100 Years

North Sydney Boys High School

1912 ~ 2012

MATHEMATICS

2013 HSC Course Assessment Task 3 (Trial Examination)

June 19, 2013

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12M2A – Mr Lowe

◯ 12M2B – Mrs Juhn

◯ 12M3A – Mr Lam

◯ 12M3B – Mr Berry

◯ 12M3C – Mr Lin

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	**Total**
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer sheet provided.

Questions **Marks**

1. Which of the following exists at the point $(1,1)$ on the graph of $y=(x-1)^3+1$? **1**

(A) A local minimum.

(B) A local maximum.

(C) A stationary point of inflexion.

(D) None of the above.

2. The midpoint of the line joining $(0,-5)$ to $(d,0)$ is **1**

(A) $\left(\dfrac{d-5}{2},0\right)$

(B) $\left(\dfrac{d}{2},-\dfrac{5}{2}\right)$

(C) $\left(0,\dfrac{5-d}{2}\right)$

(D) $\left(\dfrac{5+d}{2},0\right)$

3. Which of the following is the derivative of **1**

$$y=\log_e(f(x))$$

with respect to x?

(A) $\dfrac{f(x)}{f'(x)}$

(B) $\dfrac{1}{f'(x)}$

(C) $\dfrac{f'(x)}{f(x)}$

(D) $\dfrac{1}{f(x)}$

4. What is the period of the function $y=4\sin\left(\dfrac{x}{3}\right)$? **1**

(A) 6π

(B) $\dfrac{2\pi}{3}$

(C) 4

(D) $\dfrac{1}{4}$

5. The graph with equation $y = x^2$ is translated 3 units down and 2 units to the right. Which equation represents the resulting graph? **1**

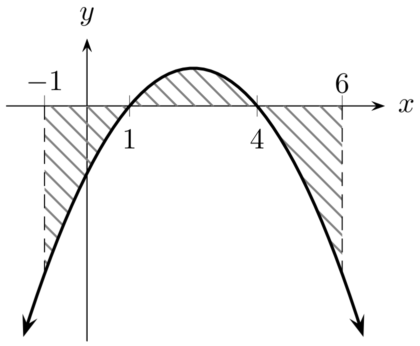
(A) $y = (x-2)^2 + 3$

(B) $y = (x-2)^2 - 3$

(C) $y = (x+2)^2 + 3$

(D) $y = (x+2)^2 - 3$

6. Which of the following expressions gives the total area of the shaded region in the diagram? **1**



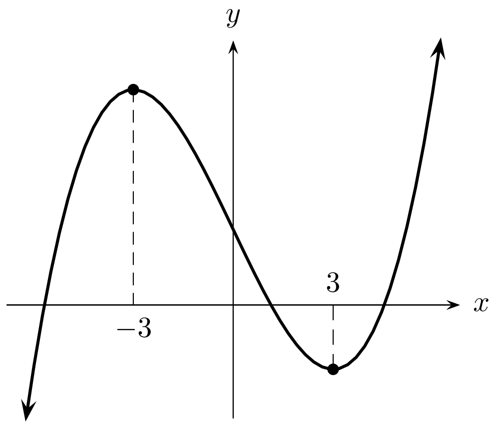
(A) $\displaystyle\int_{-1}^{6} f(x)\,dx$

(B) $\displaystyle-\int_{-1}^{0} f(x)\,dx + \int_{0}^{6} f(x)\,dx$

(C) $\displaystyle-\int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx - \int_{4}^{6} f(x)\,dx$

(D) $\displaystyle\int_{1}^{4} f(x)\,dx + 2\int_{4}^{6} f(x)\,dx$

7. From the graph of $y = f(x)$, when is $f'(x)$ negative? **1**



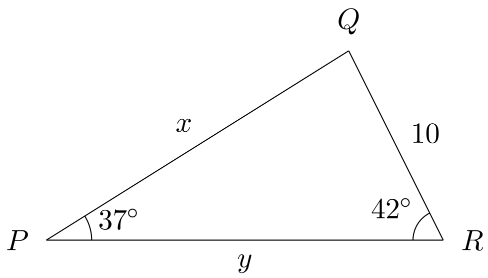
(A) $x < -3$ or $x > 3$

(B) $-3 < x < 3$

(C) $x \leq -3$ or $x \geq 3$

(D) $-3 \leq x \leq 3$

8. $\triangle PQR$ has side lengths x, y and 10 as shown. $\angle RPQ = 37^\circ$ and $\angle QRP = 42^\circ$. **1**



Which of the following expressions is correct for $\triangle PQR$?

