicated
   (a) \(w = z - 2, |z| = 3\)
   (b) \(w = \frac{z - 2}{z}, |z| = 1\)
   (c) \(w = \frac{z - 2i}{z}, |z| = 2\)
   (d) \(w = \frac{z - 2 + i}{z + 2 - i}, |z| = 1\)

3. Find the locus of \(z\) if
   (a) \(w = \frac{z - i}{z}\) and \(w\) is purely real
   (b) \(w = \frac{z - i}{z^2}\) and \(w\) is purely imaginary
   (c) \(w = \frac{z - 2}{z + 2}\) and \(\arg w = \frac{\pi}{3}\)

4. Sketch on an Argand diagram the locus of the point \(P\) representing \(z\), given that \(|z|^2 = z + \bar{z} + 1\).

5. \(|z + i| \leq 2\) and \(0 \leq \arg(z + 1) \leq \frac{\pi}{4}\). Sketch the region in the Argand diagram which contains the point \(P\) representing \(z\).

6. \(|z - 1| \leq |z - i|\) and \(|z - 2 - 2i| \leq 1\). Sketch the region in the Argand diagram which contains the point \(P\) representing \(z\). If \(P\) describes the boundary of this region, find the value of \(z\) when \(\arg(z - 1) = \frac{\pi}{4}\).

7. \(|z - 1| = 1\). Sketch the locus of the point \(P\) representing \(z\) on an Argand diagram. Hence deduce that \(\arg(z - 1) = \arg(z^3)\).

8. \(\arg(z + 3) = \frac{\pi}{3}\). Sketch the locus of the point \(P\) representing \(z\) on an Argand diagram. Find the modulus and argument of \(z\) when \(|z|\) takes its least value. Hence find, in the form \(a + ib\), the value of \(z\) for which \(|z|\) is a minimum.

9. \(z = x + iy\) is such that \(\frac{z - i}{z + 1}\) is purely imaginary. Find the equation of the locus of the point \(P\) representing \(z\) and show this locus on an Argand diagram.

10. \(\text{Re}\left(\frac{z - 1}{z}\right) = 0\). Find the equation of the locus of the point \(P\) representing \(z\) on an Argand diagram and sketch this locus.

11. Find the locus of \(z\) if
    (a) \(\left|\frac{z - i}{z + 2}\right| = 1\)
    (b) \(\arg\left(\frac{z - i}{z + 2}\right) = \frac{\pi}{2}\)
    (c) \(\left|\frac{z - 2}{z + 2}\right| \leq 1\)
    (d) \(2(z + \bar{z}) - 5i(z - \overline{z}) = 21\)
    (e) \(z\bar{z} - (2 + i)z - (2 - i)\bar{z} \leq 4\)
(f) \( \arg \left( \frac{z-i}{z+2} \right) = 0 \)

(g) \( |z + 3i|^2 + |z - 3i|^2 = 90 \)