SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2010

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time – 180 minutes.
- Write using black or blue pen.
  Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** question in a separate answer booklet.

Total Marks - 120 Marks

- Attempt questions 1 – 10
- All questions are of equal value.

Examiner: *E. Choy*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Total marks – 120
Attempt Questions 1 – 10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Evaluate $\dfrac{\sqrt{a^2+b^2}}{c}$, if $a = 1\cdot 23$, $b = 0\cdot 8$ and $c = 4\cdot 81$. **1**
Leave your answer correct to 2 decimal places.

(b) Factorise $3m^2 - 13m + 4$ **1**

(c) If $\dfrac{5}{2+\sqrt{3}} = a + b\sqrt{3}$, for rational a and b, by rationalising the denominator find a and b. **1**

(d) Solve $|\,2x - 1\,| > 5$ and graph the solution on a number line **2**

(e) Solve the following equations simultaneously **2**

$$3x + y = 6$$
$$6x - 2y = -8$$

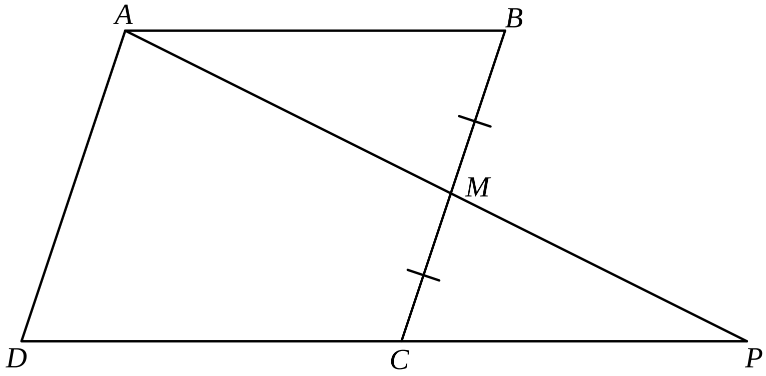
(f) Find a primitive of $5 + \sin x$. **2**

(g) Express $\dfrac{3x-1}{4} - \dfrac{x-2}{3}$ as a single fraction in its simplest form. **1**

(h) Given $\log_a 3 = 0\cdot 6$ and $\log_a 2 = 0\cdot 4$, find $\log_a 18$. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

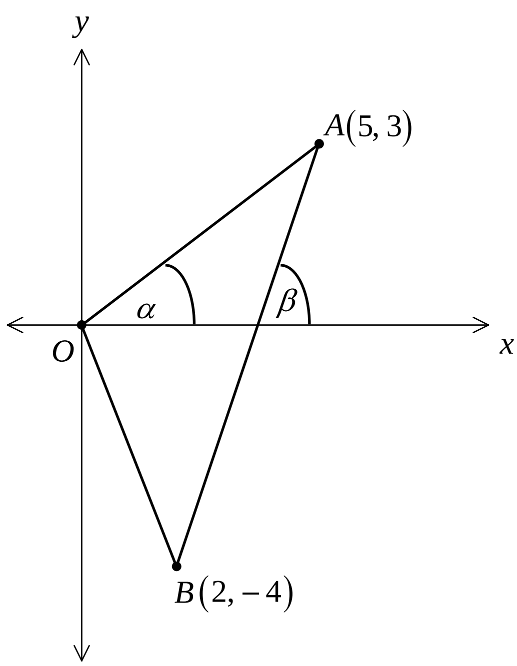
(a) The diagram below shows the parallelogram *ABCD* with *M* the midpoint of *BC*. The intervals *AM* and *DC* are produced to meet at *P*.



(i) Prove that $\Delta ABM \equiv \Delta PCM$ **2**

(ii) Hence prove that *ABPC* is a parallelogram. **2**

(b) The diagram below shows ΔAOB with *A* and *B* the points $(5, 3)$ and $(2, -4)$ respectively.
The angle of inclination of *OA* is α and the angle of inclination of *AB* is β.



(i) Write down the gradients of *OA* and *AB*. **2**

(ii) Hence, find α and β, both correct to the nearest degree. **2**

(iii) Find the length of *OA* **1**

(iv) Find the length of *AB*. **1**

(v) Find the area of ΔAOB. **2**
Give your answer correct to two significant figures.

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Differentiate with respect to x

(i) $\left(e^{x}-2\right)^{5}$ **2**

(ii) $\dfrac{x^{3}}{\tan x}$ **2**

(b) Find $\displaystyle\int \frac{3x}{x^{2}-1}\,dx$ **2**

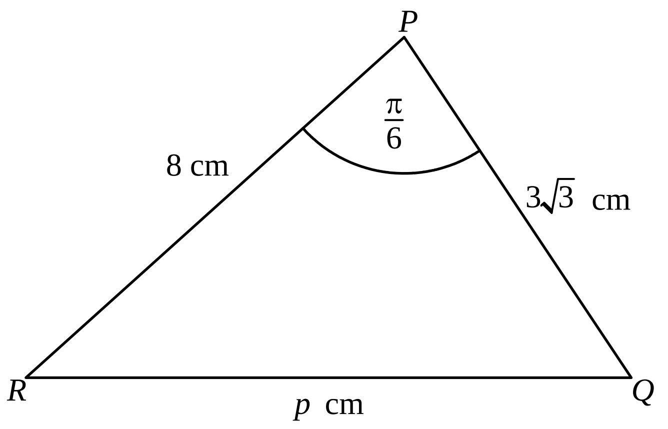
(c) Evaluate $\displaystyle\int_{0}^{2} e^{-x}\,dx$. **2**

Leave your answer in *exact* form.

(d) Solve $\tan x = -\sqrt{3}$ for $0 \le x \le 2\pi$ **1**

(e) Sketch the graph of $x^{2}+y^{2}=7$, showing all intercepts. **1**

(f) In ΔPQR below, $\angle PRQ = \dfrac{\pi}{6}$, PR = 8 cm, $PQ = 3\sqrt{3}$ cm and RQ = p cm.



Find the value of p in exact form. **1**

(g) Given that $\sin\theta = \dfrac{3}{4}$ and $\tan\theta < 0$, find the exact value of $\cos\theta$. **1**

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the equation of the normal to $y = \log_e(3x-2)$ at the point $(1, 0)$ **2**

(b) Consider the quadratic equation $x^2 - kx + k + 3 = 0$, for k real.

(i) Find the discriminant and write it in simplest form. **1**

(ii) For what values of k does the quadratic equation have no real roots. **1**

(iii) If the product of the roots is equal to three times the sum of the roots, find the value of k. **1**

(c) The table below shows the values of the function $f(x)$ for five values of x.

x	4	$4\cdot5$	5	$5\cdot5$	6
$f(x)$	$1\cdot3$	$2\cdot9$	$0\cdot7$	$-0\cdot2$	$-1\cdot1$

Use Simpson's rule with these five function values to find an estimate for **2**

$$\int_4^6 f(x)\,dx.$$

Give your answer correct to one decimal place.