(A) $x = 10 \times \dfrac{\sin 42^\circ}{\sin 37^\circ}$

(B) $y = 10 \times \dfrac{\sin 37^\circ}{\sin 101^\circ}$

(C) $x = \dfrac{10}{\sin 37^\circ}$

(D) $y = \dfrac{10}{\tan 37^\circ}$

9. If M is decreasing at an increasing rate, what does this suggest about $\dfrac{dM}{dt}$ and $\dfrac{d^2M}{dt^2}$? **1**

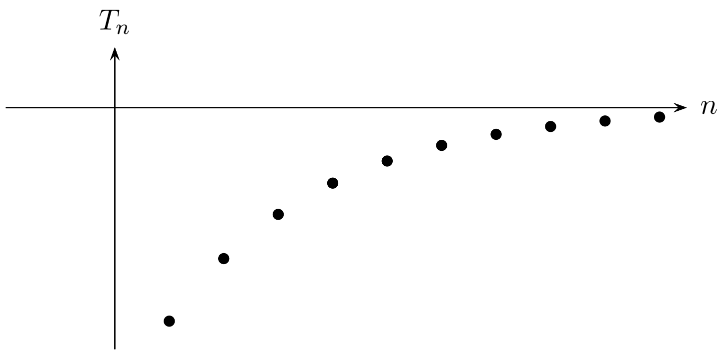
(A) $\dfrac{dM}{dt} < 0$ and $\dfrac{d^2M}{dt^2} < 0$

(B) $\dfrac{dM}{dt} > 0$ and $\dfrac{d^2M}{dt^2} < 0$

(C) $\dfrac{dM}{dt} < 0$ and $\dfrac{d^2M}{dt^2} > 0$

(D) $\dfrac{dM}{dt} > 0$ and $\dfrac{d^2M}{dt^2} > 0$

10. The graph shows consecutive terms of a sequence. Which of the following statements best describes the sequence? **1**



(A) geometric, $|r| \geq 1$.

(B) arithmetic, $|d| \geq 1$.

(C) arithmetic, $|d| < 1$.

(D) geometric, $|r| < 1$.

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Fully factorise $4x^2 - 36$. **2**

(b) Solve $|2x - 3| < 13$. **2**

(c) Solve for x: $4^x - 9 \times 2^x + 8 = 0$. **2**

(d) For the parabola $(x-2)^2 = 4y$

i. Find the coordinates of the vertex. **1**

ii. State the equation of the directrix of the parabola. **1**

(e) Write down the domain of $f(x) = \dfrac{1}{(x-3)(2-x)}$. **2**

(f) Evaluate $\displaystyle\lim_{x \to 2} \frac{x-2}{x^2 + x - 6}$. **2**

(g) Find the equation of the tangent to the curve $y = 2\sin 2x$ at the point $\left(\dfrac{\pi}{8}, \sqrt{2}\right)$. **3**

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) Differentiate with respect to x:

i. $e^{\tan x}$. **2**

ii. $\dfrac{3x}{x^2+1}$. **2**

(b) For the equation $3x^2 - 2x + 7 = 0$, evaluate:

i. $\alpha + \beta$ **1**

ii. $\alpha\beta$ **1**

iii. $\alpha^2 + \beta^2$ **2**

(c) A function is defined by $f(x) = x^3 - 3x^2 - 9x + 22$.

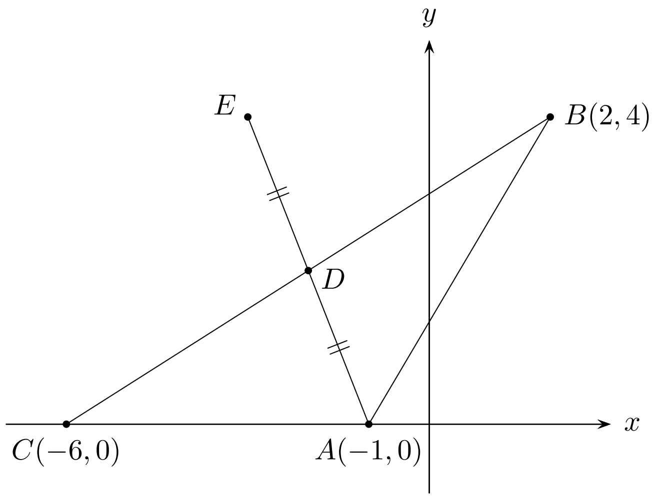
i. Find the coordinates of the turning points of the graph $y = f(x)$ and determine their nature. **3**

ii. Find the coordinates of the point(s) of inflexion. **2**

iii. Hence sketch the graph of $y = f(x)$, showing the turning points, the point(s) of inflexion and the y intercept. **2**

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) In the diagram A, B, C and D are the points $(-1,0)$, $(2,4)$, $(-6,0)$, $(-2,2)$ respectively. D is also the midpoint of AE.



