GOSFORD HIGH SCHOOL

2010

TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS

Time Allowed - 3 hours

(plus 5 minutes reading time)

All necessary working should be shown.

Full marks may not be awarded for unnecessarily untidy work or work that is poorly organized.

Students must begin each new question on a new page.

Questions will be collected separately at the conclusion of the assessment task.

All questions are to be attempted.

QUESTION 1 (12 marks)

(a) Use shading to represent the region satisfying $x+2y \le 6$ (3)

(b) Solve $x^2 - 3x - 10 = 0$ (2)

(c) Given $x = 3$ is a root of the quadratic equation $kx^2 - 20x + 2(k-14) = 0$

(i) Find the value of k. (2)

(ii) Find the other root. (1)

(d) If $\cos\alpha = 0 \cdot 64$ and $0 \le \alpha \le 2\pi$,
find all possible values of α (give answers correct to 3 significant figures) (2)

(e) Given that $\tan\theta = -\frac{5}{\sqrt{5}}$ and $\sin\theta < 0$ find the **exact value** of $\sec\theta$ (2)

QUESTION 2 (12 marks)

(a) Find $\int \frac{dx}{x^2}$ (1)

(b) Evaluate $\int_{-1}^{1} e^{2x+2}\,dx$ (2)

(c) Find the gradient of the tangent to the curve $y = \frac{x}{x+1}$ at the point $x = 2$ (2)

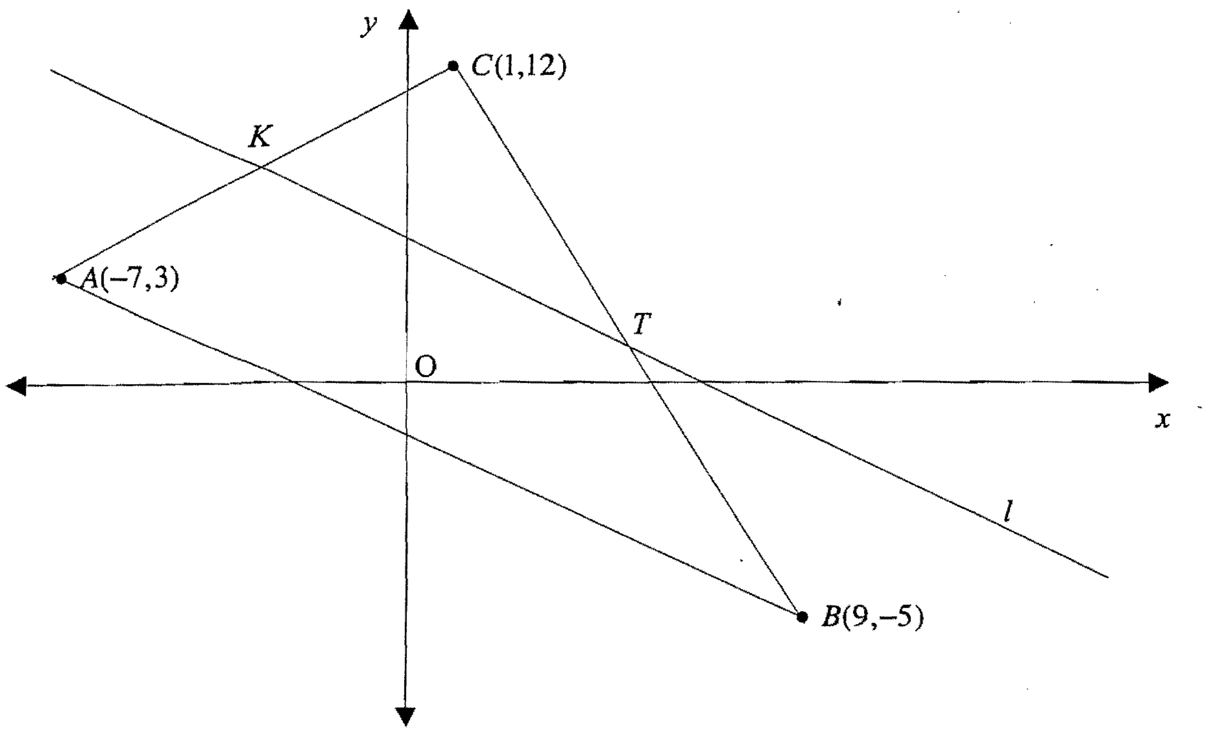
(d) State the period and amplitude of the curve $y = 3\sin(2\theta)$ (2)

(e) Find the equation of the parabola with focus $(6,-2)$ and directrix the line $x = -4$ (3)

(f) For a particular series the sum to n terms S_n is given by $S_n = 2^{n+1} - 2$.