(d) The equation of a parabola is given by $(x-1)^2 = 8y$

(i) Write down the coordinates of the vertex. **1**

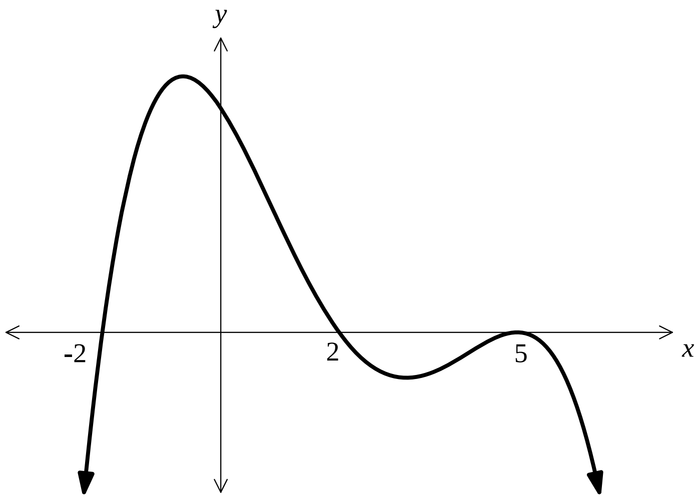
(ii) Write down the focal length **1**

(iii) Sketch the graph of the parabola, clearly showing the focus and directrix. **1**

(e) An infinite geometric series has a limiting sum of 24. **2**
If the first term is 15, find the common ratio.

Question 5 (12 marks)   Use a SEPARATE writing booklet   **Marks**

(a) The graph of $y = f'(x)$ is shown below.

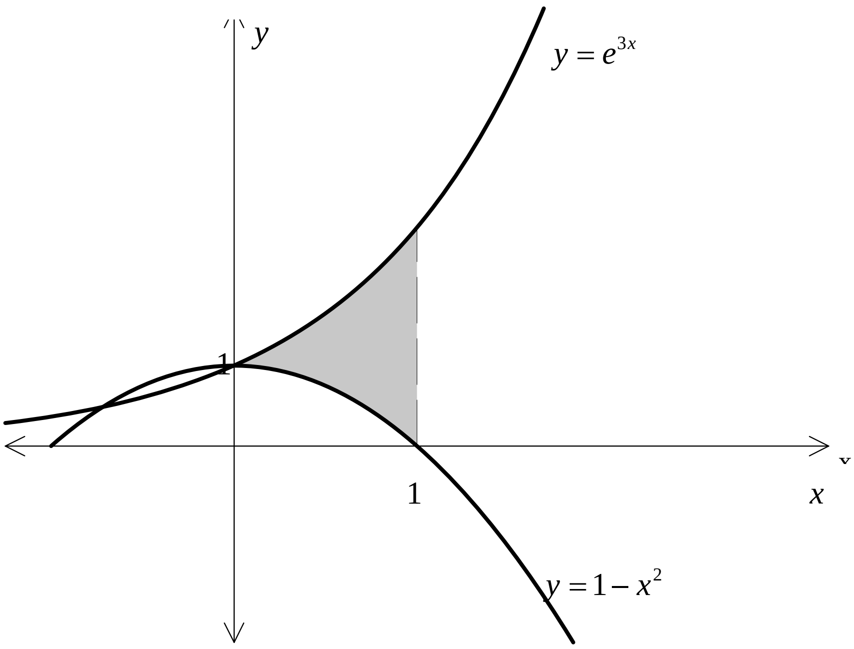


Sketch the graph of $y = f(x)$, given that $f(0) = 0$ and $f(5) = -3$. **3**
Show clearly any turning points or points of inflexion.

(b) Differentiate $\log_e(\cos x)$ and express your answer in simplest form **2**

(c) Solve the equation $1 + \log_2 x = \log_2 \sqrt{x}$ **2**

(d) The diagram below shows the region enclosed between the two curves, $y = e^{3x}$ and $y = 1 - x^3$, and the line $x = 1$.



Find the area of the shaded region. **3**

(e) Sketch the graph of the function $y = 2\tan x$ for $0 \le x \le \frac{\pi}{2}$. **2**
State the range.

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle P is moving in a straight line so that its velocity v metres per second after t seconds is given by $v = 12 - 4t$.

Initially, P is 3 metres to the right of the origin O.

(i) Find the initial velocity and acceleration of P. **1**

(ii) If the displacement of P from O is x metres, find an expression for x in terms of t. **1**

(iii) Find when and where P is at rest. **1**

(iv) Sketch the graph of $v = 12 - 4t$ for $0 \le t \le 5$. **1**

(v) Hence, or otherwise, find the total distance travelled by P during the first 5 seconds. **2**

(b) A function is defined by $f(x) = \frac{x^3}{4}(x - 8)$

(i) Find the coordinates of the stationary points of the graph of $y = f(x)$ and determine their nature. **2**

(ii) Sketch the graph of $y = f(x)$ showing all its essential features including stationary points and intercepts. **3**

(iii) For what values of x is the curve increasing? **1**

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) The function $f(x) = e^x + e^{-x}$ is defined for all real values of x.

(i) Show that $f(x) = e^x + e^{-x}$ is an even function. **1**

(ii) Find the stationary point and its nature. **3**
Hence sketch the graph of $y = f(x)$.

(iii) The region bounded by the curve $y = e^x + e^{-x}$, the x-axis and the line $x = -2$ and $x = 2$ is rotated about the x-axis. **3**
Find the volume of the solid of revolution, correct to one decimal place.

(b) The population N of a certain species at time t is given by $N = N_0 e^{-0 \cdot 03t}$, where t is in days and N_0 is the initial population of the species.

(i) Show that $N = N_0 e^{0 \cdot 03t}$ is a solution of the differential equation **1**

$$\frac{dN}{dt} = -0 \cdot 03N$$

(ii) How long, to the nearest day, will it take for the population to halve? **1**

(iii) Find, in terms of N_0, the rate of change of the population at the time when the population has halved. **1**

(iv) Find the number of days, to the nearest whole number, for the species' population to fall just below 5% of the initial number present **2**

Question 8 (12 marks) Use a SEPARATE writing booklet **Marks**

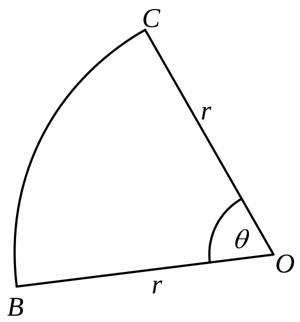
(a) A couple plan to buy a home and they wish to save a deposit of \$40 000 over five years. They agree to invest a fixed amount of money at the beginning of each month during this time. Interest is calculated at 12% per annum compounded monthly.

(i) Let \$*P* be the monthly investment. Show that the total investment \$*A* after five years is given by **2**

$$A = P\left(1\cdot01 + 1\cdot01^2 + ... + 1\cdot01^{60}\right)$$

(ii) Find the amount \$*P* needed to be deposited each month to reach their goal. Answer correct to the nearest dollar. **2**

(b) The diagram below shows a sector *OBC* of a circle with centre *O* and radius *r* cm. The arc *BC* subtends an angle θ radians at *O*.



(i) Show that the perimeter of the sector is $r(2+\theta)$ **2**

(ii) Given that the perimeter of the sector is 36 cm, show that its area is given by **2**

$$A = \frac{648\theta}{(\theta+2)^2}$$

(iii) Hence show that the maximum area of the sector is 81 cm^2 **4**

Question 9 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) An underground storage tank is in the shape of a rectangular prism with a floor area of 12 m^2 and a ceiling height of 2 m.

At 2 p.m. one Sunday, rain water begins to enter the storage tank.
The rate at which the volume V of the water changes over time t hours is given by

$$\frac{dV}{dt} = \frac{24t}{t^2 + 15}$$

where $t = 0$ represents 2 p.m. on Sunday and where V is measured in cubic metres.
The storage tank is initially empty.

