



2013 Trial Examination

FORM VI

MATHEMATICS 2 UNIT

Thursday 1st August 2013

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 98 boys

Examiner

BDD

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The value of $\int_0^2 (6x^2 + 1) dx$ is:

- (A) 17
- (B) 18
- (C) 24
- (D) 66

QUESTION TWO

The line intersecting the x -axis at $x = -1$ and passing through the point $A(1, -4)$ is represented by which of the following equations?

- (A) $x + 2y - 1 = 0$
- (B) $x + 2y + 1 = 0$
- (C) $2x - y - 2 = 0$
- (D) $2x + y + 2 = 0$

QUESTION THREE

The quadratic equation $2x^2 + 12x - 9 = 0$ has roots α and β . The value of $\alpha^2\beta + \alpha\beta^2$ is:

- (A) -108
- (B) -27
- (C) 27
- (D) 108

QUESTION FOUR

What is the sum of the first ten terms of the series $96 - 48 + 24 - 12 + \dots$?

- (A) 63.9375
- (B) 191.8125
- (C) -32.736
- (D) 98.208

QUESTION FIVE

Which of the following does $\frac{d}{dx}(e^3)$ equal?

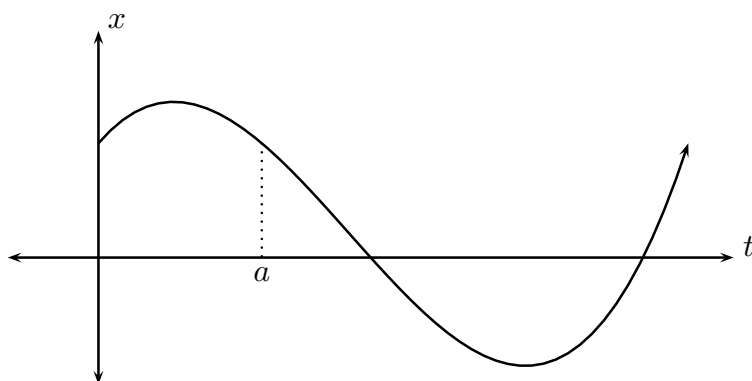
- (A) $3e^2$
- (B) e^3
- (C) 0
- (D) $\frac{1}{4}e^4$

QUESTION SIX

Which of the following statements is INCORRECT?

- (A) $\log a^n = n \log a$
- (B) $\log ab = \log a + \log b$
- (C) $\log(a - b) = \frac{\log a}{\log b}$
- (D) $\log e = 1$

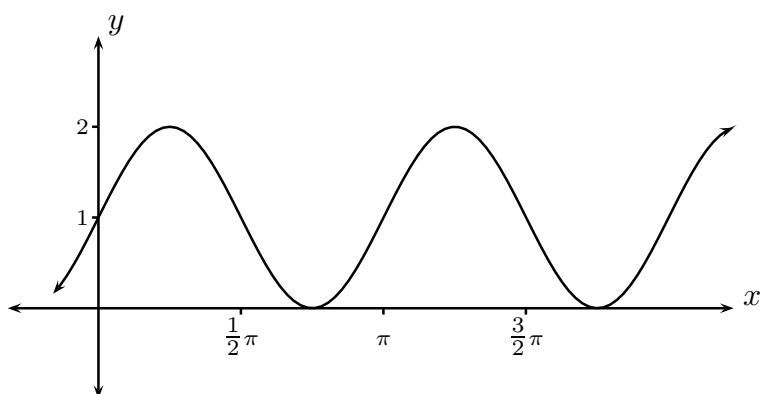
QUESTION SEVEN



A particle's motion is described by the cubic graph above. Which of the following statements is NOT true of the particle at time $t = a$?

- (A) The particle's velocity is negative.
- (B) The particle has positive acceleration.
- (C) The particle is moving towards the origin.
- (D) The particle has returned to its initial position.

QUESTION EIGHT



The equation of the graph sketched above could be:

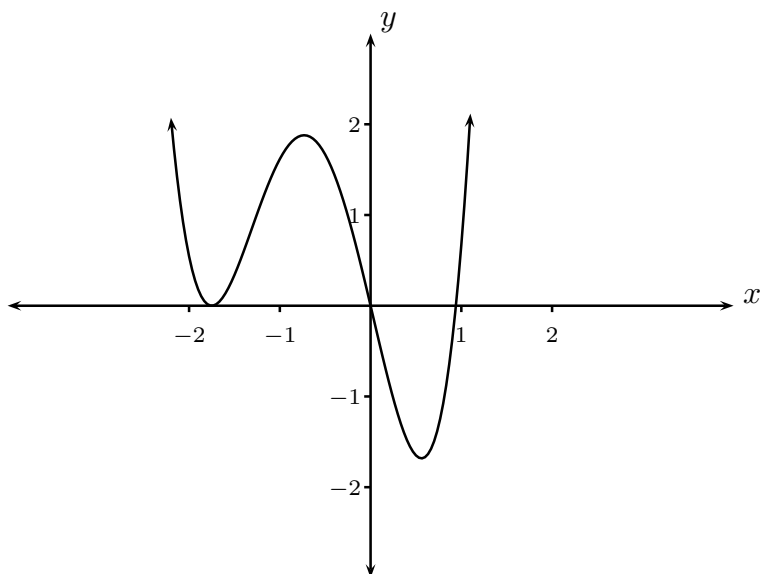
- (A) $y = 1 + \sin 2x$
- (B) $y = 1 - \sin 2x$
- (C) $y = 1 + 2 \sin 2x$
- (D) $y = 1 - 2 \sin x$

QUESTION NINE

Which of the following statements is NOT true of the function $y = x^4 + 4x^2$?

- (A) It is even.
- (B) It has a single stationary point at $x = 0$.
- (C) It has a single x -intercept at $x = 0$.
- (D) It has a single point of inflexion at $x = 0$.

