BAULKHAM HILLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2010

MATHEMATICS

EXTENSION 2

GENERAL INSTRUCTIONS:

- Reading time − 5 minutes.
- Working time − 3 hours.
- Write using blue or black pen.
- Start each question on a new page.
- Write your Student Number at the top of each page.
- Calculators may be used.
- A table of standard integrals is provided.
- **ALL** necessary working should be shown in every question.

QUESTION 1 (15 marks)

Marks

(a) Find each of the following integrals

(i) $\int x^2(1+2x^3)^{-5}dx$ 2

(ii) $\int tan^4 x dx$ 3

(iii) $\int \frac{dx}{3+2\cos x}$ 4

(b) Find $\int \frac{e^x+e^{2x}}{1+e^{2x}}dx$ 3

(c) (i) Show that $\sin(A+B)+\sin(A-B)=2sinAcosB$ 1

(ii) Hence calculate $\int sin5xcos4xdx$ 2

QUESTION 2 (15 marks)

(a) For $z_1=2-3i$ and $z_2=1+5i$ find, in the form *a*+*ib*, the values of

(i) $z_1+\overline{z}_2$ 2

(ii) z_1z_2 2

(iii) $\frac{z_1}{z_2}$ 2

(b) (i) Solve $(x+iy)^2=6i$ 2

(ii) Hence or otherwise solve $z^2-(1-i)z-2i=0$ 3

(c) (i) Express $z=1-\sqrt{3}i$ in modulus-argument form 2

(ii) Hence express z^6 in the form $a+ib$ 2

QUESTION 3 (15 marks)

Marks

(a) The hyperbola H has the equation $\frac{x^2}{4} - \frac{y^2}{12} = 1$

Find (i) its eccentricity 2

(ii) the coordinates of its foci 1

(iii) the equations of its directrices 1

(iv) the equations of its asymptotes 1

(b) (i) Show that the gradient of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point P$(a\cos\theta, b\sin\theta)$ is $\frac{-b\cos\theta}{a\sin\theta}$ 2

(ii) Hence show that the equation of the tangent is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ 3

(iii) Show that the x-intercept of this tangent is $(\frac{a}{\cos\theta}, 0)$ 1

(iv) Hence, or otherwise, find the points on the curve $4x^2 + 3y^2 = 12$ whose tangent passes through (2,0) 4

QUESTION 4 (15 marks)

(a) OABC is a square on the Argand diagram and is labeled in an anticlockwise direction. A represents $z = a + ib$ and B represents $4 + 7i$.

(i) Find, in terms of a and b, the complex number represented by C. 2

(ii) Hence evaluate a and b. 2

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. Find the equation with roots:

(i) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ 2

(ii) α^2, β^2 and γ^2 2

(c) (i) Show that 2 is a root of multiplicity 3 for $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$ 2

(ii) Hence solve $P(x) = 0$ 2

(d) Draw on separate argand diagrams the following loci:

(i) $z\bar{z} = 3$ 1

(ii) $\arg\left(\frac{z}{z-1}\right) = \frac{\pi}{3}$ 2

QUESTION 5 (15 marks)

Marks

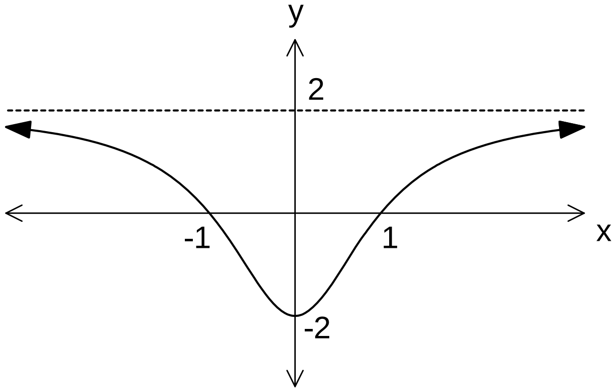
(a) Suppose $x>0, y>0, z>0$

(i) Prove $x^2 + y^2 + z^2 \geq xy + yz + xz$ 2

(ii) Given $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$
prove $x^3 + y^3 + z^3 \geq 3xyz$ 1

(iii) Hence show $a + b + c \geq 3(abc)^{\frac{1}{3}}$ 1

(b) Given below is the graph of $f(x) = 2 - \frac{4}{x^2+1}$.