(i) Show that the volume of water in the tank at time t is given by **3**

$$V = 12\log_e\left(\frac{t^2+15}{15}\right), \; t \geq 0$$

(ii) Find the time when the tank will be completely filled with water if the water continues to enter the tank at the given rate. **3**
Express your answer to the nearest minute.

(iii) The owners return to the house and manage to simultaneously stop the water entering the tank and start the pump in the tank. **3**
This occurs at 6 p.m. on Sunday.
The rate at which the water is pumped out of the tank is given by

$$\frac{dV}{dt} = \frac{t^2}{k} \text{ where } k \text{ is a constant}$$

At exactly 8 p.m. the tank is emptied of water.
Find the value of k.
Express your answer correct to 4 significant figures.

(b) The captain of the submarine, the HMAS Yddap, spots a freighter on the horizon. He knows that a single torpedo has a probability of $\frac{1}{4}$ of sinking the freighter, $\frac{1}{2}$ of damaging it and $\frac{1}{4}$ of missing it.
He also knows that 2 damaging shots will sink the freighter.
If two torpedoes are fired independently, find the probability of

(i) sinking the freighter with 2 damaging shots; **1**

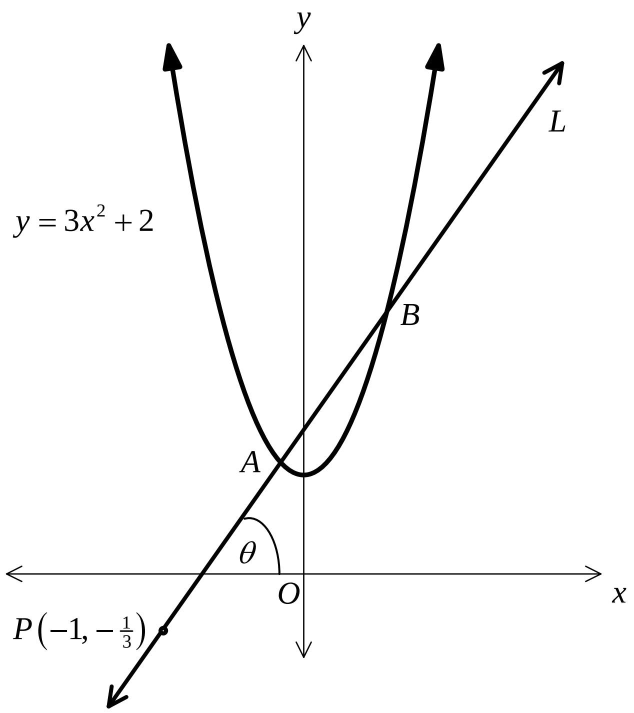
(ii) sinking the freighter. **2**

Question 10 (12 marks) Use a SEPARATE writing booklet **Marks**

Let L be the straight line passing through $P\left(-1, -\frac{1}{3}\right)$ with angle of inclination θ to the x-axis. It is known that the coordinates of any point Q on L are in the form $\left(-1 + r\cos\theta, -\frac{1}{3} + r\sin\theta\right)$, where r is a real number.

(a) Show that $PQ = |\, r \,|$. **2**

(b) In the figure below, L cuts the parabola $y = 3x^2 + 2$ at point A and B.
Let $PA = r_1$ and $PB = r_2$.



(i) By considering the fact that the points A and B lie both on the line L **and** the parabola $y = 3x^2 + 2$, show that r_1 and r_2 are the roots of the equation **2**

$$9r^2\cos^2\theta - 3r\left(\sin\theta + 6\cos\theta\right) + 16 = 0$$

(ii) Using b (i), show that $AB^2 = \dfrac{\left(\sin\theta - 2\cos\theta\right)\left(\sin\theta + 14\cos\theta\right)}{9\cos^4\theta}$. **3**

(iii) Let L_1 be a tangent to the parabola $y = 3x^2 + 2$ from P, with point of contact R. Using the above results, find the two possible slopes of L_1. **2**

(iv) Show that $PR = \frac{4\sqrt{5}}{3}$ when one of the slopes of L_1 has a value of 2. **3**

End of paper

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

RU TRIAL 2010

Question 1

(a) $\dfrac{\sqrt{1.23^2+0.8^2}}{4.81} \doteq 0.31$ 1

(b) $3m^2+3m+4$

$=(3m-1)(m-4)$

$3m \quad -1$

$m \quad -4$ 1

(c) $\dfrac{5}{2+\sqrt{3}} = \dfrac{5}{2+\sqrt{3}} \times \dfrac{2-\sqrt{3}}{2-\sqrt{3}}$

$= \dfrac{10-5\sqrt{3}}{4-3}$

$= 10-5\sqrt{3}$ 1

$\therefore a=10, \; b=-5$

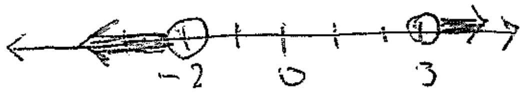
(d) $|2x-1|>5$

$-5>2x-1$ or $2x-1>5$

$0>2x+4 \qquad 2x>6$

$-4>2x \qquad x>3$

$x<-2$ 2



(e) $3x+y=6$ — ①

$6x-2y=-8$ — ②

$2\times①: 6x+2y=12$ — ③

②+③: $12x=4$

$x=\frac{1}{3}$

in ① $1+y=6$

$y=5$ 2

(f) $\int 5+\sin x \, dx = 5x-\cos x+$ 2

(g) $\dfrac{3x-1}{4} - \dfrac{x-2}{3}$

$= \dfrac{3(3x-1)-4(x-2)}{12}$

$= \dfrac{9x-3-4x+8}{12}$

$= \dfrac{5x+5}{12}$ 1

(h) $\log_a 3 = 0.6; \; \log_a 2 = 0.4$

$\log_a 18 = \log_a(3^2 \times 2)$

$= 2\log_a 3 + \log_a 2$

$= 2\times(0.6)+0.4$

$= 1.2+0.4$

$= 1.6$ 2

Q2 (a)

(i) $\angle BAM = \angle MPC$ (alt $\angle$s $AB \parallel DP$)
$\angle AMB = \angle CMP$ (vert. opp.)
$BM = MC$ (given).
$\therefore \Delta ABM \equiv \Delta PCM$ (AAS).

(ii) $AM = MP$ (corresponding sides congruent Δs)
$\therefore$ ABPC is a parallelogram since diagonals bisect each other.

(b) (i) $m_{OA} = \frac{5}{3}$ $\quad m_{AB} = \frac{7}{3}$

(ii) $\alpha = 59°2'$ $\quad \beta = 66°48'$

(iii) $OA = \sqrt{25+9}$
$= \sqrt{34}$.

(iv) $AB^2 = (5-2)^2 + (3+4)^2$

$AB^2 = 9 + 49$
$= 58$
$AB = \sqrt{58}$.

(v) $\angle OAB = \beta - \alpha$.
$= 7°46'$

$Area = \frac{1}{2} \times \sqrt{34} \times \sqrt{58} \times \sin(\angle OAB)$.

$= 3$.