QUESTION TEN



The diagram above shows the graph of a function $y = f(x)$. A pupil draws the graph of $y = 2 - |x|$ on the diagram in order to determine the number of solutions to the equation $f(x) = 2 - |x|$. His answer should be:

- (A) 0
- (B) 1
- (C) 2
- (D) 4

_____ End of Section I _____

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

- (a) Find the value of $\frac{e^x}{1+x^2}$ when $x = -3$. Give your answer correct to 3 decimal places. **1**
- (b) Differentiate:
- (i) $y = \cos 2x$ **1**
- (ii) $y = \ln(3x + 1)$ **1**
- (iii) $y = e^{3x}$ **1**
- (c) Find the exact value of $\tan \frac{2\pi}{3}$. **1**
- (d) Rationalise the denominator of $\frac{1}{3 - \sqrt{5}}$. **1**
- (e) Find the following integrals:
- (i) $\int (3x^2 + 4x) dx$ **1**
- (ii) $\int \frac{5}{x} dx$ **1**
- (iii) $\int (2x + 1)^5 dx$ **1**
- (f) Find the area of a sector subtending an angle of 6 radians at the centre of a circle of radius 3 cm. **1**
- (g) Find the one-hundredth term of the arithmetic sequence with first term 8 and common difference 3. **1**
- (h) Solve $2 \cos \theta - 1 = 0$, for $0 \leq \theta \leq 2\pi$. **2**
- (i) Draw a one-third page sketch of the parabola $x^2 = -8y$, carefully marking the focus and directrix. **2**

QUESTION TWELVE (15 marks) Use a separate writing booklet.

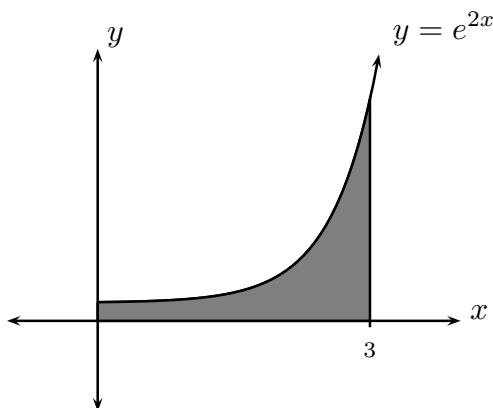
Marks

(a) Find the equation of the tangent to $y = x^2 + 4x$ at $x = 1$.

2

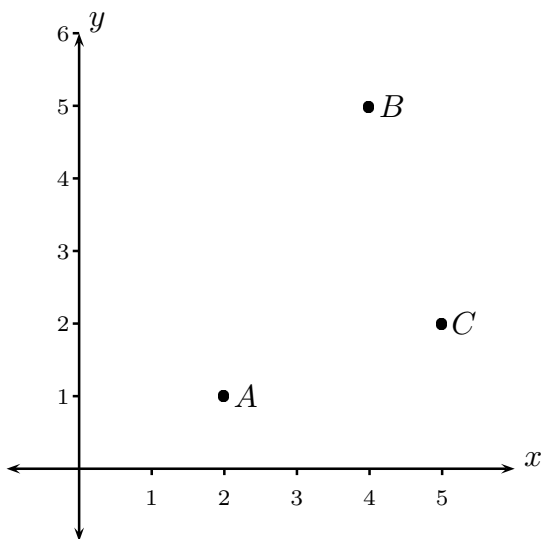
(b)

2



The graph above shows the area bounded by the curve $y = e^{2x}$, the line $x = 3$ and the coordinate axes. Find the exact shaded area.

(c)



The points $A(2, 1)$, $B(4, 5)$ and $C(5, 2)$ have been marked in the coordinate plane above.

(i) Show that the equation of the line passing through A and B is $2x - y - 3 = 0$.

2

(ii) Determine the length of interval AB .

1

(iii) Find the perpendicular distance from the point C to the line AB .

1

(iv) Hence find the area of triangle ABC .

1

(d) Find the domain and range of $y = \sqrt{2x - 6}$.

2

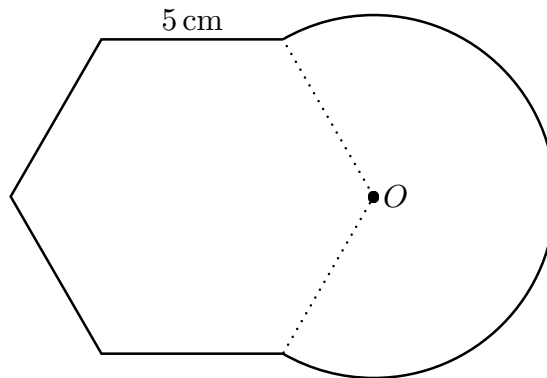
(e) Solve the quadratic inequation $x^2 + 2x - 3 < 0$.

2

QUESTION TWELVE (Continued)

(f)

2

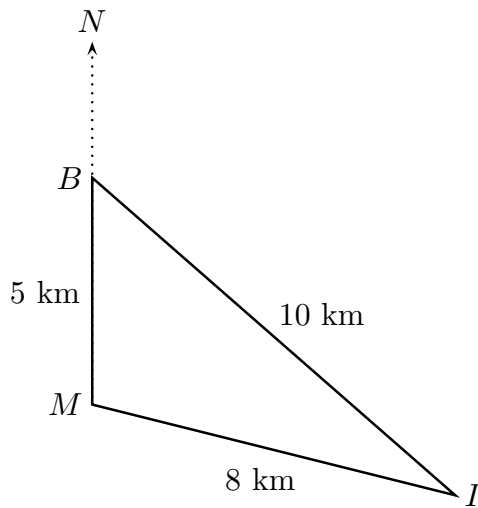


The diagram above shows a regular hexagon joined to the radii of a sector. The side length of the hexagon is 5 cm. Find the exact perimeter of the resulting shape.

QUESTION THIRTEEN (15 marks) Use a separate writing booklet.

Marks

(a)



An island I , a buoy B and the mainland M lie on the vertices of a triangle, as in the diagram above. The distance from M to B is 5 km, from B to I is 10 km and from I to M is 8 km. The buoy is directly north of the mainland.

(i) Use the cosine rule to find $\angle MBI$, correct to the nearest minute.