Use the graph of $y = f(x)$ to sketch, on separate axes, the graphs of

(i) $y = |f(x)|$ 2

(ii) $y = [f(x)]^2$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = \frac{1}{f(x)}$ 2

(c) For the curve $x^3 + 3x^2y - 2y^3 = 16$

(i) Show that $\frac{dy}{dx} = \frac{x^2+2xy}{2y^2-x^2}$ 1

(ii) Find the coordinates of the stationary points on the curve 2

QUESTION 6 (15 marks)

(a) Find, using slices, the volume generated when the area bounded by $y = x^2$ and the line $y = 3$ is rotated about the line $y = 3$. 4

(b) Find, using cylindrical shells, the volume obtained by revolving about the y-axis the region bounded by the curve $y = sinx$, for $0 \leq x \leq \pi$, and the x-axis. 4

(c) A solid has a semi-circular base whose equation is $= \sqrt{4 - x^2}$. Vertical cross-sections, perpendicular to the diameter, are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.

(i) Draw a neat diagram, including a typical slice, representing this information. 1

(ii) By slicing at right angles to the x-axis, show that the volume of the solid is given by $\int_0^2 (4 - x^2)^{\frac{3}{2}} dx$. 3

(iii) Hence calculate this volume. 3

QUESTION 7 (15 marks)

Marks

(a) Use the method of partial fractions to show that $\int_0^1 \frac{6x+4}{(x^2+1)(x+1)} dx = \frac{5\pi}{4} - \frac{1}{2}\log_e 2$ 4

(b) Let $P(z) = z^4 + bz^2 + d$ where b and d are real numbers and $d \neq 0$. *P(z)*has a double zero α.

(i) Prove $P'(z)$ is odd. 2

(ii) Prove that $-\alpha$ is also a double zero of *P(z)*. 2

(c) A mass of 35 kg is dropped from a balloon falling at 30 m/s. The mass experiences air resistance measuring 70*v* Newtons, where *v* m/s is its velocity. Take *g* as 10m/s^2.

(i) Show that the velocity of the mass *t* seconds after being dropped, but before hitting the ground, is given by $v = 5 + 25e^{-2t}$. 3

(ii) Describe what happens to the velocity as $t \to \infty$. 1

(iii) If the mass was dropped from 400m above the ground, how close to the ground will it be after 1 minute? 3

QUESTION 8 (15 marks)

(a) Given $I_n = \int_0^1 x^n e^{-x} dx$

(i) Calculate I_0 2

(ii) Prove $I_n = nI_{n-1} - \frac{1}{e}$ 2

(iii) Hence find $\int_0^1 x^3 e^{-x} dx$ 2

(b) Two particles of equal mass are attached to the ends A and B of a light inextensible string which passes through a small hole at the apex C of a hollow right circular cone fixed with its axis vertical and apex on top. The semi-vertical angle of the cone is θ. The particle at A, where AC is a units, moves in a horizontal circle with constant angular velocity ω on the smooth surface of the cone, while the other particle at B hangs at rest inside the cone.

(i) Represent this information on a diagram showing relevant forces. 1

(ii) Show that $\omega^2 = \frac{g}{a(1+\cos\theta)}$ 2

(iii) Hence, or otherwise, deduce that $\frac{g}{2\omega^2} < a < \frac{g}{\omega^2}$ 2

(c) If $x>0$, prove $x - \frac{1}{3}x^3 < \tan^{-1} x < x - \frac{1}{3}x^3 + \frac{1}{3}x^5$ 4

END OF EXAM

Q1 (a)(i) $I = \int x^2(1+2x^3)^{-5}\,dx$

let $u = 1+2x^3 \therefore \frac{du}{dx} = 6x^2$

$\therefore \frac{du}{6} = x^2\,dx$

$\therefore I = \frac{1}{6}\int u^{-5}\,du$ — 1

$= \frac{1}{6}\frac{u^{-4}}{-4} + c$

$= -\frac{1}{24}(1+2x^3)^{-4} + c$ — 1 (2)

(ii) $I = \int \tan^4 x\,dx$

$= \int \tan^2 x \tan^2 x\,dx$

$= \int \tan^2 x(\sec^2 x - 1)\,dx$ — 1

$= \int \tan^2 x \sec^2 x - \tan^2 x\,dx$ — 1

$= \frac{1}{3}\tan^3 x - \int \sec^2 x - 1\,dx$ — 1

$= \frac{1}{3}\tan^3 x - \tan x + x + c$ (3)

(iii) $I = \int \frac{dx}{3+2\cos x}$

$= \int \frac{\frac{2dt}{1+t^2}}{3+2\left(\frac{1-t^2}{1+t^2}\right)}$ — 1

$= \int \frac{2dt}{3+3t^2+2-2t^2}$

$= \int \frac{2dt}{5+t^2}$ — 1

$= \frac{2}{\sqrt5}\tan^{-1}\left(\frac{t}{\sqrt5}\right) + c$ — 1

$= \frac{2}{\sqrt5}\tan^{-1}\left(\frac{1}{\sqrt5}\tan\frac{x}{2}\right) + c$ — 1 (4)

b) $I = \int \frac{e^x}{1+e^{2x}}dx + \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}dx$