Question 3:

a) i) $(e^x - 2)^5$

$\frac{d}{dx} = 5(e^x - 2)^4 \times e^x$ (1) (1)

$= 5e^x(e^x - 2)^4$

ii) $\frac{x^3}{\tan x}$ — u, — v

$u = x^3$ $\quad v = \tan x$

$u' = 3x^2$ $\quad v' = \sec^2 x$ (1)

$\frac{d}{dx} = \frac{\tan x \times 3x^2 - x^3 \times \sec^2 x}{\tan^2 x}$

$= \frac{3x^2 \tan x - x^3 \sec^2 x}{\tan^2 x}$ (1)

b) $\int \frac{3x}{x^2 - 1}\, dx$

$= \frac{3}{2}\int \frac{2x}{x^2 - 1}\, dx$ (1)

$= \frac{3}{2}\log_e(x^2 - 1) + C$ (1)

c) $\int_0^2 e^{-x}\, dx$

$= \left[-e^{-x}\right]_0^2$ (1)

$= -e^{-2} - -e^0$

$= -e^{-2} + 1$

$= 1 - e^{-2}$ (1)

d) $\tan x = -\sqrt{3}$ $\quad$ S | A / T | C (S and C crossed out)

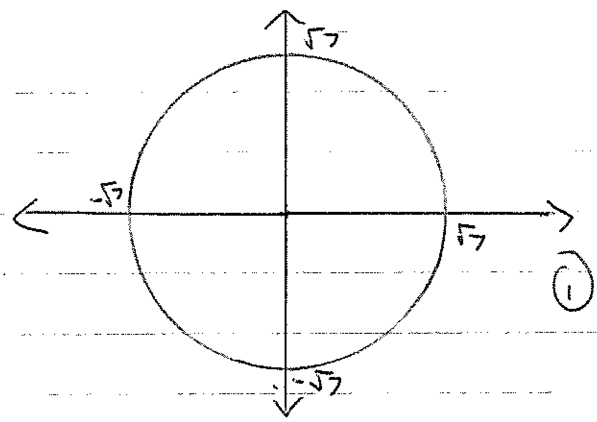
$\therefore x = -\pi/3$

for $0 \le x \le 2\pi$

$x = \frac{2\pi}{3}, \frac{5\pi}{3}$ (1)

e) $x^2 + y^2 = 7$

$\therefore$ circle radius $\sqrt{7}$



(1)

f) $p^2 = 8^2 + (3\sqrt{3})^2 - 2 \times 8 \times 3\sqrt{3} \times \cos \pi/6$

$= 91 - (48\sqrt{3} \times \sqrt{3}/2)$

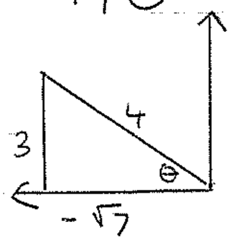
$= 91 - 72$

$= 19$

$p = \sqrt{19}$ cm (1)

g) $\sin\theta = 3/4$ $\quad \tan < 0$

S | A / T | C (S crossed out) $\quad \therefore 90° < \theta < 180°$



$\therefore \cos\theta = -\frac{\sqrt{7}}{4}$.

(1)

2010 MATHEMATICS (2U) TRIAL HSC.

QUESTION FOUR

a) $y = \ln(3x-2)$

gradient tangent =

$$\frac{dy}{dx} = \frac{3}{3x-2}$$

gradient normal =

$$-\frac{dx}{dy} = \frac{2-3x}{3}$$

at $(1,0)$ $\quad -\frac{dx}{dy} = \frac{-3+2}{3} = \frac{-1}{3}$

Eqtn of normal

$$y - y_1 = -\frac{dx}{dy}(x - x_1)$$

$$y - 0 = -\frac{1}{3}(x-1)$$

$$3y = 1 - x$$

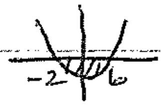
$$x + 3y - 1 = 0.$$

b)(i) $x^2 - kx + k + 3 = 0$

$$\Delta = b^2 - 4ac$$
$$= (-k)^2 - 4(1)(k+3)$$
$$= k^2 - 4k - 12.$$

ii) For no real roots $\Delta < 0$

$$k^2 - 4k - 12 < 0$$
$$(x-6)(x+2) < 0.$$
$$-2 < x < 6$$



b(iii) Let the root be $\alpha + \beta$.

$$\alpha\beta = \frac{c}{a} = k+3.$$

$$\alpha + \beta = -\frac{b}{a} = -(-k) = k.$$

NOW $\quad \alpha\beta = 3(\alpha+\beta).$
$$k + 3 = 3k.$$
$$k = 3/2$$

c) Simpson's Rule.

$$A \doteqdot \frac{h}{3}\left[y_1 + y_{n+1} + 4(y_{odd}) + 2(y_{even})\right]$$

$$h = \frac{b-a}{n} = \frac{6-4}{4} = \frac{1}{2}.$$

$$A \doteqdot \frac{1}{6}\left[1\cdot3 - 1\cdot1 + 2(0\cdot7) + 4(2\cdot9 - 0\cdot2\ldots\right]$$

$$\doteqdot \frac{1}{6}\left[0\cdot2 + 1\cdot4 + 10\cdot8\right]$$
$$\doteqdot 2\cdot1 \text{ units}^2.$$

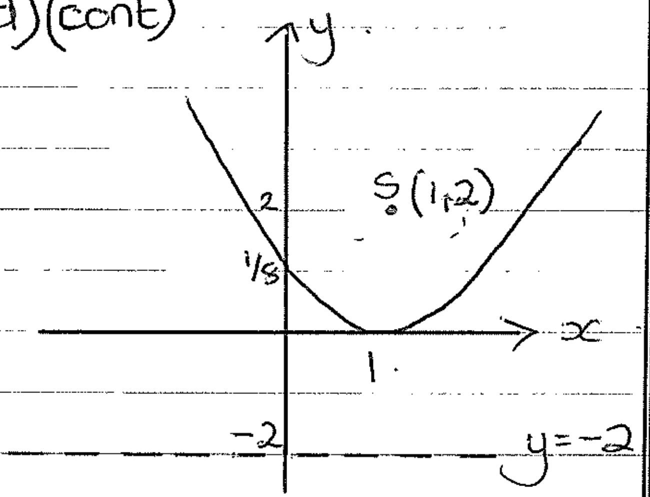
$$\therefore \int_4^6 f(x)\,dx \doteqdot 2\cdot1 \text{ units}^2$$

d) $(x-1)^2 = 8y$
$$(x-1)^2 = 4(2)y.$$

vertex $= (1,0)$.
Focal length $= 2$

Question 4

d) (cont)



e) Limiting sum = 24

$|r|<1$

$$\frac{a}{1-r} = 24$$

$a = 15$.

$$\frac{15}{1-r} = 24.$$

$15 = 24 - 24r$.