2

(ii) What is the true bearing of the island from the buoy?

1

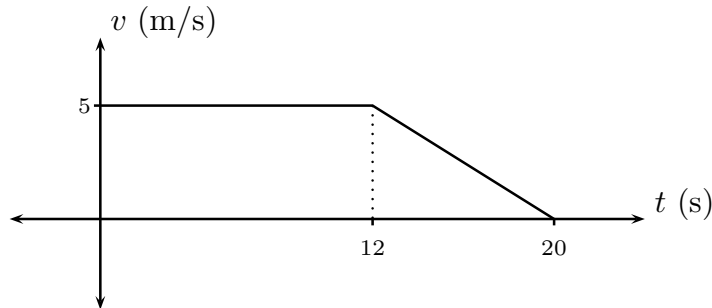
(b) Differentiate $y = \frac{e^{5x} + 1}{e^x}$.

2

QUESTION THIRTEEN (Continued)

- (c) Consider the region bounded by the curve $y = \sec x$, the x -axis and the lines $x = 0$ and $x = \frac{\pi}{3}$. Find the volume obtained by rotating this region about the x -axis. 2

- (d) 1



The velocity-time graph of a particle is shown above. Find the distance travelled in the first 20 seconds.

- (e) A particle travelling in one dimension has velocity function $v = 6 - 2t$, where v is in metres per second and t is in seconds. The particle is initially seven metres to the right of the origin. Assume that the positive direction is to the right.
- (i) Find the particle's acceleration function. 1
 - (ii) Find the particle's displacement function. 1
 - (iii) When is the particle at rest and what is its displacement at this time? 2
 - (iv) Draw a one-third page sketch of the particle's displacement function, showing the intercepts with the axes and the vertex of this parabola. 2
 - (v) What is the total distance travelled over the first eight seconds? 1

QUESTION FOURTEEN (15 marks) Use a separate writing booklet.

Marks

(a) Consider the function $f(x) = x^4 - 4x^3 + 5$.

(i) Find the coordinates of the stationary points of $y = f(x)$.

3

(ii) Determine the nature of the stationary points.

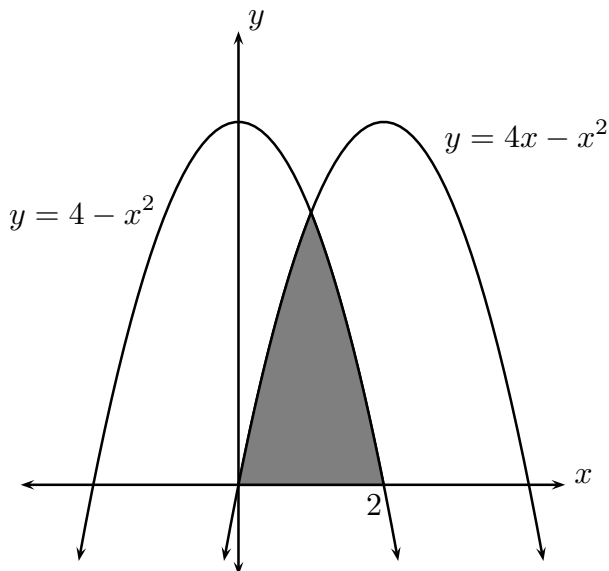
2

(iii) Sketch the graph of $y = f(x)$, showing the stationary points and y -intercept.
You need not find any x -intercepts.

2

(b)

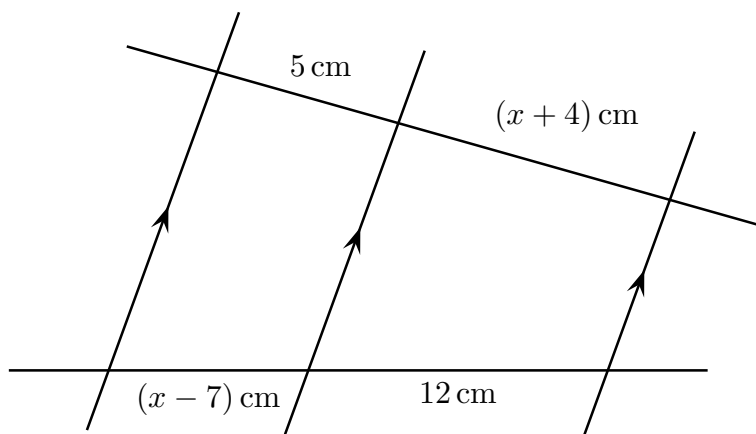
3



Find the area of the region shaded in the diagram above.

(c)

2



Find the value of x in the diagram above, giving a reason.

(d) A point $P(x, y)$ moves so that its distance from $A(6, 1)$ is twice its distance from $B(-3, 4)$.

(i) Show that the locus of P is a circle.

2

(ii) Find the centre and radius of the circle.

1

QUESTION FIFTEEN (15 marks) Use a separate writing booklet.

Marks

- (a) Find the exact value of $\cos \theta$ given that $\tan \theta = 7$ and $\sin \theta < 0$. 2
- (b) (i) Show that $3x^2 + 4x + 5$ is positive definite. 1
- (ii) Explain why the function $y = x^3 + 2x^2 + 5x + 7$ is always increasing. 1
- (c) Prove that $\sin \theta \tan \theta + \cos \theta = \sec \theta$. 2
- (d) Prove that $f(x) = \frac{2x}{x^2 + 1}$ is an odd function. 1
- (e) (i) Differentiate xe^x . 1
- (ii) Hence find $\int xe^x dx$. 1
- (f) The rate of elimination $\frac{dQ}{dt}$ of a drug by the kidneys is given by the equation

$$\frac{dQ}{dt} = -kQ$$

where k is a constant and Q is the quantity of drug present in the blood. In this question, t is measured in minutes and Q in milligrams.