Show that $T_n = 2^n$ (2)

QUESTION 3 (12 marks)



In the diagram A, B, and C represent the points $(-7,3)$, $(9,-5)$ and $(1,12)$. The line l passes through the midpoint K of AC and is parallel to AB. The line l meets BC at T

(i) Find the midpoint of AC (1)

(ii) Find the coordinates of T (1)

iii) Find the gradient of BC (1)

(iv) Find the equation of BC, writing your answer in general form (2)

(v) Find the length of BC (1)

(vi) Find the perpendicular distance from A to BC (2)

(vii) ~~(viii)~~ Find the area of triangle ABC (2)

(viii) (ix) State the area of triangle KTC (2)

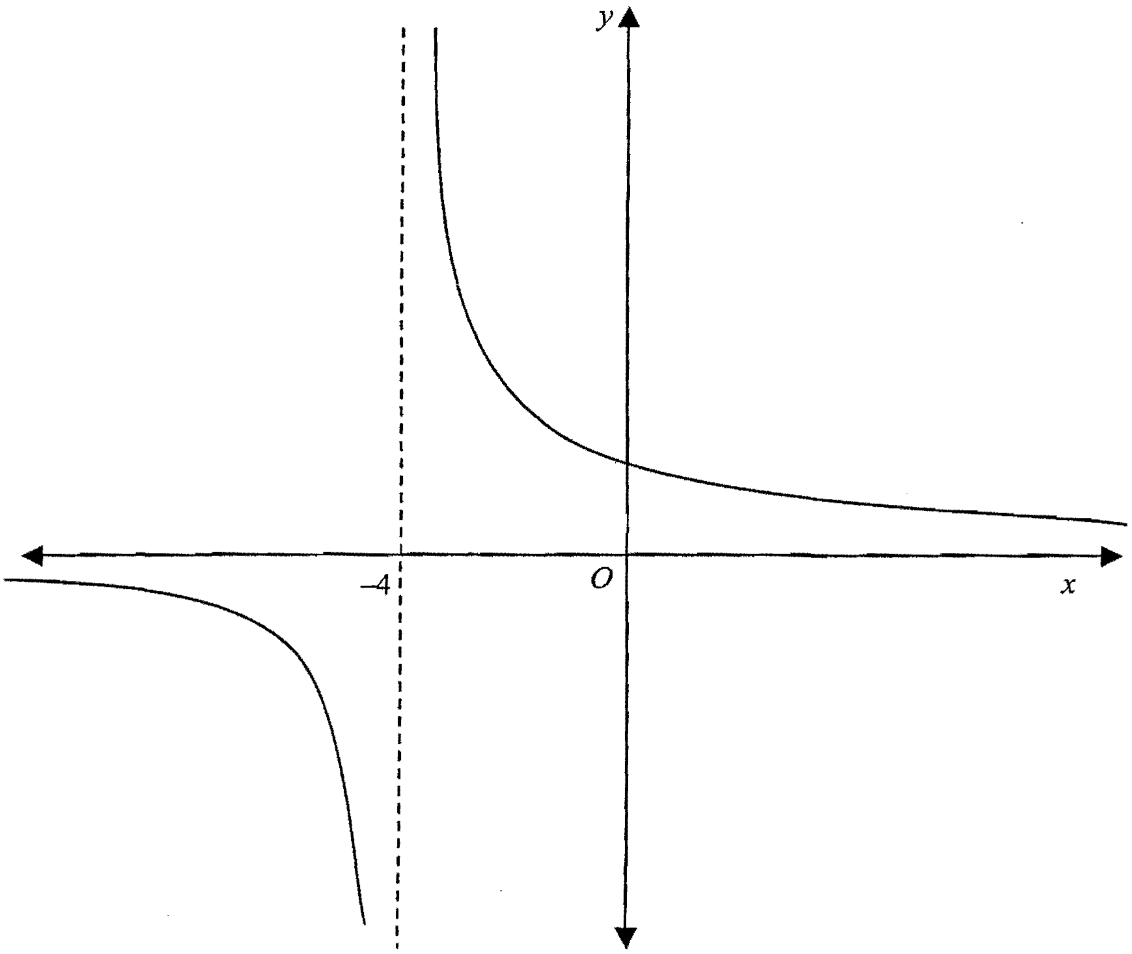
QUESTION 4 (12 marks)

(a) Use Simpson's rule with five function values to calculate $\int_0^4 2^x dx$ (4)

(b) The curve $y = \dfrac{1}{x+4}$ has been drawn on the number plane below.