$24r = 9$.

$$r = \frac{9}{24} = \frac{3}{8}.$$

Q5

2U

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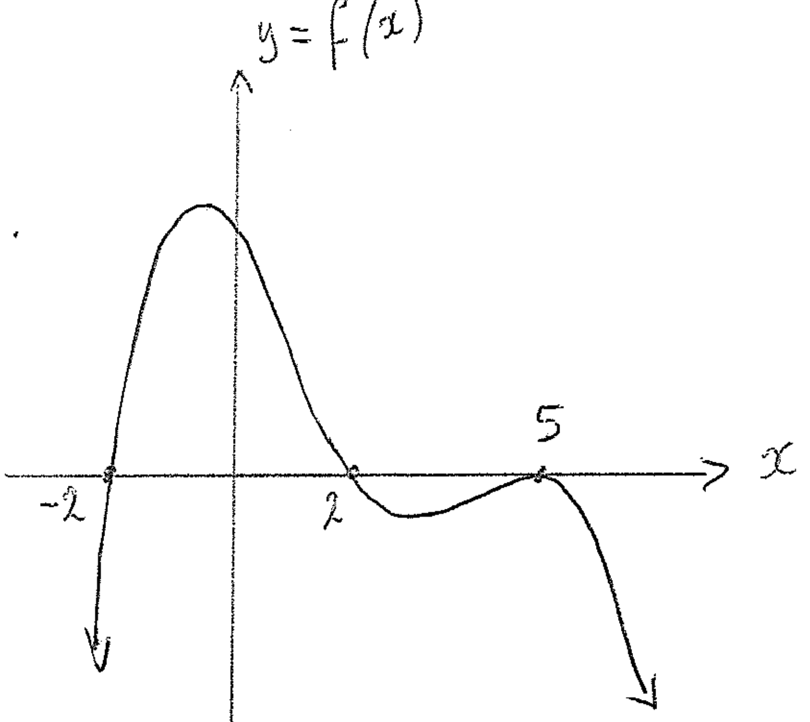
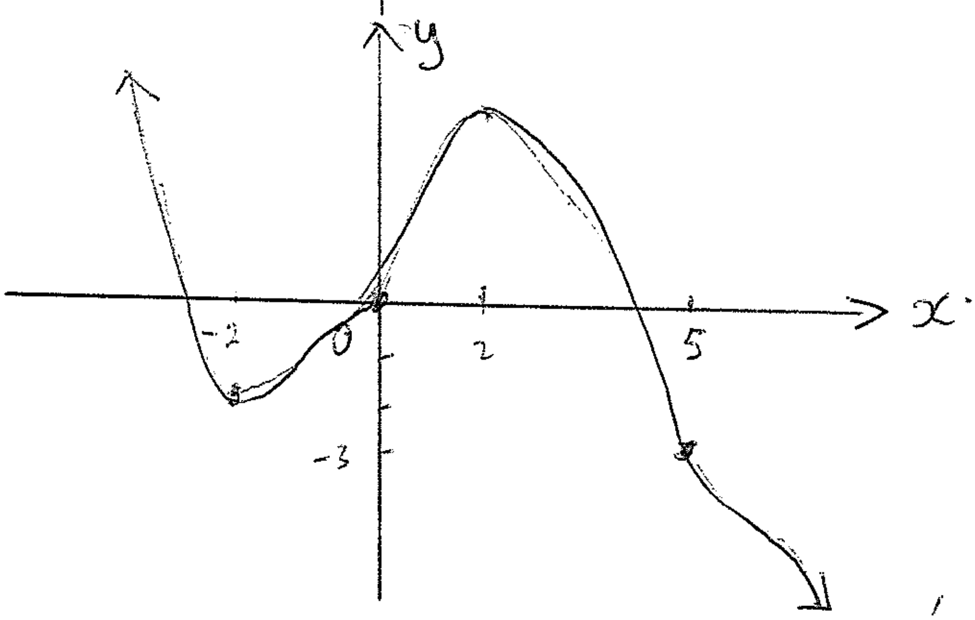
(a)





at $x = -2$, $y' = 0$.

$x = -2 - \varepsilon$ $y' < 0$ } min

$x = -2 + \varepsilon$ $y' > 0$ } t. pt.

at $x = 2$, $y' = 0$

$x = 2 - \varepsilon$ $y' > 0$ } max

$x = 2 + \varepsilon$ $y' < 0$ } t. pt

at $x = 5$, $y' = 0$

$x = 5 - \varepsilon$ $y' < 0$ } inflexion

$x = 5 + \varepsilon$ $y' < 0$ } pt.

Also data $f(0) = 0$

$f(5) = -3$.

(3)

(5) (b) $\frac{d}{dx}(\log_e \cos x) = \frac{1}{\cos x} \times -\sin x$

$= \frac{-\sin x}{\cos x} = -\tan x$

(2)

(c) $1 + \log_2 x = \log_2 \sqrt{x}$

$1 + \log_2 x = \frac{1}{2}\log_2 x$

$1\log_2 x - \frac{1}{2}\log_2 x = -1$

$\frac{1}{2}\log_2 x = -1$

$\log_2 x = -2$

so $2^{-2} = x$

$x = \frac{1}{4}$ (0.25) (2)

(d) $A = \int_0^1 (e^{3x} - (1-x^2))\, dx$

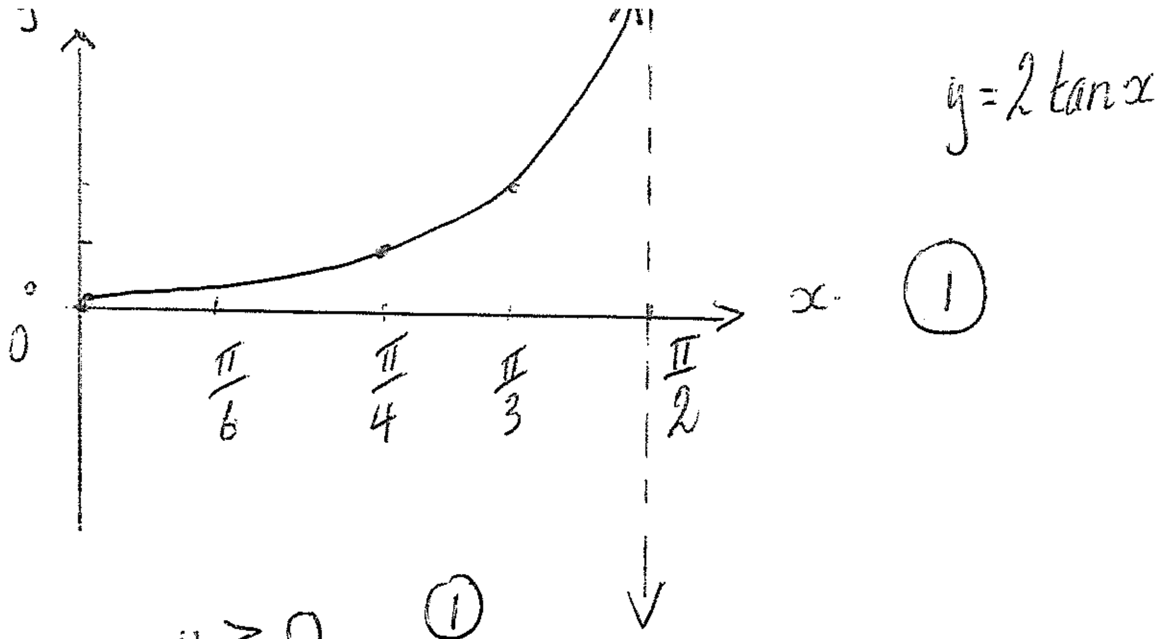
$= \int_0^1 (e^{3x} - 1 + x^2)\, dx$

$= \left. \frac{1}{3}e^{3x} - x + \frac{x^3}{3} \right]_0^1$

$= \left(\frac{1}{3}e^3 - 1 + \frac{1}{3}\right) - \left(\frac{1}{3}e^0 - 0 + \frac{0}{3}\right)$

$= \frac{1}{3}e^3 - 1 + \frac{1}{3} - \frac{1}{3} = \frac{1}{3}e^3 - 1$ units^2. (3)

(e)



(1)

Range. $y \geq 0$ (1)

6(a)