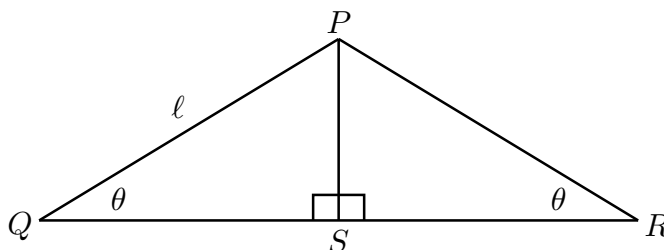
- (i) Show that $Q = Q_0 e^{-kt}$ satisfies the equation $\frac{dQ}{dt} = -kQ$. 1
- (ii) The initial quantity of drug present was measured to be 100 mg and at time $t = 20$ minutes, the quantity was 74 mg. Find the values of Q_0 and k . Give k correct to five decimal places and Q_0 to the nearest mg. 2
- (iii) What is the initial rate of elimination of the drug? Give your answer correct to one decimal place. 1
- (iv) How long is it until only half the original quantity of drug remains? Give your answer correct to the nearest minute. 2

The exam continues on the next page

QUESTION SIXTEEN (15 marks) Use a separate writing booklet.

Marks

- (a) In the diagram below, $\angle PQS = \angle PRS = \theta$ and $PQ = \ell$.



- (i) Prove that $\triangle PQS \equiv \triangle PRS$.
- (ii) Give a reason why $QS = RS$.
- (iii) Show that $QR = 2\ell \cos \theta$.
- (iv) Show that the area of $\triangle PQR$ is given by

$$A = \ell^2 \cos \theta \sin \theta.$$

1
1
1
1

- (v) Use calculus to find the value of θ that gives the maximum area of $\triangle PQR$.

3

- (b) A university student is planning to use a cash account containing \$50 000 to help fund his expenses. The account earns interest at 6% per annum, compounded monthly. At the end of each month interest is added to the account balance and then the student withdraws \$1500. Let A_n be the amount of money remaining in the account at the end of the n th month, following the student's withdrawal.

- (i) Find an expression for A_1 .
- (ii) Find expressions for A_2 and A_3 .
- (iii) After how many months will the account have a balance of zero dollars?
Give your answer to the nearest month.

1
2
2

- (c) Solve the following equation, for $0 \leq \theta \leq 2\pi$:

$$3 \sin^2 \theta + 3 \cos^2 \theta + 3 \tan^2 \theta + 3 \cot^2 \theta + 3 \sec^2 \theta + 3 \operatorname{cosec}^2 \theta = 29$$

3

_____ End of Section II _____

END OF EXAMINATION

B L A N K P A G E

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$



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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER:

Question One

A B C D

Question Two

A B C D

Question Three

A B C D

Question Four

A B C D

Question Five

A B C D

Question Six

A B C D

Question Seven

A B C D

Question Eight

A B C D

Question Nine

A B C D

Question Ten

A B C D

2 Unit Trial HSC Solutions 2013

MCQ

1. B

2. D

3. C

4. A

5. C

6. C

7. B

8. A

9. D

10. D

QUESTION 11

(a) $\frac{e^{-3}}{1+(-3)^2} \approx 0.005$ ✓

(b) (i) $\frac{dy}{dx} = -2\sinh 2x$ ✓

(ii) $\frac{d}{dx} \ln(3x+1) = \frac{3}{3x+1}$ ✓

(iii) $\frac{d}{dx} e^{3x} = 3e^{3x}$ ✓

(c) $\tan \frac{2\pi}{3} = \tan \left(\pi - \frac{\pi}{3} \right)$
 $= -\tan \frac{\pi}{3}$
 $= -\sqrt{3}$ ✓

(d) $\frac{1}{3-\sqrt{5}} = \frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$
 $= \frac{3+\sqrt{5}}{9-5}$
 $= \frac{3+\sqrt{5}}{4}$ ✓

(e) (i) $\int (3x^2 + 4x) dx = x^3 + 2x^2 + c$ ✓

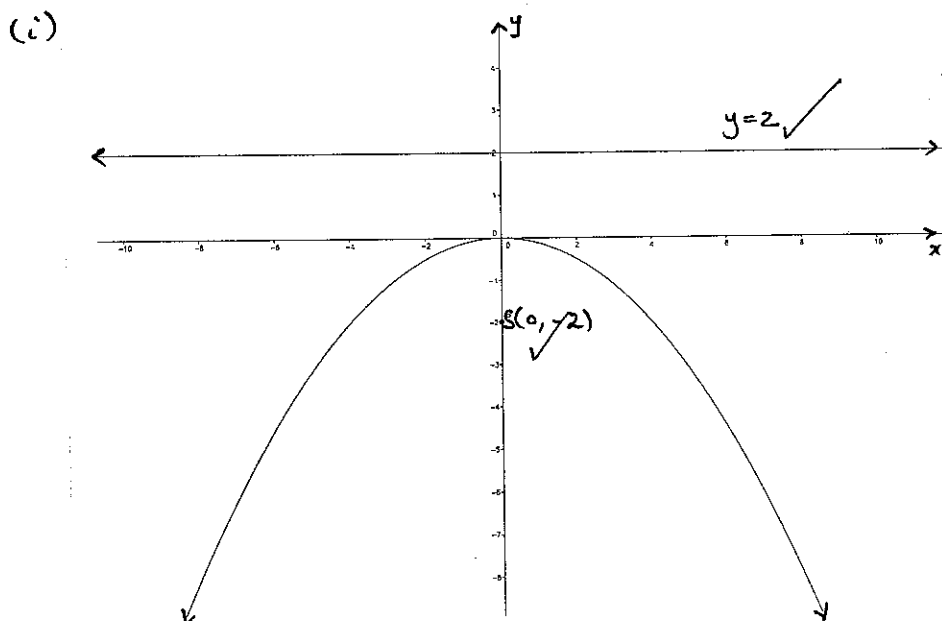
(ii) $\int \frac{5}{x} dx = 5 \log x + c$ ✓