(i) Show that the line $x + 4y = 0$ is a tangent to $y = \dfrac{1}{x+4}$ (2)

(ii) Calculate the **exact** area of the region enclosed by the curve, the x axis and the ordinates at $x = -2$ and $x = 4$. (3)



(c) Solve $|12 - x| = 3x$ (3)

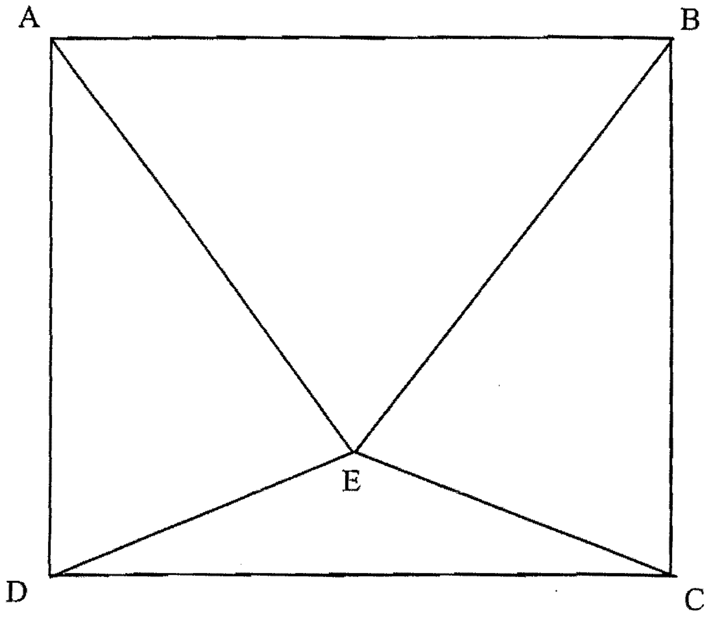
QUESTION 5 (12 marks)

(a) In the diagram below ABCD is a square and triangle ABE is equilateral.

(i) Prove $\Delta ADE \equiv \Delta BCE$. (3)

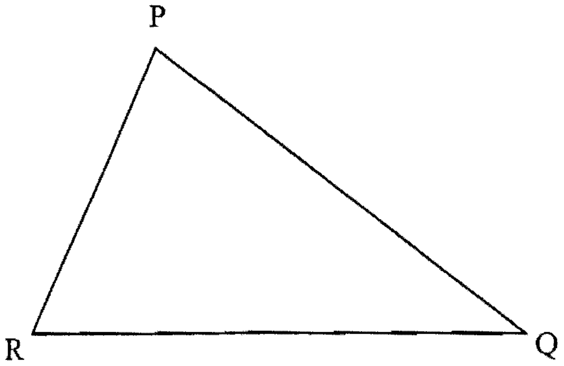
(ii) Hence, or otherwise, state why $\angle AED = \angle BEC$. (1)

(iii) Find the size of $\angle DEC$ giving reasons (2)



NOT DRAWN TO SCALE

(b) In the ΔPQR, $PR = \sqrt{6}$, $PQ = 2\sqrt{3}$ and $RQ = 3+\sqrt{3}$.
Use the Cosine Rule to show, **without use of your calculator**, that $\angle PQR = 30°$ (3)



NOT DRAWN TO SCALE

(c) Find the equation of the normal to the curve $y = \tan\left(\dfrac{x}{2}\right)$ at the point on the curve where $x = \dfrac{\pi}{2}$ (3)

QUESTION 6 (12 marks)

(a) Find $f'(e)$ if $f(x)=\sqrt{3+\ln x}$ (3)

(b) Evaluate $\sum_{n=1}^{\infty}\frac{3}{2^n}$ (3)

(c) Find the values of m for which the quadratic equation $x^2+(m-2)x+4=0$ has no real roots. (3)

(d) A particle moves along the x axis with acceleration at time t given by

$$\frac{d^2x}{dt^2}=12\cos(2t)$$

Find x in terms of t given that when $t=0$, $x=0$ and $\frac{dx}{dt}=0$ (3)

QUESTION 7 (12 marks)

(a) The area of a sector AOB of a circle centre O and radius length 6cm is 27 cm^2.