i. Find the length of the interval AB. **1**

ii. Find the equation of the circle with centre at B which passes through the point A. **1**

iii. Find the size of $\angle CAB$ to the nearest degree. **2**

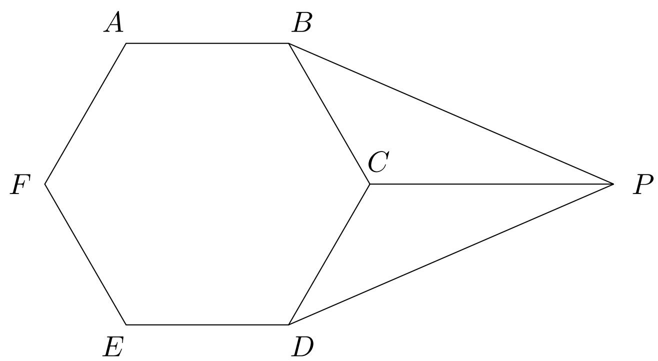
iv. Find the midpoint of BC. **1**

v. Show that the equation of the line BC is $x - 2y + 6 = 0$. **1**

vi. Find the perpendicular distance of A from the line BC in simplest exact form. **2**

vii. What type of quadrilateral is $ABEC$? Give reasons for your answer. **2**

(b) $ABCDEF$ Is a regular hexagon, and $CP \parallel AB$.



i. Find the size of $\angle BCP$, giving reasons. **2**

ii. Prove that $\triangle BCP \equiv \triangle DCP$. **3**

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) For what value(s) of k does the equation **2**

$$x^2 + (k+2)x + 4 = 0$$

have equal roots?

(b) Evaluate the following integrals:

i. $\displaystyle\int \frac{1}{x\sqrt{x}}\,dx$ **2**

ii. $\displaystyle\int (\sin x + \cos x)\,dx$ **2**

(c) Evaluate $\displaystyle\int_2^4 \frac{3x}{x^2-1}\,dx$, leaving your answer in the simplest exact form. **3**

(d) Consider the function $y = e^{x^2}$.

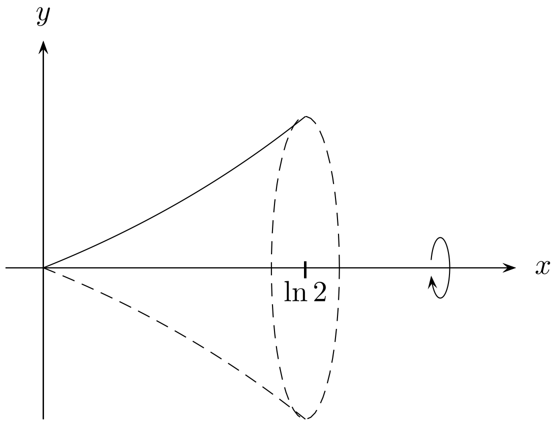
x	0	0.5	1.0	1.5	2.0
e^{x^2}	1.00		2.72		

i. Copy the above table of values on to your page and supply the missing values (correct to 2 decimal places) **1**

ii. Using Simpson's rule with five function values, find an estimate for the definite integral, correct to 2 decimal places: **2**

$$\int_0^2 e^{x^2}\,dx$$

(e) The part of the curve $y = e^x - 1$ between $x = 0$ and $x = \ln 2$ is rotated about the x-axis. **3**

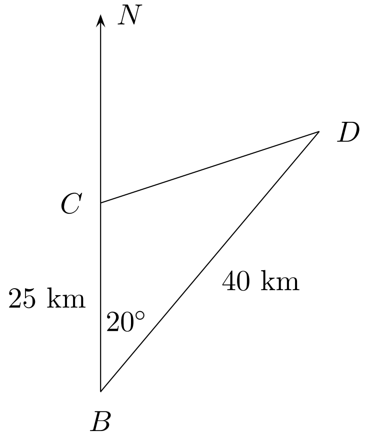


Find the exact volume (in simplest form) of the solid obtained.

Question 15 (15 Marks) Commence a NEW page. **Marks**

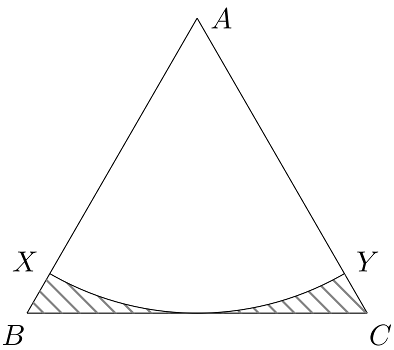
(a) A town C is located 25 km due north of town B. Another town D is 40 km on a bearing of 020° from B. Towns B, C and D are connected by straight roads. **3**

Determine how much shorter it is for a man to travel from town D directly to town B, rather than through town C (give your answer correct to 1 decimal place).