$= I_1 + \frac{1}{2}\ln(1+e^{2x}) + c$

In I_1, let $u = e^x \therefore du = e^x dx$

$\therefore I = \int \frac{du}{1+u^2} + \frac{1}{2}\ln(1+e^{2x}) + c$ — 1 — 1

$= \tan^{-1} u + \frac{1}{2}\ln(1+e^{2x}) + c$ — 1

$= \tan^{-1} e^x + \frac{1}{2}\ln(1+e^{2x}) + c$ (3)

c) LHS $= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A\cos B =$ RHS — 1

$I = \frac{1}{2}\int 2\sin 5x\cos 4x\,dx$ — 1

$= \frac{1}{2}\int \sin 9x + \sin x\,dx$

$= \frac{1}{2}\left(-\frac{1}{9}\cos 9x - \cos x\right) + c$ — 1 (3)

Q2 (a)(i) $z_1 + \bar{z}_2 = 2-3i+1-5i$

$= 3-8i$ (2)

(ii) $z_1 z_2 = (2-3i)(1+5i)$

$= 2+10i-3i+15$

$= 17+7i$ (2)

(iii) $\frac{z_1}{z_2} = \frac{2-3i}{1+5i} \times \frac{1-5i}{1-5i}$

$= \frac{2-10i-3i-15}{1+25}$

$= \frac{-13-13i}{26}$

$= -\frac{1}{2} - \frac{1}{2}i$ (2)

b) (i) $x^2 - y^2 + 2ixy = 6i$

$\therefore x^2 - y^2 = 0$ and $2xy = 6$

$\therefore x^2 - \frac{9}{x^2} = 0 \qquad \therefore y = \frac{3}{x}$

$x^4 = 9$

$x^2 = \pm 3$ But $x^2 > 0$

$\therefore x^2 = 3$

$x = \pm\sqrt3,\ y = \pm\sqrt3$ (2)

$\therefore z = \pm(\sqrt3 + \sqrt3 i)$

(ii) $z = \frac{(1-i) \pm \sqrt{[-(1-i)]^2 - 4\times1\times -2i}}{2\times1}$

$= \frac{1-i \pm \sqrt{1-2i-1+8i}}{2}$

$= \frac{1-i \pm \sqrt{6i}}{2}$ — 1

$= \frac{1-i \pm (\sqrt3+\sqrt3 i)}{2}$ — 1 — 1

$= \frac{1+\sqrt3+(\sqrt3-1)i}{2}, \frac{1-\sqrt3+(-\sqrt3-1)i}{2}$ (3)

c) (i) $z = 2\left(\frac{1}{2} - \frac{\sqrt3}{2}i\right) = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)$ (2)

(ii) $z^6 = \left[2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)\right]^6$

$= 2^6(\cos -2\pi + i\sin -2\pi)$

$= 64(1+0i)$

$= 64$ (2)

Q3(a)(i) $a = 2,\ b = \sqrt{12} = 2\sqrt{3}$

15

$b^2 = a^2(e^2 - 1)$ — 1

$12 = 4(e^2 - 1)$

$e^2 - 1 = 3$ ✓1

$e^2 = 4 \quad \underline{e = 2}$ (2)

(ii) Foci $= (\pm ae, 0) = \underline{(\pm 4, 0)}$ (1)

(iii) Dir. are $x = \pm\frac{a}{e}$ ∴ $\underline{x = \pm 1}$ (1)

(iv) Asym. are $y = \pm\frac{bx}{a}$

∴ $y = \pm\frac{2\sqrt{3}}{2}x$ ∴ $\underline{y = \pm\sqrt{3}x}$ (1)

(b)(i) Diff. Impl.

$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$

$\frac{2yy'}{b^2} = -\frac{2x}{a^2}$

$y' = -\frac{b^2x}{a^2y}$ ✓1 ✓1

∴ $m_{a\cos\theta} = -\frac{b^2 a\cos\theta}{a^2 b\sin\theta} = \underline{-\frac{b\cos\theta}{a\sin\theta}}$ (2)

(ii) Tangent is

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ — 1

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$ 1

$\frac{bx\cos\theta}{ab} + \frac{ay\sin\theta}{ab} = \frac{ab}{ab}$ — 1

$\underline{\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1}$ (3)

(iii) x-int when $y = 0$

∴ $\frac{x\cos\theta}{a} = 1$

$x = \frac{a}{\cos\theta}$

∴ x-int is $\left(\frac{a}{\cos\theta}, 0\right)$ (1)

(iv) $4x^2 + 3y^2 = 12$

$\frac{x^2}{3} + \frac{y^2}{4} = 1$ — 1

∴ $a = \sqrt{3},\ b = 2$

since $a\cos\theta = 2$, $\cos\theta = \frac{2}{\sqrt{3}}$

∴ $\theta = \frac{\pi}{6}$ or $-\frac{\pi}{6}$ in Q_1 and Q_4 1

∴ Points of contact are:

$P(a\cos\theta, b\sin\theta)$

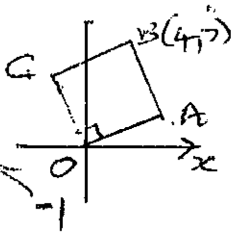
∴ $P_1 = \left(\sqrt{3}\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}\right) = \underline{\left(1\tfrac{1}{2}, 1\right)}$ ✓1

$P_2 = \left(\sqrt{3}\cos-\frac{\pi}{6}, 2\sin-\frac{\pi}{6}\right) = \underline{\left(1\tfrac{1}{2}, -1\right)}$ ✓1

(4)

Q4(a)(i) $\omega_c = (a + ib) \times i$ — 1

$\underline{= -b + ai}$ — 1



(ii) $-b + ai + a + ib = 4 + 7i$

$(a - b) + (b + a)i = 4 + 7i$ — 1

∴ $a - b = 4$

$b + a = 7$

∴ $2a = 11$ and $2b = 3$

$\underline{a = 5\tfrac{1}{2}} \quad \underline{b = 1\tfrac{1}{2}}$ — 1 (4)

b)(i) Eqn is of the form

$\left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right) - 1 = 0$ — 1

$1 + 2x^2 - x^3 = 0$

or $\underline{x^3 - 2x^2 - 1 = 0}$ — 1 (2)

(ii) Eqn is of the form

$(\sqrt{x})^3 + 2\sqrt{x} - 1 = 0$ — 1

$\sqrt{x}(x + 2) = 1$

Sq. both sides

$x(x + 2)^2 = 1$

$x(x^2 + 4x + 4) = 1$ ✓1 (2)

$\underline{x^3 + 4x^2 + 4x - 1 = 0}$

c)(i) $P(x) = x^4 - 3x^3 - 6x^2 + 28x - 24$

$P'(x) = 4x^3 - 9x^2 - 12x + 28$

$P''(x) = 12x^2 - 18x - 12$

$= 6(2x^2 - 3x - 2)$

$= 6(2x + 1)(x - 2)$ ✓1

$= 0$ when $x = -\frac{1}{2}$ or 2

$P(-\frac{1}{2}) = \frac{1}{16} + \frac{3}{8} - \frac{3}{2} - 14 - 24 = -39\frac{1}{16} \neq 0$

$P(2) = 16 - 24 - 24 + 56 - 24 = 0$ — 1 (2)

∴ $x = 2$ is a root of multiplicity 3

(ii) $P(x) = (x - 2)^3(ax + b)$ — 1

Coeff of $x^4 = 1$ ∴ $a = 1$

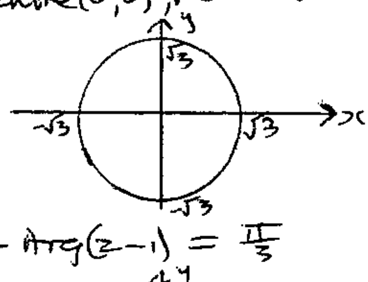
const. term $= -24$ ∴ $-8b = -24$

$b = 3$ ✓1

∴ $P(x) = (x - 2)^3(x + 3)$ ∴ $\underline{x = 2, -3}$ (2)

d)(i) $z\bar{z} = 3$ ∴ $x^2 + y^2 = 3$

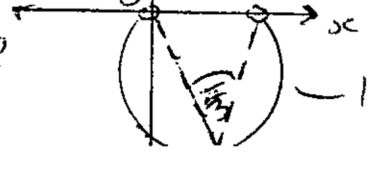
Circle, centre (0,0), radius $\sqrt{3}$



(1)

(ii) $\operatorname{Arg} z - \operatorname{Arg}(z - 1) = \frac{\pi}{3}$

Major Arc, excluding (0,0), (1,0)



— 1 (2)

Q5(a)(i) $(x-y)^2 \ge 0$ for all x, y

$x^2-2xy+y^2 \ge 0$

$x^2+y^2 \ge 2xy$ — 1

Also $y^2+z^2 \ge 2yz$

$y^2+x^2 \ge 2zx$ — 1

Adding: $2(x^2+y^2+z^2) \ge 2(xy+yz+xz)$

$\therefore x^2+y^2+z^2 \ge xy+yz+xz$

(ii)

$x^3+y^3+z^3-3xyz=(x+y+z) \times (x^2+y^2+z^2-xy-yz-zx)$

$\ge (x+y+z)(xy+yz+xz-xy-yz-zx)$ (from part (i)) — 1

$\ge (x+y+z) \times 0$

≥ 0

$\therefore x^3+y^3+z^3 \ge 3xyz$

(iii) In (ii)