(i) $t = 0$ $\boxed{v = 12}$

$a = \frac{dv}{dt} = \boxed{-4}$

(ii) $x = 12t - 2t^2 + c$ $t=0$ $x=3$

$c = 3$

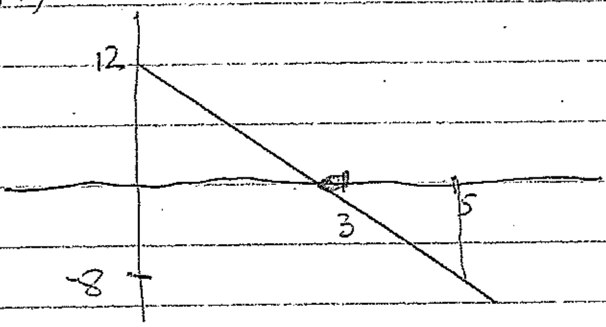
$x = 12t - 2t^2 + 3$

(iii) $v = 0$ $\boxed{t = 3}$

$x = (12)(3) - 2(9) + 3$

$\boxed{x = 21}$

(iv)



$t = 0$ $x = 3$

$t = 3$ $x = 21$ 18 m travelled

$t = 5$ $x = 13$ 8 "

total 26 m trav

or area under vel curve

$\frac{1}{2} \times 12 \times 3 + \frac{1}{2} \times 2 \times 8$

$\boxed{= 26 \text{ m}}$

6(b)

$f(x) = \frac{x^3}{4}(x-8) = \frac{x^4}{4} - 2x^3$

$f'(x) = x^3 - 6x^2 = x^2(x-6)$

St. pts $x = 0$ 6

$y = \quad -108$

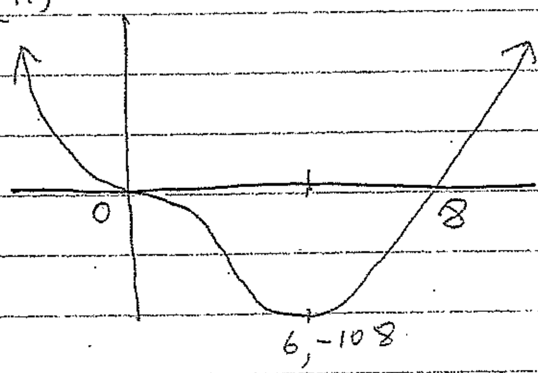
$x = -1$ $f'(x) < 0$ $x = 1$ $f'(x) < 0$

horizontal pt. of inflexion

$x = 5$ $f'(x) < 0$ $x = 7$ $f'(x) > 0$

min. turn. pt

(ii)



slope $1\frac{1}{2}$

pts $1\frac{1}{2}$

(iii) increasing $x > 6$

Question 7

a) i) $f(x) = e^{x} + e^{-x}$

$f(-x) = e^{-x} + e^{-(-x)}$

$= e^{x} + e^{-x}$

since $f(-x) = f(x)$

$f(x)$ is an even function

ii) $f'(x) = e^{x} - e^{-x}$

let $f'(x) = 0$

$e^{x} - \frac{1}{e^{x}} = 0$

$e^{2x} - 1 = 0$

$e^{2x} = 1$

$2x = \ln 1$

$2x = 0$

$x = 0$

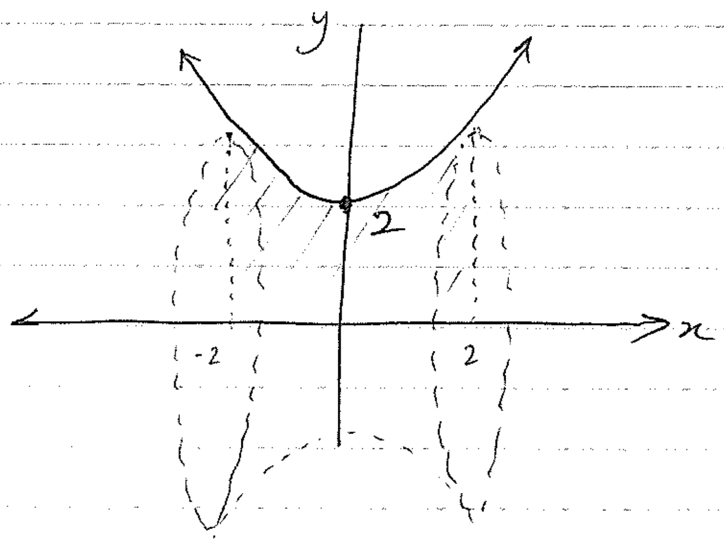
$f''(x) = e^{x} + e^{-x}$

> 0 for all x ↺

$f(0) = e^{0} + e^{-0}$

$= 2$

$\therefore$ Minimum Turning Point at $(0,2)$



iii) $V = \pi \int_a^b y^2 \, dx$

$V = \pi \int_{-2}^{2} (e^x + e^{-x})^2 \, dx$

$V = 2\pi \int_0^2 (e^{2x} + 2 + e^{-2x}) \, dx$ (since even)

$V = 2\pi \left[\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} \right]_0^2$

$V = 2\pi \left[\frac{1}{2} e^{2(2)} + 2(2) - \frac{1}{2} e^{-2(2)} - \left(\frac{1}{2} e^{2(0)} + 2(0) - \frac{1}{2} e^{-2(0)} \right) \right]$

$V = 2\pi \left[\frac{1}{2} e^4 - \frac{1}{2} e^{-4} + 4 \right]$

$V = \pi \left[e^4 - e^{-4} + 8 \right]$

$V \approx 196.6$ units^2

b) i) $N = N_0 e^{-0.03t}$

$\frac{dN}{dt} = -0.03 N_0 e^{-0.03t}$

$\frac{dN}{dt} = -0.03N$

ii) when $N = \frac{N_0}{2}$

$\frac{N_0}{2} = N_0 e^{-0.03t}$

$e^{-0.03t} = \frac{1}{2}$

$-0.03t = \ln\left(\frac{1}{2}\right)$

$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.03}$

$t = 23$ days (to nearest day)

iii) when $N = \frac{N_0}{2}$

$$\frac{dN}{dt} = -0.03 \frac{N_0}{2}$$

$$\frac{dN}{dt} = -0.015 N_0$$

iv) when $N < 0.05 N_0$

$$N_0 e^{-0.03t} < 0.05 N_0$$

$$e^{-0.03t} < 0.05$$

$$\ln e^{-0.03t} < \ln 0.05$$

$$-0.03t < \ln 0.05$$

$$t > \frac{\ln 0.05}{-0.03}$$

$$t > 99.85$$

$t = 100$ days (to nearest whole number)

QUESTION 8. (2U)

(a) (i) $n = 60$ (ie 5×12 months.

$i\% = 0.1$ (interest per. month)

Total $A = P(1.01)^{60} + P(1.01)^{59} + \dots + P(1.01)^{1}$

$= P(1.01 + 1.01^2 + 1.01^3 + \dots + 1.01^{60})$

(ii). $40\,000 = \dfrac{P(1.01(1.01^{60} - 1)}{1.01 - 1.}$

$\therefore P = \dfrac{40\,000 \times 0.01.}{1.01(1.01^{60} - 1)}$

$= \$485.$

(b) (i) $P = r + r + l$ (where $l = r\theta$)

$= r + r + r\theta$

$= 2r + r\theta$

$= r(2 + \theta)$

(ii) $A = \frac{1}{2}r^2\theta$ where $36 = r(2+\theta)$

$= \frac{1}{2} \times \left(\frac{36}{2+\theta}\right)^2 \times \theta$ $\quad \therefore r = \dfrac{36}{2+\theta}$

$= \dfrac{648\theta}{(2+\theta)^2}$

(iii) $A' = \dfrac{(2+\theta)^2 \times 648 - 648\theta \times 2(2+\theta)^1}{(2+\theta)^4}$

$= \dfrac{648(2+\theta) - 1296\theta}{(2+\theta)^3}$

$= \dfrac{1296 - 648\theta}{(2+\theta)^3}$

If $A' = 0$ $\theta = 2$ & $A = \dfrac{648 \times 2}{16}$

$= 81 \text{ cm}^2$

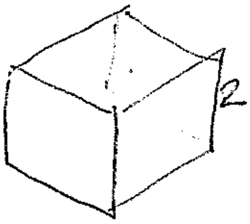
Test

θ	1.5	2	2.5
A'	$\frac{324}{(3.5)^2}$ >0	0	$\frac{-324}{(4.5)^2}$ <0

$\therefore$ MAX.