NOT TO SCALE

(b) In the diagram, $\triangle ABC$ is an equilateral triangle with sides of length 6 cm. An arc with centre A and BC as tangent, cuts AB and AC at X and Y respectively.



i. Show that the radius of the arc is $3\sqrt{3}$ cm. **2**

ii. Find in exact form, the area of the shaded region. **3**

(c) i. Differentiate $\log_e(\cos x)$ with respect to x, writing your answer in simplest form. **2**

ii. Sketch the curve $y = \tan x$ for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. **2**

iii. Hence or otherwise, find the area bounded by the curve $y = \tan x$, the x axis and the line $x = \dfrac{\pi}{3}$, leaving your answers in simplest exact form. **3**

Question 16 (15 Marks) Commence a NEW page. **Marks**

(a) In order to reduce the 'bill shock' that occurs whenever he receives his mobile phone bill, Mr Lam decides to reduce his mobile data usage by 15% of the previous day's usage, starting from the first day of the billing cycle.

On the first day, he used 200 megabytes (MB) of mobile data.

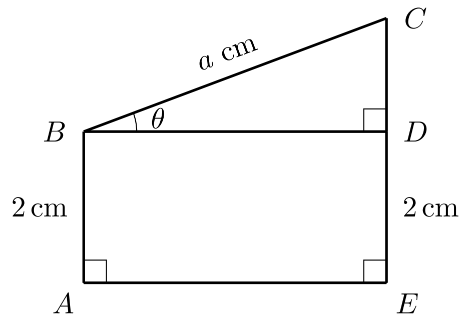
i. Find the amount of data (in MB) used on the fifth day, correct to 2 decimal places. **1**

ii. Find the total amount of data (in MB) used after 7 days, correct to 2 decimal places. **2**

iii. Show that a mobile phone plan which offers 1500MB of data per month would be sufficient for his usage. **2**

(b) If $\tan\alpha = \dfrac{1}{2}$ and α is acute, find the *exact* value of $\sin\alpha$ and $\cos\alpha$. **2**

(c) The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. D is a point on the line EC with $AB = ED = 2\,\text{cm}$, and $BC = a$ cm, where $a > 0$.



Also, $\angle BAE = \angle CEA = \frac{\pi}{2}$, and $\angle CBD = \theta$, where $0 < \theta < \frac{\pi}{2}$.

i. Find BD and CD in terms of a and θ. **2**

ii. Find the length L, of the wire in the frame (which includes the length BD), in terms of a and θ. **1**

iii. Find $\dfrac{dL}{d\theta}$, and hence show that $\dfrac{dL}{d\theta} = 0$ when $\tan\theta = \dfrac{1}{2}$. **2**

iv. Given that $a = 3\sqrt{5}$, find the maximum length of wire in the frame. **3**

End of paper.

STANDARD INTEGRALS

$\int x^n\,dx$	$= \frac{1}{n+1}x^{n+1} + C,$	$n \neq -1; \quad x \neq 0$ if $n < 0$
$\int \frac{1}{x}\,dx$	$= \ln x + C,$	$x > 0$
$\int e^{ax}\,dx$	$= \frac{1}{a}e^{ax} + C,$	$a \neq 0$
$\int \cos ax\,dx$	$= \frac{1}{a}\sin ax + C,$	$a \neq 0$
$\int \sin ax\,dx$	$= -\frac{1}{a}\cos ax + C,$	$a \neq 0$
$\int \sec^2 ax\,dx$	$= \frac{1}{a}\tan ax + C,$	$a \neq 0$
$\int \sec ax \tan ax\,dx$	$= \frac{1}{a}\sec ax + C,$	$a \neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a} + C,$	$a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$= \sin^{-1}\frac{x}{a} + C,$	$a > 0, -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2-a^2}\right) + C,$	$x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$= \ln\left(x + \sqrt{x^2+a^2}\right) + C$	

NOTE: $\ln x = \log_e x, \ x > 0$

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

◯ 12M2A – Mr Lowe

◯ 12M2B – Mrs Juhn

◯ 12M3A – Mr Lam

◯ 12M3B – Mr Berry

◯ 12M3C – Mr Lin

1 – Ⓐ Ⓑ Ⓒ Ⓓ
2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
6 – Ⓐ Ⓑ Ⓒ Ⓓ
7 – Ⓐ Ⓑ Ⓒ Ⓓ
8 – Ⓐ Ⓑ Ⓒ Ⓓ
9 – Ⓐ Ⓑ Ⓒ Ⓓ
10 – Ⓐ Ⓑ Ⓒ Ⓓ