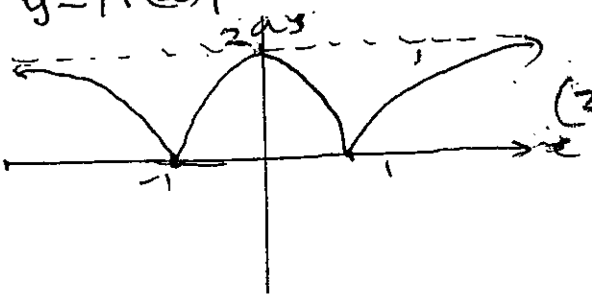
let $x^3=a,\ y^3=b,\ z^3=c$

$\therefore a+b+c \ge 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$ — 1

$\ge 3(abc)^{\frac{1}{3}}$

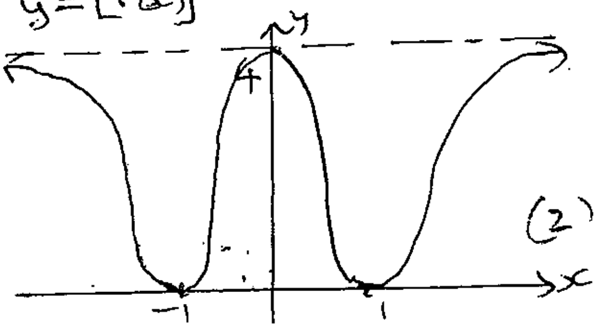
(4)

b)(i) $y=|f(x)|$



(2)

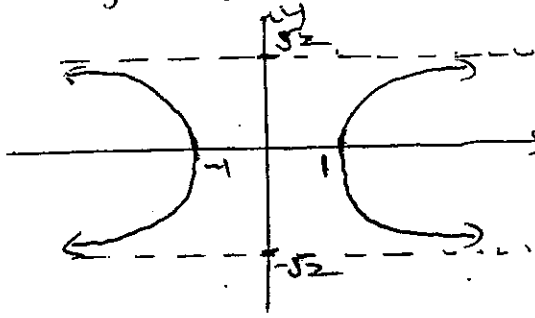
(ii) $y=[f(x)]^2$



(2)

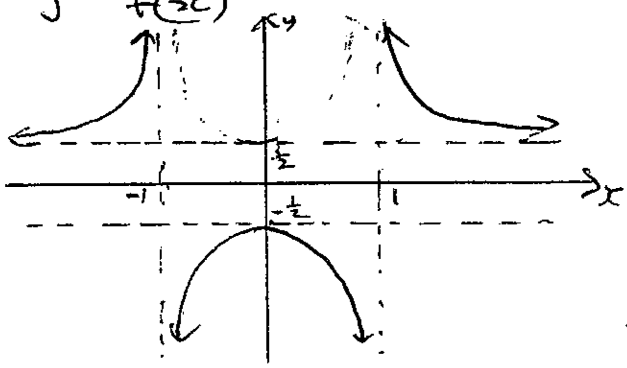
(iii) $y^2=f(x)$

$\therefore y=\pm\sqrt{f(x)}$



(2)

(iv) $y=\frac{1}{f(x)}$



c) (i) $x^3+3x^2y-2y^3=16$

Diff. imp.

$3x^2+6xy+3x^2y'-6y^2 \cdot y'=0$ 1

$y'(6y^2-3x^2)=3x^2+6xy$

$\therefore y'=\frac{3(x^2+2xy)}{3(2y^2-x^2)}$

$=\frac{x^2+2xy}{2y^2-x^2}$ (1)

(ii) St pts when $y'=0$

$x^2+2xy=0$

$x(x+2y)=0$

$\therefore x=0,\ y=-2$ 1

or $x=-2y$

$\therefore (-2y)^3+3(-2y)^2y-2y^3=16$

$-8y^3+12y^3-2y^3=16$

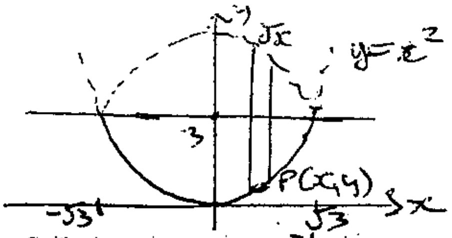
$2y^3=16$

$y=2$ 1

$\therefore x=-4$

$\therefore$ St. pts are $(-4,2)$ and $(0,-2)$ (2)

Q6.(a)



— 1

Vol. of slice $\delta V = \pi r^2 h$

$= \pi(3-y)^2 \delta x$

$= \pi(3-x^2)^2 \delta x$ — 1

Total Vol., $V = \lim_{\delta x \to 0} \sum_{x=-\sqrt{3}}^{x=\sqrt{3}} \pi(3-x^2)^2 \delta x$