(iii) $\int (2x+1)^5 dx = \frac{(2x+1)^6}{12} + c$ ✓

$$\begin{aligned}
 (f) \quad A &= \frac{1}{2} r^2 \theta \\
 &= \frac{1}{2} \cdot 3^2 \text{ cm}^2 \cdot 6 \\
 &= 27 \text{ cm}^2 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 (g) \quad t_n &= 8 + 3(n-1) \\
 \text{so } t_{100} &= 8 + 3 \times 99 \\
 &= 305 \quad \checkmark
 \end{aligned}$$

$$\begin{aligned}
 (h) \quad 2 \cos \theta - 1 &= 0, \quad 0 \leq \theta \leq 2\pi. \\
 \cos \theta &= \frac{1}{2} \\
 \text{so } \theta &= \frac{\pi}{3} \checkmark \text{ or } \frac{5\pi}{3} \checkmark
 \end{aligned}$$



QUESTION 12

$$(a) \text{ For } y = x^2 + 4x, \quad \frac{dy}{dx} = 2x + 4.$$

$$\text{At } x=1, \quad \frac{dy}{dx} = 6 \quad \& \quad y = 5.$$

$$\text{Point: } (1, 5); \quad \text{gradient: } 6$$

$$\text{Equation of tangent: } y - 5 = 6(x - 1)$$

$$\therefore y = 6x - 1$$

$$\begin{aligned}
 (b) \text{ Area given by } \int_0^3 e^{2x} dx &= \left. \frac{1}{2} e^{2x} \right|_0^3 \\
 &= \frac{1}{2} (e^6 - 1)
 \end{aligned}$$

$$(c) \quad A(2, 1) \quad B(4, 5) \quad C(5, 2)$$

$$(i) \text{ Eq}^n \text{ of line: } y - 1 = \frac{5-1}{4-2} (x-2)$$

$$y = 2x - 3$$

$$\text{so } 2x - y - 3 = 0, \quad \text{as required.}$$

OR substitute coordinates of A, B to show points lie on the line.

$$\begin{aligned}
 \text{(ii)} \quad AB^2 &= (4-2)^2 + (5-1)^2 \\
 &= 4 + 16 \\
 &= 20
 \end{aligned}$$

$$\text{so } AB = 2\sqrt{5}$$

$$\begin{aligned}
 \text{(iii)} \quad d_{\perp} &= \frac{|2(5) - (2) - 3|}{\sqrt{4+1}} \\
 &= \frac{|15|}{\sqrt{5}} \\
 &= \sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad A_{\Delta ABC} &= \frac{1}{2} (AB) d_{\perp} \\
 &= \frac{1}{2} \cdot 2\sqrt{5} \cdot \sqrt{5} \\
 &= 5 \text{ u}^2
 \end{aligned}$$

(d) For $\sqrt{2x-6}$, domain is all real x such that $2x-6 \geq 0$; i.e. all real x : $x \geq 3$.

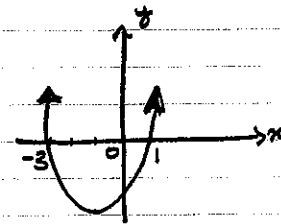
The function $\sqrt{2x-6}$ is increasing for all $x \geq 3$, hence its minimum value lies at the lower bound of its domain ($x=3$):

$$\min \sqrt{2x-6} = \sqrt{2(3)-6} = 0$$

Since $y = \sqrt{2x-6}$ increases without bound for all $x \geq 3$, the range is then all real $y \geq 0$.

$$\begin{aligned}
 \text{(e)} \quad x^2 + 2x - 3 &< 0 \\
 \text{i.e. } (x+3)(x-1) &< 0
 \end{aligned}$$

Solution is $-3 < x < 1$.



(f) Regular hexagon composed of six congruent equilateral triangles of side length 2cm.

\therefore internal angle is $2 \times 60^\circ = 120^\circ$.

Reflex $\angle \therefore 360^\circ - 120^\circ = 240^\circ$

$$\text{(u) (Arc length)} = \frac{240^\circ}{360^\circ} \times 2\pi \times 2\text{cm} = \frac{8\pi}{3} \text{ cm}$$

$$(13) \text{ (Hexagonal contribution)} = 4 \times 2 \text{ cm} = 8 \text{ cm}.$$

$$\therefore \text{ (perimeter)} = 8 \text{ cm} + \frac{8\pi}{3} \text{ cm}$$

$$= 8 \left(1 + \frac{\pi}{3} \right) \text{ cm}.$$

QUESTION 13

$$(a) \text{ (i)} \quad 8^2 = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos(\angle MBI)$$

$$\Rightarrow \cos(\angle MBI) = 0.61$$

$$\text{so } \angle MBI \doteq 52^\circ 25'$$

$$(ii) \text{ True bearing: } 180^\circ - 52^\circ 25' = 127^\circ 35' \text{ T}$$

$$(b) \quad y = \frac{e^{5x} + 1}{e^x}$$

$$= e^{4x} + e^{-x}$$

$$\text{so } \frac{dy}{dx} = 4e^{4x} - e^{-x}$$

(c)

$$V = \pi \int_0^{\pi/3} y^2 dx$$

$$= \pi \int_0^{\pi/3} \sec^2 x dx$$

$$= \pi \left[\tan x \right]_0^{\pi/3}$$

$$= \pi \left[\sqrt{3} - 0 \right]$$

$$= \sqrt{3} \pi \text{ u}^3.$$

$$\begin{aligned}
 \text{(d)} \quad |x| &= \int_0^{12} v_1(t) dt + \int_{12}^{20} v_2(t) dt \\
 &= (12-0)(5) + \frac{1}{2}(20-12)(5) \\
 &= 60 + 20 \\
 &= 80 \text{ m}
 \end{aligned}$$

$$\text{(e)} \quad v = 6 - 2t \text{ m/s}; \quad x(0) = 7 \text{ m}.$$

$$\text{(i)} \quad a = \frac{dv}{dt} = -2 \text{ m/s}^2.$$

$$\begin{aligned}
 \text{(ii)} \quad x &= \int 6 - 2t dt \\
 &= 6t - t^2 + C
 \end{aligned}$$

When $t=0$, $x=7$, so $C=7$.