Suggested Solutions

Section I

1. (C) **2.** (B) **3.** (C) **4.** (A) **5.** (B)
6. (C) **7.** (B) **8.** (A) **9.** (A) **10.** (D)

Question 11 (Juhn)

(a) (2 marks)

$$\begin{aligned} 4x^2 - 36 &= 4(x^2 - 9) \\ &= 4(x-3)(x+3) \end{aligned}$$

(b) (2 marks)

$$\begin{aligned} |2x - 3| &< 13 \\ -\underset{+3}{13} < 2x - &\underset{+3}{3} < \underset{+3}{13} \\ -10 < 2x &< 16 \\ -5 < x &< 8 \end{aligned}$$

(c) (2 marks)

$$2^{2x} - 9 \times 2^x + 8 = 0$$

Let $u = 2^x$,

$$\begin{aligned} u^2 - 9u + 8 &= 0 \\ (u-8)(u-1) &= 0 \\ \therefore u &= 8, 1 \\ \therefore 2^x &= 8, 1 \\ \therefore x &= 0, 3 \end{aligned}$$

(d) $(x-2)^2 = 4y$

i. (1 mark) – $V(2, 0)$.

ii. (1 mark) – $y = -1$

(e) (2 marks)

$$D = \{x : x \neq 2, x \neq 3\}$$

(f) (2 marks)

$$\begin{aligned} \lim_{x \to 2} \frac{x-2}{x^2 + x - 6} &= \lim_{x \to 2} \frac{\cancel{x-2}}{\cancel{(x-2)}(x+3)} \\ &= \lim_{x \to 2} \frac{1}{x+3} = \frac{1}{5} \end{aligned}$$

(g) (3 marks)

$$\begin{aligned} y &= 2\sin 2x \\ \frac{dy}{dx} &= 4\cos 2x \Big|_{x=\frac{\pi}{8}} \\ &= 4\cos\frac{\pi}{4} = \frac{4}{\sqrt{2}} \end{aligned}$$

Equation of tangent will be in the form $y = mx + b$:

$$y = \frac{4}{\sqrt{2}}x + b$$

When $x = \frac{\pi}{8}$, $y = \sqrt{2}$,

$$\begin{aligned} \sqrt{2} &= \frac{4}{\sqrt{2}} \times \frac{\pi}{8} + b \\ \therefore b &= \sqrt{2} - \frac{\pi}{2\sqrt{2}} \\ \therefore y &= \frac{4}{\sqrt{2}}x + \left(\sqrt{2} - \frac{\pi}{2\sqrt{2}}\right) \end{aligned}$$

Question 12 (Lam)

(a) i. (2 marks)

$$\frac{d}{dx}\left(e^{\tan x}\right) = \left(\sec^2 x\right) e^{\tan x}$$

ii. (2 marks)

$$\begin{aligned} y &= \frac{3x}{x^2+1} \\ u = 3x &\quad v = x^2 + 1 \\ u' = 3 &\quad v' = 2x \\ \frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{3(x^2+1) - 2x(3x)}{(x^2+1)^2} \\ &= \frac{3x^2 + 3 - 6x^2}{(x^2+1)^2} = \frac{-3x^2+3}{(x^2+1)^2} \end{aligned}$$

(b) $3x^2 - 2x + 7 = 0$

i. (1 mark)

$$\alpha + \beta = -\frac{b}{a} = \frac{2}{3}$$

ii. (1 mark)

$$\alpha\beta = \frac{c}{a} = \frac{7}{3}$$

iii. (2 marks)

$$\begin{aligned}\alpha^2+\beta^2 &= (\alpha+\beta)^2-2\alpha\beta\\ &= \left(\frac{2}{3}\right)^2-2\left(\frac{7}{3}\right)\\ &= \frac{4}{9}-\frac{14}{3}=-\frac{38}{9}\end{aligned}$$

(c) i. (3 marks)

$$\begin{aligned}y &= x^3-3x^2-5x+22\\ y' &= 3x^2-6x-9\\ &= 3\left(x^2-2x-3\right)\\ &= 3(x-3)(x+1)\end{aligned}$$

Stationary points occur when $y'=0$:

$$\therefore x=-1,3$$

When $x=-1$,

$$y=(-1)^3-3(-1)^2-9(-1)+22=27$$

When $x=3$,

$$y=3^3-3(3^2)-9(3)+22=-5$$

Finding the second derivative,

$$\begin{aligned}y' &= 3x^2-6x-9\\ y'' &= 6x-6\end{aligned}$$

When $x=-1$,

$$y'=6(-1)-6<0$$

$\therefore$ $(-1,27)$ is a local max. When $x=3$,

$$y'=6(3)-6>0$$

$\therefore$ $(3,-5)$ is a local min.