$= 2\pi \int_0^{\sqrt{3}} (3-x^2)^2 dx$ — 1

$= 2\pi \int_0^{\sqrt{3}} 9 - 6x^2 + x^4 \, dx$

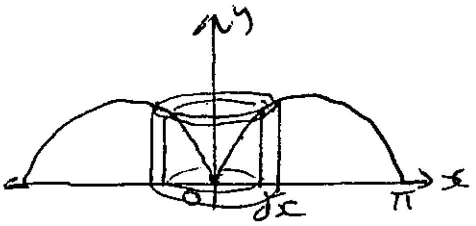
$= 2\pi \left[9x - 2x^3 + \frac{x^5}{5}\right]_0^{\sqrt{3}}$

$= 2\pi \left[9\sqrt{3} - 6\sqrt{3} + \frac{9\sqrt{3}}{5}\right]$

$= \frac{2\pi}{5}\left[45\sqrt{3} - 30\sqrt{3} + 9\sqrt{3}\right]$

$= \frac{48\pi\sqrt{3}}{5} u^3$ — 1 (4)

b)



Vol_{shell} = in. circ. × height × thick.

$= 2\pi x y \delta x$ — 1

Vol. $= \lim_{\delta x \to 0} \sum_{x=0}^{x=\pi} 2\pi x y \delta x$

$= 2\pi \int_0^{\pi} x \sin x \, dx$ — 1

$= 2\pi \left[uv - \int v u'\right]$

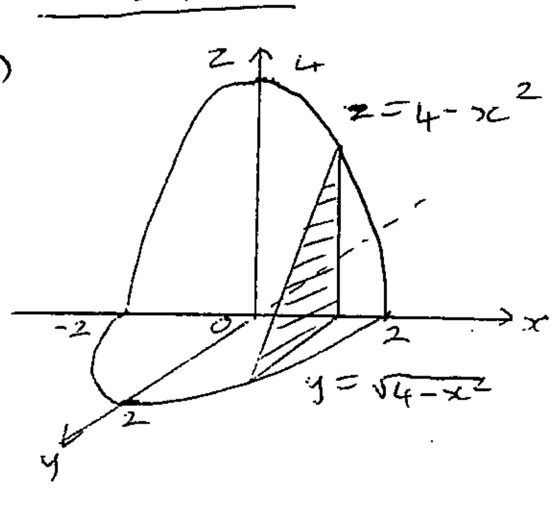
$= 2\pi \left\{\left[-x\cos x\right]_0^{\pi} - \int_0^{\pi} -\cos x \, dx\right\}$ — 1

$= 2\pi \left\{-\pi\cos\pi + \left[\sin x\right]_0^{\pi}\right\}$

$= 2\pi\left[\pi - 0\right]$

$= 2\pi^2 u^3$ — 1 (4)

c) (i)



— (1)

(ii) $Vol_{slice} = \left(\frac{1}{2} b h\right) H$

$= \frac{1}{2} y z \delta x$ — 1

$= \frac{1}{2}\sqrt{4-x^2}(4-x^2)\delta x$

$= \frac{1}{2}(4-x^2)^{\frac{3}{2}} \delta x$ — 1

Vol. $= \lim_{\delta x \to 0} \sum_{x=-2}^{x=2} \frac{1}{2}(4-x^2)^{\frac{3}{2}} \delta x$

$= 2\int_0^2 \frac{1}{2}(4-x^2)^{\frac{3}{2}} dx$ — 1

$= \int_0^2 (4-x^2)^{\frac{3}{2}} dx$ (3)

(iii) $V = \int_0^2 (4-x^2)^{\frac{3}{2}} dx$

Let $x = 2\sin\theta \therefore dx = 2\cos\theta \, d\theta$

When $x = 0$, $\theta = 0$

When $x = 2$, $\theta = \frac{\pi}{2}$ — 1

$\therefore V = \int_0^{\frac{\pi}{2}} (4 - 4\sin^2\theta)^{\frac{3}{2}} \cdot 2\cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} 8(1-\sin^2\theta)^{\frac{3}{2}} \cdot 2\cos\theta \, d\theta$

$= 16\int_0^{\frac{\pi}{2}} \cos^3\theta \cdot \cos\theta \, d\theta$

$= 16\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta$ — 1

$= 16\int_0^{\frac{\pi}{2}} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$

$= 4\int_0^{\frac{\pi}{2}} 1 + 2\cos 2\theta + \cos^2 2\theta \, d\theta$

$= 4\left[\theta + \sin 2\theta\right]_0^{\frac{\pi}{2}} + 4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 4\theta) \, d\theta$

$= 4 \times \frac{\pi}{2} + 2\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$

$= 2\pi + \pi - 0$

$= 3\pi \ u^3$ — 1 (3)