Find (i) the angle AOB in radians (2)

(ii) the length of the minor arc AB (2)

(b) The fifth term of an Arithmetic sequence is 14 and the sum of the first ten terms is 165.

Find the first term and the common difference of the sequence. (4)

(c) In 1810 Lily, an early settler of Gosford, left a will in which she established a fund of $500 for its future citizens to spend on such things as schools, hospitals etc. Her instructions were that this money was to be invested at 6% p.a. compounded yearly.

(i) If Lily's instructions were followed, how much would have been in the fund 100 years after it was established? (1)

(ii) Suppose that at the beginning of **each subsequent year** after the establishment, a further $500 had been added to the fund and had also earned 6% interest, compounded annually.

Express the amount of money ($$M$) in the fund after 100 years as a Geometric series and hence derive the value of M correct to the nearest dollar. (3)

QUESTION 8 (12 marks)

(a) A tank is emptied by a tap from which the water flows so that, until the flow ceases, the rate of flow after t minutes is R litres/minute where $R = -(t-6)^2$

(i) What is the initial rate of flow? (1)

(ii) How long does it take to empty the tank? (1)

(iii) How long will it take (to the nearest second) for the flow to drop to 20 litres/minute (2)

(iv) How much water was in the tank initially? (2)

(b) After its engine is cut, a biplane travels in a straight line with negative acceleration proportional to its velocity (V) i.e. $\dfrac{dV}{dt} = -kV$

The plane is travelling at 400m/s when the engine is cut and 10 seconds later the velocity is 250m/s.

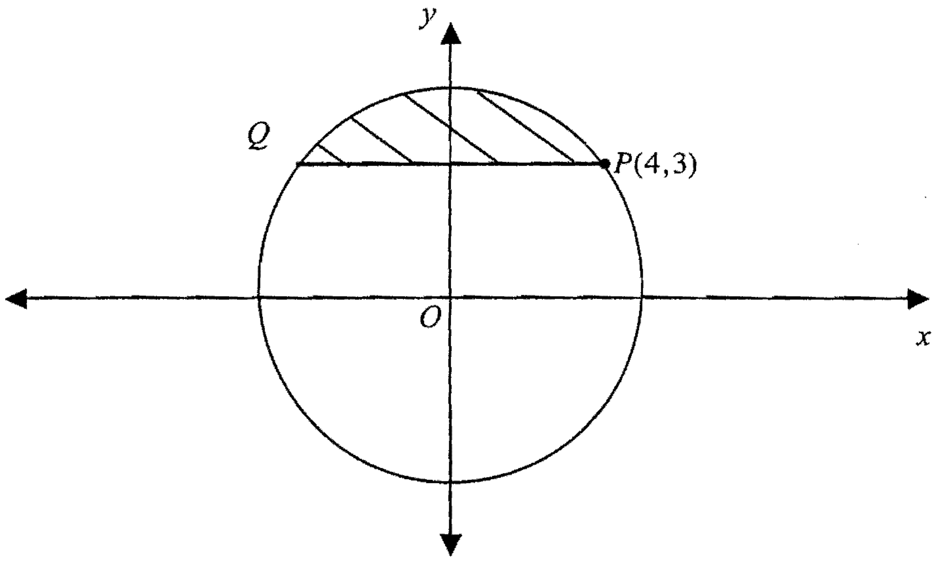
(i) Show that this equation is satisfied by the equation $V = V_0 e^{-kt}$ (1)

(ii) Find the value of V_0 and k (to 3 decimal places) (3)

(iii) Find how long it takes for the velocity to fall to 100m/s (answer to 1 decimal place) (2)

QUESTION 9 (12 marks)

(a) P and Q lie on the circle with centre the origin O.