Alternatively , use table of variations:

x	-2	-1	0	3	4
$\frac{dy}{dx}$	$+$ 15	0	$-$ -9	0	$+$ 15
y	↗	$(-1,27)$	↘	$(3,-5)$	↗

Hence $(-1,27)$ is a local max, and $(3,-5)$ is a local min.

ii. (2 marks)

$$\begin{aligned}y' &= 3x^2-6x-9\\ y'' &= 6x-6\end{aligned}$$

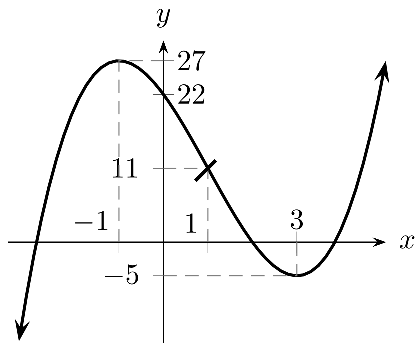
Pt of inflexion occurs when $y''=0$:

$$\begin{aligned}6x-6&=0\\ x&=1\end{aligned}$$

x	0	1	2
$\frac{d^2y}{dx^2}$	$-$ -6	0	$+$ 6
y	⌢		⌣

When $x=1$, $y=1-3-9+22=11$. Hence $(1,11)$ is a point of inflexion as a change of sign of the 2nd derivative also occurs.

iii. (2 marks)



Question 13 (Berry)

(a) i. (1 mark)

$$AB=\sqrt{(2+1)^2+(4-0)^2}=5$$

ii. (1 mark)

$$(x-2)^2+(y-4)^2=25$$

iii. (2 marks)

$$m_{AB} = \tan\theta = \frac{4}{3}$$
$$\therefore \theta = 53.13^\circ \cdots$$
$$\therefore \angle CAB = 180^\circ - 53.13^\circ \approx 127^\circ$$

iv. (1 mark)

$$M\left(\frac{2-6}{2}, \frac{4-0}{2}\right) = \left(-\frac{4}{2}, 2\right) = (-2, 2)$$

v. (1 mark)

$$\frac{y-0}{x+6} = \frac{4-0}{2+6} = \frac{1}{2}$$
$$\therefore y = \frac{1}{2}(x+6) = \frac{1}{2}x + 3$$
$$2y = x + 6$$
$$x - 2y + 6 = 0$$

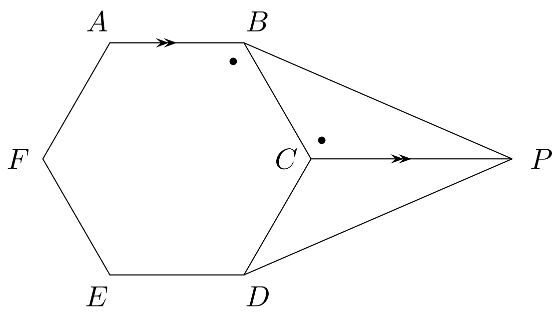
vi. (2 marks)

$$(-1, 0) \to x - 2y + 6 = 0$$
$$d_\perp = \frac{|1(-1) - 2(0) + 6|}{\sqrt{1^2 + 2^2}}$$
$$= \frac{5}{\sqrt{5}} = \sqrt{5}$$

vii. (2 marks)

- As D is the midpoint of AE as well, $ABEC$ is a parallelogram as the diagonals bisect each other.
- But as $AC = 5$ as well as $AB = 5$, then $ABEC$ is a rhombus as adjacent sides are equal and also has the properties of a parallelogram.

(b) i. (2 marks)



Interior angle sum of polygon: $S = 180(n-2)$

$$S = 180(6-2) = 720^\circ$$
$$\therefore \theta = \frac{720}{6} = 120^\circ$$

Hence $\angle ABC = 120^\circ$ and $\angle BCP = 120^\circ$ (alternate $\angle$ equal only if $AB \parallel CP$)

ii. (3 marks)

- Similarly, $\angle DCP = 120^\circ$.

In $\triangle BCP$ and $\triangle DCP$:

- $BC = CD$ (sides of a regular hexagon)
- $\angle BCP = \angle DCP$ (previously proven)
- CP is common

$\therefore \triangle BCP \equiv \triangle DCP$ (SAS)

Question 14 (Ziaziaris)

(a) (2 marks)

$$x^2 + (k+2)x + 4 = 0$$
$$\Delta = b^2 - 4ac$$
$$= (k+2)^2 - 4(1)(4)$$
$$= (k+2)^2 - 16$$
$$= (k+2-4)(k+2+4)$$
$$= (k-2)(k+6)$$

Equal roots occur when $\Delta = 0$, i.e. when $k = 2$ or -6.