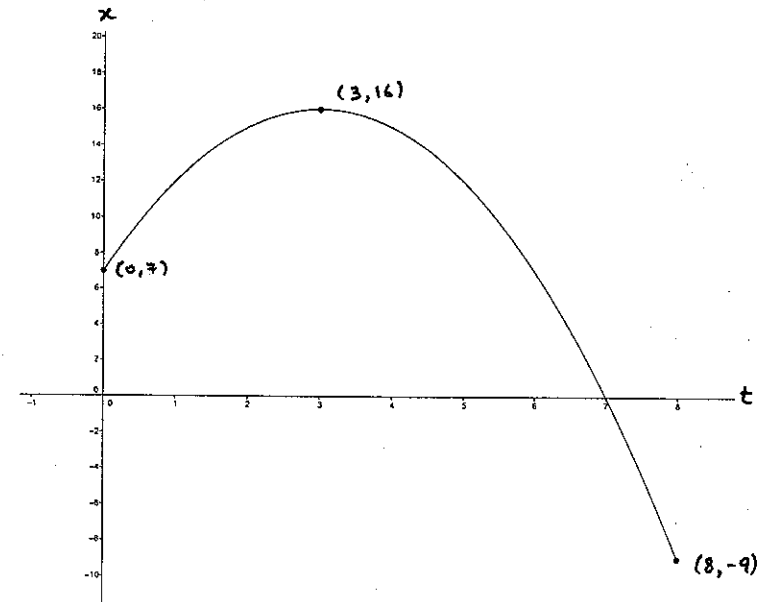
$$\therefore x(t) = 6t - t^2 + 7.$$

(iii) Particle at rest when $v(t) = 0$. Then

$$6 - 2t = 0 \Rightarrow t = 3 \text{ s}.$$

$$\begin{aligned}
 \text{When } t=3, \quad x(3) &= 6(3) - (3)^2 + 7 \\
 &= 16 \text{ m}.
 \end{aligned}$$

(iv)



(v) Total distance: from $t=0$ to $t=3$, distance is $16-7=9\text{m}$;
 from $t=3$ to $t=7$, distance is 16m ;
 from $t=7$ to $t=8$, distance is 9m .

Total $\therefore 34\text{m}$.

QUESTION 14

(a) $f(x) = x^4 - 4x^3 + 5$.

(i) Stationary points w for x : $f'(x) = 0$.

Now, $f'(x) = 4x^3 - 12x^2$, so solving

$$4x^2(x-3) = 0 \Rightarrow x = 0 \text{ or } x = 3. \checkmark$$

For $x = 0$, $f(0) = 5 \rightarrow (0, 5)$ is a stationary point. \checkmark

For $x = 3$, $f(3) = -22 \rightarrow (3, -22)$ is a stationary point. \checkmark

(ii) Nature:

$$f''(x) = 12x^2 - 24x$$

$$= 12x(x-2)$$

Now, $f''(0) = 0 \rightarrow$ indeterminate: use table.

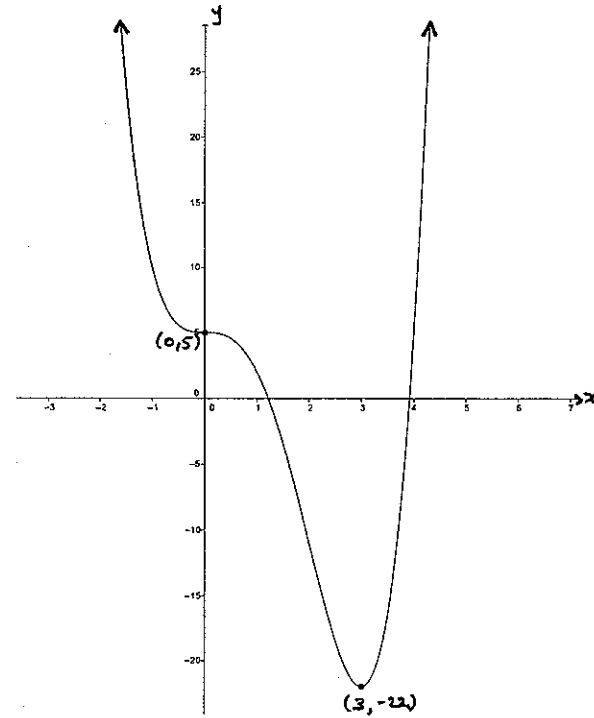
$f''(3) = 36 > 0 \rightarrow$ local minimum at $(3, -22)$. \checkmark

For $(0, 5)$:

x	-1	0	1
$f''(x)$	36	0	-12
concav.	∪	.	∩

\therefore horizontal point of inflexion at $(0, 5)$. \checkmark

(iii)



\checkmark for correct information from above

\checkmark right shape

(b) Intersection of graphs occurs for x such that

$$4 - x^2 = 4x - x^2$$

i.e. $x = 4$.

Area, A , is compound, so

$$A = \int_0^1 (4x - x^2) dx + \int_1^2 (4 - x^2) dx \checkmark$$

\checkmark at least one correct integration + substitution without simplification.

$$= 2x^2 - \frac{x^3}{3} \Big|_0^1 + 4x - \frac{x^3}{3} \Big|_1^2$$

$$= \left(2 - \frac{1}{3}\right) - (0) + \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right)$$

$$= \frac{10}{3} \text{ or } 3\frac{1}{3} \text{ u}^2. \checkmark$$

(c) By ratio of intercepts of transversals on parallel lines,

$$\frac{x+4}{5} = \frac{12}{x-7}$$

\checkmark with reason (saying "only 1 ratio of intercepts okay").

so

$$(x-7)(x+4) = 60$$

$$x^2 - 3x - 88 = 0$$

$$(x-11)(x+8) = 0$$

$$\therefore x = 11 \text{ cm or } x = -8 \text{ cm.}$$

$$x = -8 \text{ cm does not work. So } x = 11 \text{ cm. } \checkmark$$

(d) (i) (Distance of P to A) is 2 x (distance of P to B)

i.e.