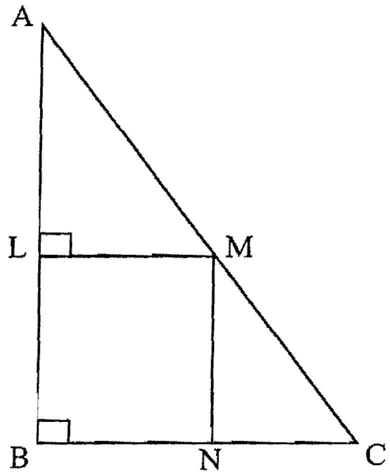
The shaded area, bounded by the circle and the line PQ, is rotated about the y axis to form a spherical cap. Find the volume of the cap. (4)

(b) In the diagram below AB = 25 m and BC = 20 m.

(i) Prove that ΔABC is similar to ΔALM. (2)

(ii) If LB $= x$ m show that the area of the rectangle LBNM is $\dfrac{4x}{5}(25-x)$ square metres (2)

(iii) Hence find the maximum possible area of the rectangle LBNM (2)



(c) If $e^{4x} = 4$ show that $x = \dfrac{1}{2}\ln 2$ (2)

QUESTION 10 (12 marks)

Consider the curve $y = \sin x(1 + \cos x)$ for $0 \le x \le 2\pi$

(i) Show that $\dfrac{dy}{dx} = 2\cos^2 x + \cos x - 1$ (2)

(ii) Show that $\dfrac{d^2y}{dx^2} = -\sin x(4\cos x + 1)$ (2)

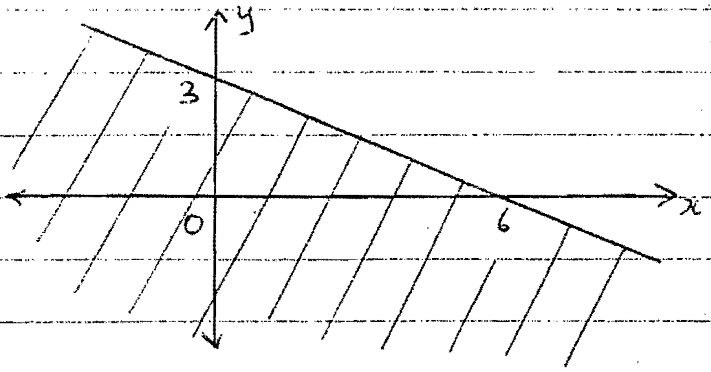
(iii) Find the stationary points on the curve (3)

(iv) Determine the nature of these stationary points. (3)

(v) Sketch the curve in the given domain (2)

Question 1

a)