(b) i. (2 marks)

$$\int \frac{1}{x\sqrt{x}}\,dx = \int x^{-\frac{3}{2}}\,dx$$
$$= -2x^{-\frac{1}{2}} + C$$

ii. (2 marks)

$$\int \sin x + \cos x\,dx = -\cos x + \sin x + C$$

(c) (3 marks)

$$\begin{aligned}\int_2^4 \frac{3x}{x^2-1} &= \frac{3}{2}\int_2^4 \frac{2x}{x^2-1}\,dx \\ &= \frac{3}{2}\left[\log_e\left(x^2-1\right)\right]_2^4 \\ &= \frac{3}{2}\left[\log_e(15)-\log_e(3)\right] \\ &= \frac{3}{2}\log_e 5\end{aligned}$$

(d) i. (1 mark)

x	0	0.5	1.0	1.5	2.0
e^{x^2}	1.00	**1.28**	2.72	**9.49**	**54.6**

ii. (2 marks)

$$\begin{aligned}&\int_0^2 e^{x^2}\,dx \\ &\approx \frac{h}{3}\left(y_0 + 4\sum y_{\text{even}} + 2\sum y_{\text{odd}} + y_\ell\right) \\ &= \frac{\frac{1}{2}}{3}\left(1 + 4(1.28+9.49) + 2(2.72) + 54.6\right) \\ &= 17.35\end{aligned}$$

(NB. If student used exact values, their answer will be 19.61)

(e) (3 marks)

$$\begin{aligned}V &= \pi\int_0^{\log_e 2} \left(e^x-1\right)^2\,dx \\ &= \pi\int_0^{\log_e 2} e^{2x}-2e^x+1\,dx \\ &= \pi\left[\frac{1}{2}e^{2x}-2e^x+x\right]_0^{\log_e 2} \\ &= \pi\left(\frac{1}{2}\left(e^{2\log_e 2}-e^0\right)-2\left(e^{\log_e 2}-e^0\right)+(\log_e 2-0)\right) \\ &= \pi\left(\frac{1}{2}(4-1)-2(2-1)+\log_e 2\right) \\ &= \pi\left(\frac{3}{2}-2+\log_e 2\right) \\ &= \pi\left(\log_e 2-\frac{1}{2}\right)\end{aligned}$$

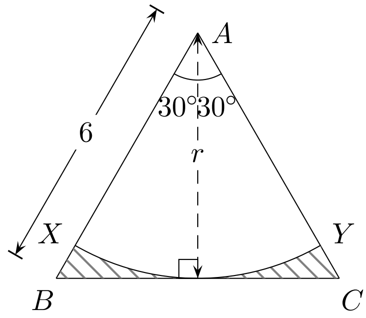
Question 15 (Lin)

(a) (3 marks)

$$\begin{aligned}CD^2 &= 25^2+40^2-2(25)(40)\cos 20^\circ \\ &= 345.61\cdots \\ \therefore CD &\approx 18.59\,\text{km}\end{aligned}$$

$DC + CB = 43.59\,\text{km}$. Hence a difference of approximately 3.6 km.

(b) i. (2 marks)



$$\frac{r}{6} = \cos 30^\circ$$

$$\therefore r = 6\cos 30^\circ = 6\times\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

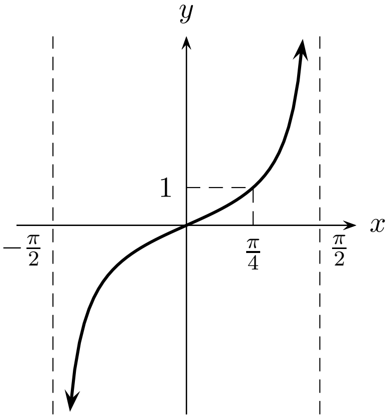
ii. (3 marks)

$$\begin{aligned}A_\triangle &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}\times 6^2\times\sin 60^\circ \\ &= 18\times\frac{\sqrt{3}}{2} = 9\sqrt{3} \\ A_{\text{sect}} &= \frac{1}{2}r^2\theta \\ &= \frac{1}{2}\left(3\sqrt{3}\right)^2\times\frac{\pi}{3} \\ &= \frac{1}{2}\times 27\times\frac{\pi}{3} = \frac{9\pi}{2} \\ A_{\text{shaded}} &= 9\sqrt{3}-\frac{9\pi}{2}\end{aligned}$$