Q7 (a) 15

Let $\frac{6x+4}{(x^2+1)(x+1)} = \frac{ax+b}{x^2+1} + \frac{c}{x+1}$

$\therefore 6x+4 = (ax+b)(x+1) + c(x^2+1)$ -1

If $x=-1$, $-2 = c\times 2 \therefore c = -1$

If $x=0$, $4 = b+c \therefore b = 5$ -1

Eq. coeff of x^2

$a+c=0 \quad \therefore a = 1$

$\therefore I = \int_0^1 \frac{x+5}{x^2+1} - \frac{1}{x+1}\,dx$ -1

$= \int_0^1 \frac{x}{x^2+1}dx + \int_0^1 \frac{5}{x^2+1}dx - \int_0^1 \frac{dx}{x+1}$

$= \frac{1}{2}\left[\ln(x^2+1)\right]_0^1 + 5\left[\tan^{-1}x\right]_0^1 - \left[\ln(x+1)\right]_0^1$

$= \frac{1}{2}(\ln 2 - \ln 1) + 5(\tan^{-1}1 - \tan^{-1}0) - (\ln 2 - \ln 1)$ -1

$= \frac{1}{2}\ln 2 - \ln 2 + 5\times\frac{\pi}{4}$

$= \underline{\frac{5\pi}{4} - \frac{1}{2}\ln 2}$ (4)

b) (i) $P(z) = z^4 + bz^2 + d$

$P'(z) = 4z^3 + 2bz$ -1

$P'(-z) = (-z)^3 + 2b(-z)$

$= -4z^3 - 2bz$

$= -(4z^3 + 2bz)$

$= -P'(z)$ -1

$\therefore \underline{P'(z) \text{ is odd.}}$ (2)

ii) $P(-\alpha) = (-\alpha)^4 + b(-\alpha)^2 + d$

$= \alpha^4 + b\alpha^2 + d$

$= P(\alpha)$

$= 0$ -1

$\therefore -\alpha$ is also a zero of $P(z)$

Let β = the fourth root.

$\Sigma\alpha = \alpha + \alpha + -\alpha + \beta = -\frac{b}{a} = 0$ -1

$\therefore \beta = -\alpha$

$\therefore$ $\underline{-\alpha \text{ is a double root of } P(z)}$ (2)

(c) (i) Result. Force, $ma = mg - mkv$

$35a = 350 - 70v$ -1

$a = 10 - 2v$

$\therefore \frac{dt}{dv} = \frac{1}{10-2v}$

$t = -\frac{1}{2}\ln(10-2v) + c$ -1

$2t = -\ln(10-2v) + 2c$

$\ln(10-2v) = 2c - 2t$

$10 - 2v = e^{2c-2t} = Ae^{-2t}$

$2v = 10 - Ae^{-2t}$

$v = 5 - \frac{A}{2}e^{-2t}$ -1

When $t=0$, $v=30$

$\therefore 30 = 5 - \frac{A}{2}$

$A = -50$

$\therefore \underline{v = 5 + 25e^{-2t}}$ (3)

(ii) As $t \to \infty$, $e^{-2t} \to 0$

$\therefore \underline{v \to 5\text{ ms}^{-1}}$ (1)

(iii) $\frac{dx}{dt} = 5 + 25e^{-2t}$

$x = 5t - \frac{25}{2}e^{-2t} + c$ -1

When $t=0$, $x=0 \therefore c = \frac{25}{2}$

$\therefore x = 5t - \frac{25}{2}e^{-2t} + \frac{25}{2}$

When $t = 60$

$x = 300 - \frac{25}{2}e^{-120} + \frac{25}{2}$

$= 312\frac{1}{2} - \frac{25}{2}e^{-120}$ -1

$\doteqdot 312.5$

$\therefore$ Height $= 400 - 312.5$ -1

$= \underline{87.5\text{ m}}$ (3)

Q8 (a) (i) $I_n = \int_0^1 x^n e^{-x}\,dx$

15

$I_0 = \int_0^1 x^0 e^{-x}\,dx$

$= \int_0^1 e^{-x}\,dx$ — 1

$= [-e^{-x}]_0^1$

$= -e^{-1} - -e^0$

$= 1 - e^{-1}$ — 1 (2)

(ii) $I_n = \int_0^1 x^n e^{-x}\,dx$

Let $u = x^n \therefore u' = nx^{n-1}$

Let $v' = e^{-x} \therefore v = -e^{-x}$ — 1

$I_n = uv - \int vu'$

$= [-x^n e^{-x}]_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx$ — 1

$= -e^{-1} + nI_{n-1}$

$I_n = nI_{n-1} - \frac{1}{e}$ (2)