b) $x^2 - 3x - 10 = 0$

$(x-5)(x+2) = 0$

$x = 5, -2$

c) (i) $k(9) - 20(3) + 2(k-14) = 0$

$9k - 60 + 2k - 28 = 0$

$11k = 88$

$k = 8$

Expression becomes $8x^2 - 20x - 12 = 0$

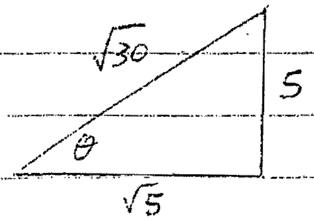
(ii) Let α be the other root

$\alpha + 3 = \frac{20}{8}$ using $\alpha + \beta = \frac{-b}{a}$

$\alpha = -\frac{1}{2}$

d) $\alpha = 0.876$ and 5.41

e) θ lies in 4th Quadrant



$\therefore \sec\theta = \sqrt{6}$

Question 2

a) $\int x^{-2}\,dx = \frac{x^{-1}}{-1} + c$

$= \frac{-1}{x} + c$

b) $\int_{-1}^{1} e^{2x+2}\,dx = \frac{1}{2}\left[e^{2x+2}\right]_{-1}^{1}$

$= \frac{1}{2}\left[e^{4} - 1\right]$

c) $y = \frac{x}{x+1}$, $y' = \frac{(x+1)\cdot 1 - x\cdot 1}{(x+1)^2}$

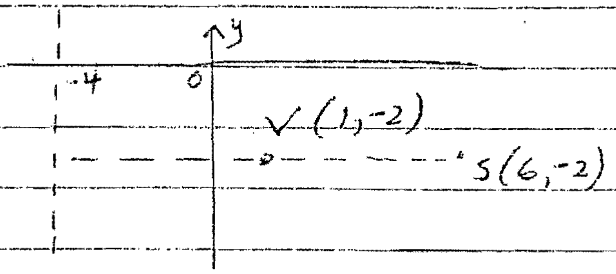
$= \frac{1}{(x+1)^2}$

$= \frac{1}{9}$ at $x = 2$

$\therefore$ Gradient of Tangent is $\frac{1}{9}$

d) Amplitude $= 3$, Period $= \pi$

e) $2a = 10 \rightarrow a = 5$



Form $(y-k)^2 = 4a(x-h)$

ie $(y+2)^2 = 20(x-1)$

f) ~~$T_n = S_n - S_{n-1}$~~

~~$= 2^{n+1} - (2^n$~~

$T_n = S_n - S_{n-1}$

$= 2^{n+1} - 2 - (2^n - 2)$

$= 2^{n+1} - 2^n$

$= 2^n(2-1)$

$= 2^n$

Question 3

(i) $K(-3, 7\frac{1}{2})$

(ii) $T(5, 3\frac{1}{2})$

(iii) $m_{BC} = \dfrac{12+5}{1-9}$

$= \dfrac{17}{-8}$

(iv) Equation of BC is $y + 5 = \frac{17}{-8}(x-9)$

$-8y - 40 = 17x - 153$

$17x + 8y - 113 = 0$

(v) $d_{BC} = \sqrt{(9-1)^2 + (-5-12)^2}$

$= \sqrt{64+289}$

$= \sqrt{353}$

(vi) $d_{\perp} = \left| \dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$

$a = 17$, $b = 8$, $c = -113$

$x_1 = -7$, $y_1 = 3$.

$= \left| \dfrac{-119 + 24 - 113}{\sqrt{353}} \right|$

$= \dfrac{208}{\sqrt{353}}$

vii) Area $= \frac{1}{2} \times \sqrt{353} \times \frac{208}{\sqrt{353}}$

$= 104$ sq units

viii) Area $= \frac{1}{2} \times \left(\frac{1}{2} \times \sqrt{353}\right) \times \left(\frac{1}{2} \times \frac{208}{\sqrt{353}}\right) = 26$ squ

Question 4.

a)

x	0	1	2	3	4
2^x	1	2	4	8	16

$$\int_0^4 2^x\,dx \doteqdot \frac{1}{3}\left[1+16+2(4)+4(2+8)\right]$$

$$\doteqdot \frac{1}{3}\times 65$$

$$= \frac{65}{3} \text{ sq units}$$

b)(i) Solving $x = {}^-4y$ and $y = \frac{1}{x+4}$

$$y = \frac{1}{-4y+4}$$

$$-4y^2+4y = 1$$

$$4y^2-4y+1 = 0$$

$$(2y-1)^2 = 0 \longrightarrow y = \tfrac{1}{2},\ x = -2$$

$\therefore$ Only one pt. of intersection $\therefore$ $x+4y=0$ is a tangent

(ii) Area $= \int_{-2}^{4} \frac{1}{x+4}\,dx.$

$$= \left[\ln(x+4)\right]_{-2}^{4}$$

$$= \ln 8 - \ln 2$$

$$= \ln 4$$

c) $|12 - x| = 3x$

$12 - x = 3x$ or $12 - x = -3x$

$12 = 4x$ $\quad$ $2x = -12$

$x = 3$ $\quad$ $x = -6$

Not a Solution (Does not satisfy)

$\therefore x = 3$ only.

Question 5

$AB = AD = BC$ (sides of square)

$AB = AE = BE$ (sides of equilateral Δ)

(i) In Δ's ADE and BCE

$AD = BC$ (sides of square)

$AE = BE$ (sides of equilateral Δ)

$\angle DAB = \angle ABC = 90^\circ$ ($\angle$'s of a square)

$\angle BAE = \angle ABE = 60^\circ$ ($\angle$'s of equilateral Δ)

$\angle DAE = \angle CBE = 30^\circ$ (adjacent complementary $\angle$'s)

$\therefore \Delta ADE \equiv \Delta BCE$ (S.A.S.)

(ii) $\angle AED = \angle BEC$ (corresponding $\angle$'s of congruent Δ's)

(iii) $\angle AEB = 60^\circ$ ($\angle$. of equilateral Δ)

$AE = AD$ (both equal to AB)

$\angle AED = \angle ADE$ (equal $\angle$'s opposite equal sides of isosceles Δ.)