2u TRIAL 2010

QUESTION 9



(a)

$$\frac{dV}{dt} = \frac{24t}{t^2+15} \qquad V=0 \text{ when } t=0$$

(i) $\int \frac{dV}{dt}\,dt = 12\int \frac{2t}{t^2+15}\,dt + C$

$$V = 12\ln(t^2+15)+C$$

Initially $0 = 12\ln(15) + C$

$$\therefore C = -12\ln(15)$$ [2]

* $\therefore V = 12\ln\left(\frac{t^2+15}{15}\right)$

ii) When full $V = 24$

$$24 = 12\ln\left(\frac{t^2+15}{15}\right)$$

$$2 = \ln\left(\frac{t^2+15}{15}\right)$$

$$\therefore \frac{t^2+15}{15} = e^2$$

$$t^2 = 15e^2 - 15$$

$$t = \sqrt{15e^2-15}$$

$$\doteqdot 9.7896 \text{ hrs}$$

[2½] → $= 9\text{hr } 47' 23''$

Time 11:47 pm [3]

* Differentiate → [1½]

(iii) At 6:00 $t = 4$ (filling)

$$V = 12\ln\left(\frac{16+15}{15}\right)$$

$$\doteqdot 8.7112 \text{ m}^3$$

Emptying

$$\frac{dV}{dt} = \frac{t^2}{k}$$

$$V = \frac{t^3}{3k} + C$$

At 6:00pm, $t=0$, $V=0$ (pumped out)

$\therefore C = 0$

So $V = \frac{t^3}{3k}$

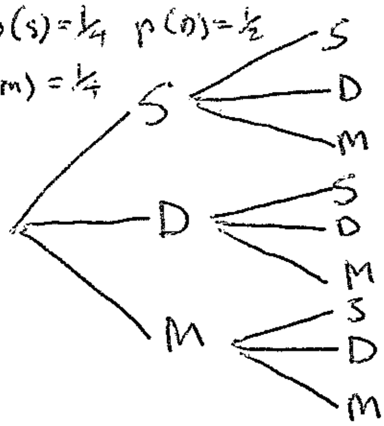
At 8:00pm, $t=2$, $V = 8.7112$ (All pumped out).

$$k = \frac{t^3}{3V}$$

$$= \frac{8}{3V}$$ [3]

$\therefore k = 0.3061$ or -0.3061

(b) $p(S) = \frac{1}{4}$ $p(D) = \frac{1}{2}$ $p(M) = \frac{1}{4}$



$p(SS) = \frac{1}{16}$
$p(SD) = \frac{1}{8}$
$p(SM) = \frac{1}{16}$
$p(DS) = \frac{1}{8}$
$p(DD) = \frac{1}{4}$
$p(DM) = \frac{1}{8}$
$p(MS) = \frac{1}{16}$
$p(MD) = \frac{1}{8}$
$p(MM) = \frac{1}{16}$

(i) $p(DD) = \frac{1}{4}$ [2]

(ii) $p(\text{Sink}) = p(Sx) + p(DS) + p(DD) + p(M\ldots$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{16}$$

$$= \frac{11}{16}$$ [2]

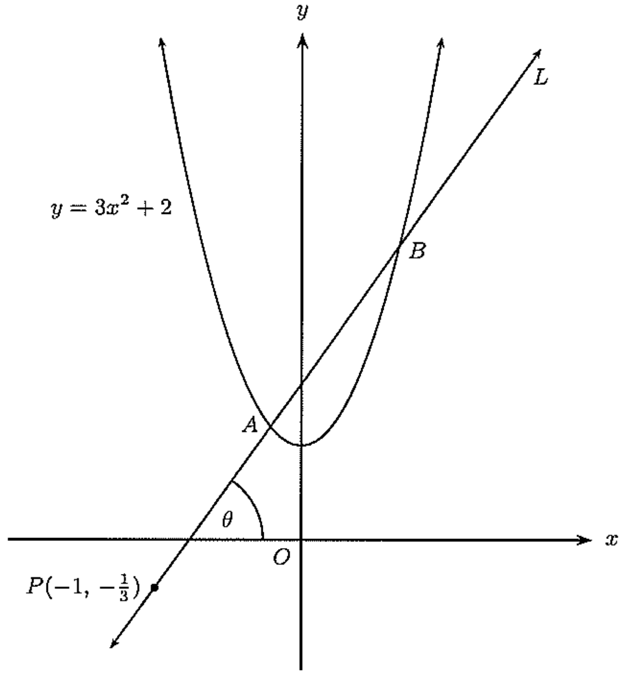
2010 Mathematics Trial HSC: **Question 10** solutions

10. Let L be the straight line passing through $P(-1, -\frac{1}{3})$ with an angle of inclination θ to the x-axis. It is known that the coördinates of any point Q on L are in the form $(-1 + r\cos\theta,\ -\frac{1}{3} + r\sin\theta)$, where r is a real number.

(a) Show that $PQ = |r|$. [1]

Solution:
$$\begin{aligned} PQ &= \sqrt{(-1 + 1 - r\cos\theta)^2 + (-\tfrac{1}{3} + \tfrac{1}{3} - r\sin\theta)^2}, \\ &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}, \\ &= \sqrt{r^2}. \\ \therefore PQ &= |r|. \end{aligned}$$

(b) In the figure below, L cuts the parabola $y = 3x^2 + 2$ at the points A and B. Let $PA = r_1$ and $PB = r_2$.



(i) By considering points A and B **lie both** on the **line** L and the **parabola** $y = 3x^2 + 2$, show that r_1 and r_2 are the roots of the equation [3]

$$9r^2\cos^2\theta - 3r(\sin\theta + 6\cos\theta) + 16 = 0.$$

Solution: Method 1—
$$\begin{aligned} \text{Where } Q \text{ cuts the parabola } y &= 3x^2 + 2: \\ -\tfrac{1}{3} + r\sin\theta &= 3(-1 + r\cos\theta)^2 + 2, \\ -1 + 3r\sin\theta &= 9(1 - 2r\cos\theta + r^2\cos^2\theta) + 6, \\ 3r\sin\theta &= 9 - 18r\cos\theta + 9r^2\cos^2\theta + 7, \\ 9r^2\cos^2\theta - 18r\cos\theta - 3r\sin\theta + 16 &= 0, \\ \textit{i.e. } 9r^2\cos^2\theta - 3r(\sin\theta + 6\cos\theta) + 16 &= 0. \end{aligned}$$