(c) i. (2 marks)

$$\frac{d}{dx}\left(\log_e\cos x\right) = \frac{-\sin x}{\cos x} = -\tan x$$

ii. (2 marks)



iii. (3 marks)

$$\begin{aligned}A &= \int_0^{\frac{\pi}{3}} \tan x\, dx \\ &= -\int_0^{\frac{\pi}{3}} -\frac{\sin x}{\cos x}\, dx \\ &= -\Big[\log_e \cos x\Big]_0^{\frac{\pi}{3}} \\ &= -\left(\log_e \cos\frac{\pi}{3} - \log_e \cos 0\right) \\ &= -\log_e \frac{1}{2} = \log_e 2\end{aligned}$$

Question 16 (Lowe)

(a) i. (1 mark)

$$\begin{aligned}a = 200 \qquad r &= 0.85 \\ T_5 &= ar^{n-1} \\ &= 200(0.85)^4 \\ &= 104.4\,\text{MB}\end{aligned}$$

ii. (2 marks)

$$\begin{aligned}S_7 &= \frac{a\left(r^n - 1\right)}{r-1} \\ &= \frac{200\left(0.85^7 - 1\right)}{0.85 - 1} \\ &= 905.9\,\text{MB}\end{aligned}$$

iii. (2 marks)

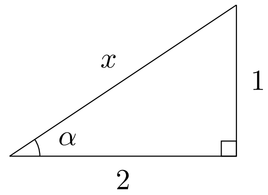
✓ [1] for significant progress towards answer.

✓ [1] for final answer.

$$\begin{aligned}S &= \frac{a}{1-r} \\ &= \frac{200}{1-0.85} = 1\,333.33\,\text{MB}\end{aligned}$$

Maximum data usage would be 1 333.33 MB. Hence a 1 500 MB data plan would be sufficient.

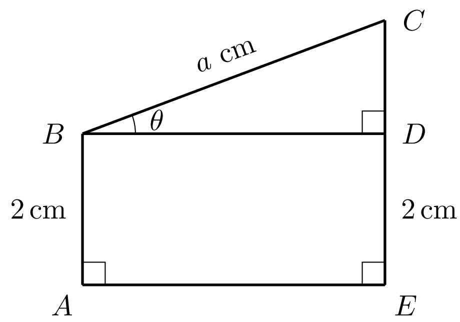
(b) (2 marks)



$$\begin{aligned}x^2 &= 1^2 + 2^2 = 5 \\ \therefore x &= \sqrt{5}\end{aligned}$$

$$\therefore \sin\alpha = \frac{1}{\sqrt{5}} \qquad \cos\alpha = \frac{2}{\sqrt{5}}$$

(c) i. (2 marks)



$$\begin{aligned}\frac{BD}{a} &= \cos\theta \\ BD &= a\cos\theta \\ \therefore CD &= a\sin\theta\end{aligned}$$

ii. (1 mark)

$$\begin{aligned}L &= 2BD + 2 + 2 + CD + a \\ &= 2a\cos\theta + a\sin\theta + 4 + a\end{aligned}$$

iii. (2 marks)

✓ [1] for correct derivative.

✓ [1] for showing $\tan\theta = \frac{1}{2}$ when $\frac{dL}{d\theta} = 0$.

$$\frac{dL}{d\theta} = -2a\sin\theta + a\cos\theta$$

When $\dfrac{dL}{d\theta} = 0$,

$$\begin{aligned} -2a\sin\theta + a\cos\theta &= 0 \\ \therefore \underset{\div a\cos\theta}{2a\sin\theta} &= \underset{\div a\cos\theta}{a\cos\theta} \\ 2\frac{\sin\theta}{\cos\theta} &= 1 \\ \therefore \tan\theta &= \frac{1}{2} \end{aligned}$$

iv. (3 marks) Maximum length of wire occurs when $\dfrac{dL}{d\theta} = 0$, i.e. $\tan\theta = \frac{1}{2}$ $(\theta \approx 0.46)$

θ	0	0.46	1
$\frac{dL}{d\theta}$	$+$ $\cos 0$	0	$-$ $-2\sin 1 + \cos 1$
L	↗	—	↘

NB. When $\theta = 1$,

$$\frac{dL}{d\theta} = a\left(-2\sin 1 + \cos 1\right) \approx -1.14a$$

As $a > 0$, $\dfrac{dL}{d\theta} < 0$.
Hence $\tan\theta = \frac{1}{2}$ produces a local maximum.

$$\begin{aligned} \therefore L &= 2\left(3\sqrt{5}\right)\cos\theta + 3\sqrt{5}\sin\theta + 4 + 3\sqrt{5} \\ &= 6 \times \cancel{\sqrt{5}} \times \frac{2}{\cancel{\sqrt{5}}} + 3\cancel{\sqrt{5}} \times \frac{1}{\cancel{\sqrt{5}}} + 4 + 3\sqrt{5} \\ &= 19 + 3\sqrt{5} \end{aligned}$$