$\angle AED = \frac{180 - 30}{2}$ ($\angle$. sum of Δ)

$= 75^\circ$

$\therefore \angle DEC = 360^\circ - 2 \times 75 - 60^\circ$ ($\angle$'s at a point)

$= 150^\circ$.

b) $\cos P\hat{Q}R = \dfrac{(2\sqrt{3})^2 + (3+\sqrt{3})^2 - (\sqrt{6})^2}{2 \times 2\sqrt{3} \times (3+\sqrt{3})}$

$= \dfrac{12 + 9 + 6\sqrt{3} + 3 - 6}{12\sqrt{3} + 12}$

$= \dfrac{18 + 6\sqrt{3}}{12\sqrt{3} + 12}$

$= \dfrac{6(3+\sqrt{3})}{4\sqrt{3}(3+\sqrt{3})}$

$= \dfrac{3}{2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{\sqrt{3}}{2}$

$\therefore P\hat{Q}R = 30^\circ.$

c) $y = \tan\left(\frac{x}{2}\right)$

when $x = \frac{\pi}{2}$, $y = 1$.

$\dfrac{dy}{dx} = \dfrac{1}{2}\sec^2\left(\dfrac{x}{2}\right)$

$= \frac{1}{2} \times 2$ at $x = \frac{\pi}{2}$.

$= 1$

Equation of Tangent is $y - 1 = 1\left(x - \frac{\pi}{2}\right)$

$y = x - \frac{\pi}{2} + 1$

Question 6

a) $f(x) = (3 + \ln x)^{\frac{1}{2}}$.

$f'(x) = \frac{1}{2}(3 + \ln x)^{-\frac{1}{2}} \times \frac{1}{x}$.

$f'(e) = \frac{1}{2}(3 + \ln e)^{-\frac{1}{2}} \times \frac{1}{e}$.

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{e}$

$= \frac{1}{4e}$

b) $\sum_{n=1}^{\infty} \frac{3}{2^n} = \frac{3}{2} + \frac{3}{2^2} + \frac{3}{2^3} + \cdots\cdots$

$= \frac{3/2}{1 - \frac{1}{2}}$ using $S = \frac{a}{1-r}$

$= 3$

c) $\Delta < 0$ for no real roots

ie $b^2 - 4ac < 0 \longrightarrow (m-2)^2 - 4(1)(4) < 0$

$m^2 - 4m + 4 - 16 < 0$

$m^2 - 4m - 12 < 0$

$(m-6)(m+2) < 0$

$-2 < m < 6$

d) $\ddot{x} = 12\cos 2t \longrightarrow \dot{x} = 6\sin 2t + c$

when $t = 0$, $\dot{x} = 0$ $\therefore c = 0$

$\therefore \dot{x} = 6\sin 2t$

$x = -3\cos 2t + c$

when $t = 0$, $x = 0$ $\therefore c = 3$

$\therefore x = 3 - 3\cos 2t$

Question 7

a) (i) $A = \frac{1}{2} r^2 \theta$.

$27 = \frac{1}{2} \times 36\,\theta$

$\theta = 1.5$ radians $= \angle AOB$

(ii) $l = r\theta$

$= 6 \times 1.5$

$= 9\text{cm}$

$\therefore$ length of arc is 9cm

b) $T_5 = a + 4d = 14 \longrightarrow a = 14 - 4d$

$S_{10} = 5(2a + 9d) = 165 \longrightarrow 2a + 9d = 33$

$\therefore 2(14 - 4d) + 9d = 33$

$28 - 8d + 9d = 33$

$d = 5 \quad \therefore a = -6$

c) (i) Amount in Fund $= 500(1.06)^{100}$

$= \$169,651.04$

(ii) Total $= 500(1.06)^{100} + 500(1.06)^{99} + \cdots + 500(1.06)$

$= 500(1.06)\left[1 + 1.06 + 1.06^2 + \cdots + 1.06^{99}\right]$

$= \dfrac{500(1.06) \times 1\,(1.06^{100} - 1)}{0.06}$.