$$\overline{AP} = 2\overline{BP}$$

$$\overline{AP}^2 = 4\overline{BP}^2$$

$$(x-6)^2 + (y-1)^2 = 4[(x+3)^2 + (y-4)^2]$$

$$x^2 - 12x + 36 + y^2 - 2y + 1 = 4[x^2 + 6x + 9 + y^2 - 8y + 16] \checkmark$$

$$= 4x^2 + 24x + 36 + 4y^2 - 32y + 64$$

which simplifies to

$$x^2 + 12x + y^2 - 10y + 21 = 0$$

$$x^2 + 12x + 6^2 + y^2 - 10y + (-5)^2 = -21 + 6^2 + (-5)^2$$

$$(x+6)^2 + (y-5)^2 = 40$$

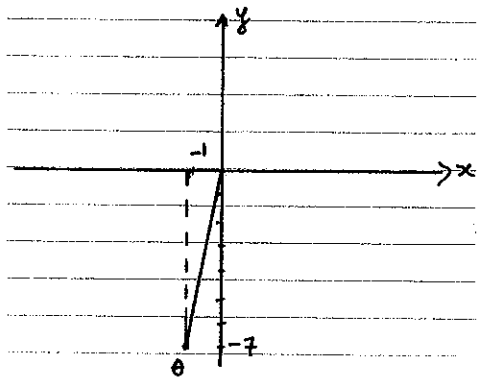
This is the equation of a circle. \checkmark

(ii) From (i), centre: (-6, 5)

radius: $2\sqrt{10}$. \checkmark

QUESTION 15

(a) Since $\tan \theta > 0$, $\sin \theta < 0$, θ lies in quadrant III.



Now, $\tan \theta = \frac{y}{x}$ for any point (x, y) on the ray given by θ .

Choose, then, $x = -1$, $y = -7$.

$$\begin{aligned} \text{Then } r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1 + 49} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \quad \checkmark \end{aligned}$$

$$\begin{aligned} \text{So } \cos \theta &= \frac{x}{r} \\ &= \frac{-1}{5\sqrt{2}} \quad \checkmark \end{aligned}$$

OR Given $\tan \theta = 7$,

$$\tan^2 \theta = 49$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} = 49$$

$$\sin^2 \theta = 49 \cos^2 \theta$$

$$1 - \cos^2 \theta = 49 \cos^2 \theta$$

$$\text{So } \cos^2 \theta = \frac{1}{50}$$

$$\text{i.e. } \cos \theta = \pm \frac{1}{5\sqrt{2}} \quad \checkmark$$

Since θ is in quadrant III,

$$\cos \theta = -\frac{1}{5\sqrt{2}} \quad \checkmark$$

(b) (i) $3x^2 + 4x + 5$ has discriminant

$$\Delta = 4^2 - 4(3)(5) < 0$$

and coefficient of x^2 is positive. \checkmark

Hence the graph of $3x^2 + 4x + 5$ lies entirely above the x -axis: the quadratic is positive definite.

(ii) $y = x^3 + 2x^2 + 5x + 7$

$$\text{so } \frac{dy}{dx} = 3x^2 + 4x + 5$$

which, by (i), is positive for all real x ;

i.e. $\frac{dy}{dx} > 0$ for all x .

Hence $y = x^3 + 2x^2 + 5x + 7$ is increasing always. \checkmark

$$\begin{aligned}
 \text{(c)} \quad \sin\theta \tan\theta + \cos\theta &= \sin\theta \cdot \frac{\sin\theta}{\cos\theta} + \cos\theta \\
 &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} \quad \checkmark \\
 &= \frac{1}{\cos\theta} \\
 &= \sec\theta \quad // \quad \checkmark
 \end{aligned}$$

(d) $f(x)$ is odd if $f(-x) = -f(x)$.

Now $f(x) = \frac{2x}{x^2+1}$, so

$$f(-x) = \frac{2(-x)}{(-x)^2+1}$$

$$= -\frac{2x}{x^2+1}$$

$$= -f(x) \quad \checkmark$$

Hence, $f(x)$ is odd. \checkmark

$$\text{(e)} \quad \text{(i)} \quad \frac{d}{dx} x e^x = x e^x + e^x \quad \checkmark$$

$$\text{(ii)} \quad \int \frac{d}{dx} x e^x dx = \int (x e^x + e^x) dx$$

$$x e^x = \int x e^x dx + \int e^x dx$$

$$= \int x e^x dx + e^x$$

$$\therefore \int x e^x dx = x e^x - e^x + C \quad \checkmark \quad \begin{array}{l} \text{May leave} \\ \text{constant} \\ \text{out} \end{array}$$

$$\text{(f)} \quad \text{(i)} \quad Q = Q_0 e^{-kt} \rightarrow \frac{dQ}{dt} = -k Q_0 e^{-kt} = -kQ \quad \checkmark //$$

(ii) At $t=0$, $Q=100$ mg, so $100 \text{ mg} = Q_0 \cdot e^0$

$\therefore Q_0 = 100 \text{ mg}$. \checkmark

At $t=20$ min, $Q=74$ mg, so $74 = 100 e^{-20k}$

$$\log\left(\frac{74}{100}\right) = -20k$$

$$\therefore k = \frac{\log\left(\frac{74}{100}\right)}{-20}$$

$$\approx 0.01506 \text{ min}^{-1} \quad \checkmark$$

(iii) Since $\frac{dQ}{dt} = -kQ$,

when $t=0$, $Q = Q_0 = 100 \text{ mg}$.

Hence the initial rate of elimination is given by

$$\begin{aligned}\frac{dQ}{dt} &= -kQ_0 \\ &= -(0.01506) \times 100 \text{ mg} \cdot \text{min}^{-1} \\ &\approx -1.5 \text{ mg} \cdot \text{min}^{-1}. \checkmark\end{aligned}$$

(iv) We need t such that

$$50 \text{ mg} = 100 \text{ mg} \cdot e^{-kt}$$

$$\text{so } \frac{1}{2} = e^{-kt} \quad \checkmark$$

$$\log \frac{1}{2} = -kt$$

$$\therefore t = \frac{\log 2}{k}$$

$$\approx 46 \text{ min}. \quad \checkmark$$

QUESTION 16

(a)(i) In $\triangle PQS$ & $\triangle PRS$:

(α) $\angle PQS = \angle PRS$ (given)

(β) $\angle PSQ = \angle PSR$ (given)

(γ) $PS = PS$ (common)

By angle-angle-side test, $\triangle PQS \equiv \triangle PRS$.