Solution: Method 2—

$$\begin{aligned}
\text{Slope of } PQ &= \frac{-1/3-(-1/3+r\sin\theta)}{-1-(-1+r\cos\theta)},\\
&= \frac{-r\sin\theta}{-r\cos\theta},\\
&= \tan\theta.\\
\therefore \text{ Equation of } PQ:\ y+1/3 &= \tan\theta(x+1).\\
\text{This intersects } y &= 3x^2+2,\\
\text{so } 3x^2+2+1/3 &= \tan\theta(x+1),\\
9x^2-3(\tan\theta)x-3\tan\theta+7 &= 0.
\end{aligned}$$

Substitute for Q, $x=-1+r\cos\theta$:

$$\begin{aligned}
9(-1+r\cos\theta)^2-3\tan\theta(-1+r\cos\theta)-3\tan\theta+7 &= 0,\\
9(1-2r\cos\theta+r^2\cos^2\theta)+3\tan\theta-3r\sin\theta-3\tan\theta+7 &= 0,\\
9-16r\cos\theta+9r^2\cos^2\theta-3r\sin\theta+7 &= 0,\\
9r^2\cos^2\theta-18r\cos\theta-3r\sin\theta+16 &= 0,\\
i.e.\ 9r^2\cos^2\theta-3r(\sin\theta+6\cos\theta)+16 &= 0.
\end{aligned}$$

(ii) Using b(i), show that $AB^2=\dfrac{(\sin\theta-2\cos\theta)(\sin\theta+14\cos\theta)}{9\cos^4\theta}$. 3

Solution: $r_1+r_2=\dfrac{3(\sin\theta+6\cos\theta)}{9\cos^2\theta}$, and $r_1r_2=\dfrac{16}{9\cos^2\theta}$,

$$\begin{aligned}
AB^2 &= (r_2-r_1)^2,\ (=r_2^2-2r_2r_1+r_1^2,)\\
&= (r_2+r_1)^2-4r_2r_1,\\
&= \left(\frac{\sin\theta+6\cos\theta}{3\cos^2\theta}\right)^2-4\times\frac{16}{9\cos^2\theta},\\
&= \frac{\sin^2\theta+12\sin\theta\cos\theta+36\cos^2\theta-64\cos^2\theta}{9\cos^4\theta},\\
&= \frac{\sin^2\theta+12\sin\theta\cos\theta-28\cos^2\theta}{9\cos^4\theta},\\
&= \frac{(\sin\theta-2\cos\theta)(\sin\theta+14\cos\theta)}{9\cos^4\theta}.
\end{aligned}$$

(iii) Let L_1 be a tangent to the parabola $y=3x^2+2$ from P, with point of contact R. Using the above results, find the two possible slopes of L_1. 2

Solution: Method 1—

$$\begin{aligned}
\text{At the point of tangency, } AB^2 &= 0,\\
i.e.\ (\sin\theta-2\cos\theta)(\sin\theta+14\cos\theta) &= 0.\\
\Rightarrow \sin\theta = 2\cos\theta,\quad &\text{or}\quad \sin\theta = -14\cos\theta,\\
\therefore \tan\theta = 2,\quad &\qquad \tan\theta = -14.
\end{aligned}$$

Hence the slopes of L_1 are 2 or -14.

Solution: Method 2—

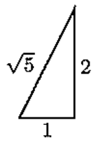
$m_{PR}=\dfrac{y+1/3}{x+1}$ and for $\begin{cases} y = 3x^2+2,\\ y' = 6x.\end{cases}$

$$\begin{aligned}
\text{At the tangent points, } 6x &= \frac{y+1/3}{x+1},\\
6x^2+6x-1/3 &= 3x^2+2,\\
9x^2+18x-7 &= 0,\\
(3x+7)(3x-1) &= 0,\\
x &= 1/3,\ -7/3.\\
\therefore y' &= 6\times 1/3,\ 6\times(-7/3),\\
&= 2,\ -14 \text{ (the slopes of } L_1).
\end{aligned}$$

(iv) Show that $PR = \frac{4\sqrt{5}}{3}$ when one of the slopes of L_1 has a value of 2. [3]

Solution: Method 1—

When the slope of $L_1 = 2$, $\sin\theta = \dfrac{2}{\sqrt{5}}$, and $\cos\theta = \dfrac{1}{\sqrt{5}}$.



$\therefore R$ is at $\left(-1 + \dfrac{r}{\sqrt{5}}, -\dfrac{1}{3} + \dfrac{2r}{\sqrt{5}}\right)$

At the point of tangency to $y = 3x^2 + 2$,

$$\frac{dy}{dx} = 6x = 2,$$
$$i.e.\ x = \frac{1}{3}.$$
$$\text{But } \frac{1}{3} = -1 + \frac{r}{\sqrt{5}},$$
$$\frac{4}{3} = \frac{r}{\sqrt{5}},$$
$$r = \frac{4\sqrt{5}}{3},$$
$$\text{Thus } PR = \frac{4\sqrt{5}}{3}.$$

Solution: Method 2—

PR is a double root of $9r^2\cos^2\theta - 3r(\sin\theta + 6\cos\theta) + 16 = 0$.

$$\text{So } PR^2 = \frac{16}{9\cos^2\theta},$$
$$PR = \sqrt{\frac{16}{9\cos^2\theta}},$$
$$= \frac{4}{3} \times \frac{1}{\cos\theta}.$$
$$\text{But, as } \tan\theta = 2,$$
$$\cos\theta = \frac{1}{\sqrt{5}},$$
$$\text{thus } PR = \frac{4\sqrt{5}}{3}.$$

Solution: Method 3—

$$\text{Equation of } PR\text{: } y + \frac{1}{3} = 2(x+1),$$
$$3y + 1 = 6x + 6,$$
$$6x - 3y + 5 = 0.$$

R is at the point of tangency to $y = 3x^2 + 2$,

$$i.e.\ 6x - 3(3x^2 + 2) + 5 = 0,$$
$$6x - 9x^2 - 6 + 5 = 0,$$
$$9x^2 - 6x + 1 = 0,$$
$$(3x - 1)^2 = 0,$$
$$\therefore x = \frac{1}{3}.$$
$$\text{But } \frac{1}{3} = -1 + \frac{r}{\sqrt{5}},$$
$$\frac{4}{3} = \frac{r}{\sqrt{5}},$$
$$r = \frac{4\sqrt{5}}{3},$$
$$\text{Thus } PR = \frac{4\sqrt{5}}{3}.$$

Solution: Method 4—

Equation of PR:
$$\begin{aligned} 3y+1 &= 6x+6, \\ 6x-3y+5 &= 0. \end{aligned}$$

R is at the point of tangency to $y = 3x^2+2$,

$$\begin{aligned} \textit{i.e. } 6x-3(3x^2+2)+5 &= 0, \\ 6x-9x^2-6+5 &= 0, \\ 9x^2-6x+1 &= 0, \\ (3x-1)^2 &= 0, \\ \therefore x &= \frac{1}{3}. \end{aligned}$$

Substitute in equation PR:
$$\begin{aligned} 6\times\frac{1}{3}-3y+5 &= 0, \\ -3y &= -7, \\ y &= \frac{7}{3}. \end{aligned}$$

So the length of
$$\begin{aligned} PR &= \sqrt{\left(-1-\frac{1}{3}\right)^2+\left(-\frac{1}{3}-\frac{7}{3}\right)^2}, \\ &= \sqrt{\frac{16}{9}+\frac{64}{9}}, \\ &= \sqrt{\frac{80}{9}}, \\ &= \frac{4\sqrt{5}}{3}. \end{aligned}$$