$= \$2\,988\,335.07$

Question 8

a) i) $R = -(t-6)^2$.

when $t=0$, $R = -36$ l/minute.

ie Water is flowing out at 36 l/minute

(ii) when $R=0$, $0 = -(t-6)^2$

$t = 6$ seconds

(iii) $R = -20$, $-20 = -(t-6)^2$

$20 = (t-6)^2$

$t-6 = \pm\sqrt{20}$

$t = 6 \pm \sqrt{20}$

But $t \le 6$, $\therefore t = 6 - \sqrt{20}$ seconds

(iv) $\frac{dV}{dt} = -(t-6)^2$.

$V = -\frac{(t-6)^3}{3} + c$.

when $t=6$, $V=0 \longrightarrow c = 0$.

$\therefore V = -\frac{(t-6)^3}{3}$

when $t=0$, $V = -\frac{(-6)^3}{3}$

$= 72$ litres.

b) $\frac{dV}{dt} = -kV$

i) $V = V_0 e^{-kt}$

$\frac{dV}{dt} = -k V_0 e^{-kt}$

$= -kV$ as required.

(ii) $400 = V_0 e^0 \rightarrow V_0 = 400$

$\therefore 250 = 400 e^{-10k}$ when $t = 10$

$0.625 = e^{-10k}$

$-10k = \ln 0.625$

$k = \frac{\ln 0.625}{-10} = 0.047$ (to 3d.p)

(iii) $100 = 400 e^{-kt}$

$0.25 = e^{-kt}$

$-kt = \ln 0.25$

$t = \cancel{\frac{\ln 0.25}{-0.047}}$

$\doteqdot 29.5$ seconds.

Question 9

a) Radius of Circle is 5 units

Equation of Circle is $x^2+y^2=25$

ie $x^2 = 25-y^2$.

$$V = \pi\int_3^5 25-y^2\,dy$$

$$= \pi\left[25y - \frac{y^3}{3}\right]_3^5$$

$$= \pi\left[125 - \frac{125}{3} - (75-9)\right]$$

$$= \frac{52\pi}{3} \text{ cubic units}$$

b) (i) In Δ's ABC and ALM

$\hat{A}$ is common

$\angle ALM = \angle ABC$ (given)

$\therefore \Delta ABC \,|||\, \Delta ALM$ (Angle Angle Test)

(ii) $AL = 25-x$

$$\frac{LM}{20} = \frac{25-x}{25}$$

$$LM = \frac{20}{25}(25-x)$$

$$= \frac{4}{5}(25-x)$$

Area of LBNM $= \frac{4x}{5}(25-x)$

(iii) $\frac{4x}{5}.(25-x)$ is a quadratic expression

Maximum value occurs at A.O.S.

ie when $x = 12.5$

$\therefore$ Max. Area $= \frac{4 \times 12.5}{5}(25-12.5)$

$= 125$ sq units

c) $e^{4x} = 4$

$4x = \ln 4$

$x = \frac{1}{4}\ln 4.$

$= \frac{1}{4}\ln 2^2$

$= \frac{1}{2}\ln 2.$

Question 10

i) $y = \sin x\,(1+\cos x)$, $0 \le x \le 2\pi$

$\frac{dy}{dx} = (1+\cos x)\cdot\cos x + \sin x\,(-\sin x)$

$= \cos x + \cos^2 x - \sin^2 x$

$= \cos x + \cos^2 x - (1-\cos^2 x)$

$= 2\cos^2 x + \cos x - 1$

(ii) $\frac{d^2y}{dx^2} = 4\cos x(-\sin x) - \sin x$

$= -\sin x\,(4\cos x + 1)$

iii) For Stationary pts $\frac{dy}{dx} = 0$

$\therefore\ 2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\therefore\ \cos x = \frac{1}{2}$ and -1.

$\therefore\ x = \frac{\pi}{3}, \frac{5\pi}{3}, \pi$

and $y = \frac{3\sqrt{3}}{4}, \frac{-3\sqrt{3}}{4}, 0$

(iv) when $x = \frac{\pi}{3}$, $\frac{d^2y}{dx^2} < 0$

$\therefore$ Maximum Turning Pt at $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{4}\right)$

when $x = \frac{5\pi}{3}$, $\frac{d^2y}{dx^2} > 0$

$\therefore$ Minimum Turning Pt at $\left(\frac{5\pi}{3}, \frac{-3\sqrt{3}}{4}\right)$

When $x = \pi$, $\frac{d^2y}{dx^2} = 0$

$\therefore$ Horizontal Pt. of inflexion at $(\pi, 0)$

* Noting concavity change through $x = \pi$