(iii) $I_3 = \int_0^1 x^3 e^{-x}\,dx$

$= 3I_2 - \frac{1}{e}$

$= 3[2I_1 - \frac{1}{e}] - \frac{1}{e}$ — 1

$= 6I_1 - \frac{4}{e}$

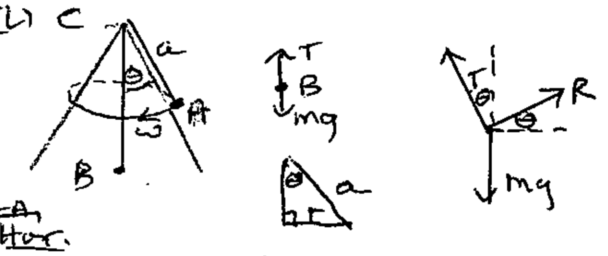
$= 6[I_0 - \frac{1}{e}] - \frac{4}{e}$

$= 6I_0 - \frac{10}{e}$

$= 6[1 - e^{-1}] - \frac{10}{e}$

$= 6 - \frac{16}{e}$ — 1 (2)

b) (i)



—(1)

i) ~~A+A~~ Hor.

Cent. Force, $mr\omega^2 = T\sin\theta - R\cos\theta$ --- (1)

Ver. $mg = T\cos\theta + R\sin\theta$ --- (2)

At B $mg = T$ — 1

∴ In (2) $mg = mg\cos\theta + R\sin\theta$

$R\sin\theta = mg(1-\cos\theta)$

$R = \frac{mg(1-\cos\theta)}{\sin\theta}$ — 1

In (1) $m.a\sin\theta\,\omega^2 = mg\sin\theta - R\cos\theta$

$ma\omega^2\sin\theta = mg\sin\theta - \frac{mg(1-\cos\theta)}{\sin\theta}.\cos\theta$

$a\omega^2\sin^2\theta = g\sin^2\theta - g(1-\cos\theta)\cos\theta$

$\sin^2\theta(g - a\omega^2) = g\cos\theta(1-\cos\theta)$

$g - a\omega^2 = \frac{g\cos\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$

$a\omega^2 = g - \frac{g\cos\theta}{1+\cos\theta}$

$= \frac{g + g\cos\theta - g\cos\theta}{1+\cos\theta}$ /2 $\Rightarrow \therefore \omega^2 = \frac{g}{1+\cos\theta}$

(ii) When R > 0

$\frac{mg(1-\cos\theta)}{\sin\theta} > 0$

$1 - \cos\theta > 0$

$\cos\theta < 1$

$\therefore 0° < \theta < 90°$

When $\theta = 0°$, $\omega^2 = \frac{g}{a(1+1)} = \frac{g}{2a}$

$\therefore a = \frac{g}{2\omega^2}$ — 1

When $\theta = 90°$, $\omega^2 = \frac{g}{a(1+0)} = \frac{g}{a}$

$\therefore a = \frac{g}{\omega^2}$ — 1

$\therefore \frac{g}{2\omega^2} < a < \frac{g}{\omega^2}$ (2)

c) $\frac{d}{dx}(x - \frac{1}{3}x^3) = 1 - x^2$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

$\frac{d}{dx}(x - \frac{1}{3}x^3 + \frac{1}{3}x^5) = 1 - x^2 + \frac{5}{3}x^4$ — 1

Now, $\frac{1}{1+x^2} - (1-x^2) = \frac{1-(1-x^4)}{1+x^2} = \frac{x^4}{1+x^2}$

> 0 for $x > 0$

$\therefore 1 - x^2 < \frac{1}{1+x^2}$ — 1

Also

$1 - x^2 + \frac{5}{3}x^4 - \frac{1}{1+x^2} = \frac{(1+x^2)(1-x^2+\frac{5}{3}x^4) - 1}{1+x^2}$

$= \frac{1 - x^2 + \frac{5}{3}x^4 + x^2 - x^4 + \frac{5}{3}x^6 - 1}{1+x^2}$

$= \frac{\frac{2}{3}x^4 + \frac{5}{3}x^6}{1+x^2} > 0$ for $x > 0$

$\therefore \frac{1}{1+x^2} < 1 - x^2 + \frac{5}{3}x^4$ — 1

$\therefore 1 - x^2 < \frac{1}{1+x^2} < 1 - x^2 + \frac{5}{3}x^4$

Integrating each exp. gives: — 1

$\int_0^x 1 - x^2\,dx < \int_0^x \frac{1}{1+x^2}\,dx < \int_0^x 1 - x^2 + \frac{5}{3}x^4\,dx$

$[x - \frac{x^3}{3}]_0^x < [\tan^{-1}x]_0^x < [x - \frac{1}{3}x^3 + \frac{1}{3}x^5]_0^x$

ie.

$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{1}{3}x^3 + \frac{1}{3}x^5$ (4)