(ii) $QS = RS$ since they are corresponding sides of two congruent triangles.

(iii) $\cos \theta = \frac{QS}{l}$

$$QS = l \cos \theta$$

Since $QS = RS$, $QR = QS + SR$

$$= 2QS$$

$$= 2l \cos \theta, \text{ as required.}$$

(iv) Area, A , given by

$$A = \frac{1}{2} \cdot l \cdot QR \cdot \sin \theta$$

$$= \frac{1}{2} \cdot l \cdot 2l \cos \theta \sin \theta$$

$$= l^2 \cos \theta \sin \theta \quad //$$

$$(v) \frac{dA}{d\theta} = l^2 [\sin\theta(-\cos\theta) + \cos\theta(\sin\theta)]$$

$$= l^2 [\cos^2\theta - \sin^2\theta]$$

Extrema for θ : $\frac{dA}{d\theta} = 0$, so solve:

$$l^2 (\cos^2\theta - \sin^2\theta) = 0$$

$$\sin^2\theta = \cos^2\theta$$

$$\tan^2\theta = 1$$

$$\therefore \tan\theta = \pm 1$$

implies $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ for $0 \leq \theta \leq \pi$.

but $\theta \neq \frac{3\pi}{4}$ since angle sum of triangle would exceed π . So $\theta = \frac{\pi}{4}$ only.

$$\text{Now, } \frac{d^2A}{d\theta^2} = l^2 [2\cos\theta(-\sin\theta) - 2\sin\theta\cos\theta]$$

$$= -4l^2 \cos\theta\sin\theta$$

$$\text{Evaluated at } \theta = \frac{\pi}{4} \rightarrow \frac{d^2A}{d\theta^2} = -4l^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} < 0$$

so $\theta = \frac{\pi}{4}$ is a minimum.

The maximum area is therefore,

$$A = l^2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{l^2}{2}$$

(b) Let $A_0 = \$50,000$, $M = \$1,500$, $R = \left(\frac{1+0.06}{12}\right) = 1.005$

$$(i) A_1 = A_0 R - M$$

$$= 50,000 (1.005) - 1,500$$

$$(ii) A_2 = A_1 R - M$$

$$= (A_0 R - M) R - M$$

$$= A_0 R^2 - MR - M$$

$$= 50,000 (1.005)^2 - 1,500 (1 + 1.005)$$

$$A_3 = A_2 R - M$$

$$= [A_0 R^2 - MR - M] R - M$$

$$= A_0 R^3 - MR^2 - MR - M$$

$$= A_0 R^3 - M(1 + R + R^2)$$

$$= 50,000 (1.005)^3 - 1,500 (1 + 1.005 + 1.005^2)$$

(ii) The pattern developed in (i) suggests for A_n :

$$A_n = A_0 R^n - M(1 + R + R^2 + \dots + R^{n-1})$$

$$= A_0 R^n - M \cdot \frac{R^n - 1}{R - 1}, \text{ since } 1 + R + R^2 + \dots + R^{n-1} \text{ is geometric.}$$

The account will reach zero for n such that $A_n = 0$.

So solve for n in the following:

$$0 = A_0 R^n - M \cdot \frac{R^n - 1}{R - 1}$$

$$= A_0 (R - 1) R^n - M (R^n - 1)$$

$$= A_0 (R - 1) R^n - M R^n + M$$

$$= [A_0 (R - 1) - M] R^n + M$$

$$\text{So } R^n = \frac{M}{M - A_0 (R - 1)}$$

$$n \log R = \log \left[\frac{M}{M - A_0 (R - 1)} \right]$$

$$n = \frac{\log \left[\frac{M}{M - A_0 (R - 1)} \right]}{\log R}$$

$$\text{So } n = \frac{\log \left[\frac{1500}{1500 - 50000 \times 0.005} \right]}{\log(1.005)}$$

$$= 36.5553 \dots$$

$$\hat{=} 37 \text{ months.}$$

$$(c) \quad 3 \sin^2 \theta + 3 \cos^2 \theta + 3 \tan^2 \theta + 3 \cot^2 \theta + 3 \sec^2 \theta + 3 \csc^2 \theta = 29$$

$$3(\sin^2 \theta + \cos^2 \theta) + 3 \tan^2 \theta + 3 \cot^2 \theta + 3(1 + \tan^2 \theta) + 3(1 + \cot^2 \theta) = 29$$

$$\cancel{3} + 3 \tan^2 \theta + 3 \cot^2 \theta + \cancel{3} + 3 \tan^2 \theta + \cancel{3} + 3 \cot^2 \theta = \cancel{29} 20$$

$$6(\tan^2 \theta + \cot^2 \theta) = 20$$

$$3 \tan^2 \theta + \frac{3}{\tan^2 \theta} = 10$$

$$3 \tan^4 \theta + 3 = 10 \tan^2 \theta$$

$$3 \tan^4 \theta - 10 \tan^2 \theta + 3 = 0$$

$$3 \tan^4 \theta - 9 \tan^2 \theta - \tan^2 \theta + 3 = 0$$

$$3 \tan^2 \theta (\tan^2 \theta - 3) - (\tan^2 \theta - 3) = 0$$

$$(3 \tan^2 \theta - 1)(\tan^2 \theta - 3) = 0$$

$$\therefore \tan^2 \theta = \frac{1}{3} \text{ or } \tan^2 \theta = 3$$

$$\text{So } \tan \theta = \pm \frac{1}{\sqrt{3}} \text{ or } \tan \theta = \pm \sqrt{3}$$

$$\text{For } \tan \theta = \pm \frac{1}{\sqrt{3}}, \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6};$$

and

$$\tan \theta = \pm \sqrt{3}, \quad \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

$$\therefore \theta \in \left\{ \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6} \right\}.$$