

# Sydney Technical High School



## Mathematics Department

TRIAL H.S.C. - MATHEMATICS 2 UNIT

AUGUST 2013

### General Instructions

- Reading time – 5 minutes
- Working Time – 180 minutes.
- Approved calculators may be used.
- Write using blue or black pen.
- A table of Standard Integrals is provided at the back of this paper.
- In Question 11-16, show relevant mathematical reasoning and/or calculations.
- Begin each question on a new side of the answer booklet.
- Marks shown are a guide and may need to be adjusted.
- Full marks may not be awarded for careless work or illegible writing.

NAME \_\_\_\_\_

TEACHER \_\_\_\_\_

Total Marks – 100

**SECTION 1** Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes.

**SECTION 2** Pages 6 – 12

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 mins.

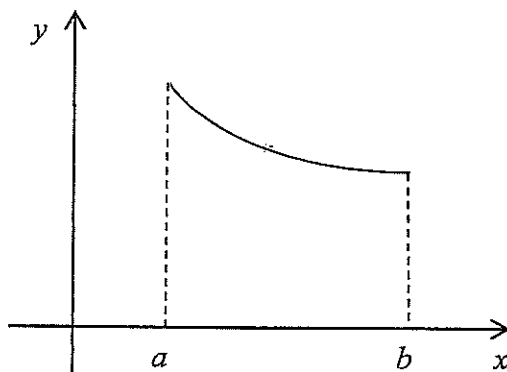
### Question 1

For what values of  $k$  does the equation  $x^2 - 6x - 3k = 0$  have real roots?

- A.  $k \geq -3$
- B.  $k \leq -3$
- C.  $k \geq 3$
- D.  $k \leq 3$

### Question 2

For the function  $y = f(x)$ ,  $a < x < b$  graphed below:



which of the following is true?

- A.  $f'(x) > 0$  and  $f''(x) > 0$
- B.  $f'(x) > 0$  and  $f''(x) < 0$
- C.  $f'(x) < 0$  and  $f''(x) > 0$
- D.  $f'(x) < 0$  and  $f''(x) < 0$

### Question 3

An infinite geometric series has a first term of 8 and a limiting sum of 12.

What is the common ratio?

- A.  $1/6$
- B.  $5/3$
- C.  $1/2$
- D.  $1/3$

**Question 4**

What are the domain and range of the function  $f(x) = \sqrt{4 - x^2}$  ?

- A. Domain:  $-2 \leq x \leq 2$ , Range:  $0 \leq y \leq 2$
- B. Domain:  $-2 \leq x \leq 2$ , Range:  $-2 \leq y \leq 2$
- C. Domain:  $0 \leq x \leq 2$ , Range:  $-4 \leq y \leq 4$
- D. Domain:  $0 \leq x \leq 2$ , Range:  $0 \leq y \leq 4$

**Question 5**

What is the maximum value of  $6 + 2x - x^2$  ?

- A. 6
- B. 1
- C. 7
- D. cannot be determined.

**Question 6**

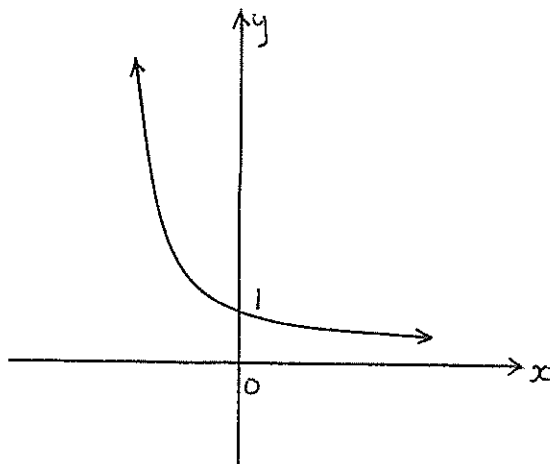
The sine curve with amplitude 3 units and period  $4\pi$  units has equation:

- A.  $y = 4 \sin 3x$
- B.  $y = 3 \sin 4x$
- C.  $y = 3 \sin 2x$
- D.  $y = 3 \sin \frac{x}{2}$

**Question 7**

The illustrated graph could be:

- A.  $y = 2^x$
- B.  $y = -2^{-x}$
- C.  $y = \left(\frac{1}{2}\right)^x$
- D.  $y = \left(\frac{1}{2}\right)^{-x}$



**Question 8**

Janet works out the sum of  $n$  terms of an arithmetic series. Her answer, which is correct, could be:

- A.  $S_n = 2(2^n - 1)$
- B.  $S_n = 9 - 2n$
- C.  $S_n = 8n - n^2$
- D.  $S_n = 7 \times 2^{n-1}$

**Question 9**

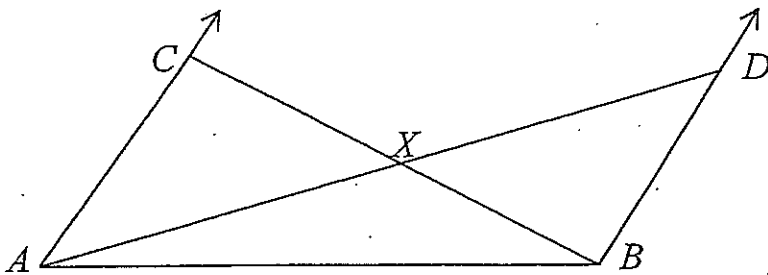


Figure not to scale

In the diagram above:  $AC \parallel BD$ ,  $\angle CAX = 2\angle BAX$ ,  $\angle DBX = 2\angle ABX$ .

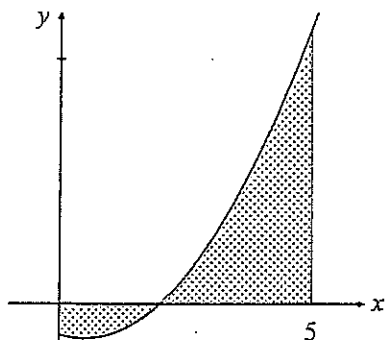
$\angle AXB = ?$

- A.  $150^\circ$
- B.  $120^\circ$
- C.  $160^\circ$
- D.  $135^\circ$

**Question 10**

Which expression below will give the area of the shaded region bounded by the curve

$y = x^2 - x - 2$ , the  $x$ -axis and the lines  $x = 0$  and  $x = 5$ ?



- A.  $A = \left| \int_0^1 (x^2 - x - 2) dx \right| + \int_1^5 (x^2 - x - 2) dx$
- B.  $A = \int_0^1 (x^2 - x - 2) dx + \left| \int_1^5 (x^2 - x - 2) dx \right|$
- C.  $A = \left| \int_0^2 (x^2 - x - 2) dx \right| + \int_2^5 (x^2 - x - 2) dx$
- D.  $A = \int_0^2 (x^2 - x - 2) dx + \left| \int_2^5 (x^2 - x - 2) dx \right|$

END OF SECTION 1

## SECTION 2

90 marks

Attempt Question 11 – 16

Allow about 2 hours 45 minutes for this section.

Answer each question in the writing book provided. Start each question on a new page. All necessary working should be shown. Full marks cannot be given for illegible writing.

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### Question 11 (15 marks)

		Marks
a)	Differentiate:	
	(i) $x \sin 2x$	2
	(ii) $e^{4x} + \frac{1}{x}$	2
	(iii) $\frac{x+1}{3+2x}$	2
b)	Find $\int (4x + 2)^6 dx$	2
c)	Solve for $x$ : $3^{1-x} = \frac{1}{\sqrt{27}}$	2
d)	Solve $(\sin x + 1)(2 \sin x + 1) = 0$ for $0 \leq x \leq 2\pi$	3
e)	Evaluate $\sum_{n=1}^{50} (2n + 3)$	2

Question 12 (15 marks)

Marks

- a) Solve  $|x + 2| = 3x$  2
- b) Use a change of base to evaluate  $\log_2 50$  correct to 2 decimal places. 1
- c) Find the gradient of the curve  $y = e^{\sin x}$  at the point where  $x = 0$ . 2
- d) If  $\alpha$  and  $\beta$  are the roots of  $x^2 + 4x + 1 = 0$ , find without solving:
- i)  $\alpha + \beta$  and  $\alpha\beta$ . 1
- ii)  $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$  2
- e) Differentiate:
- i)  $\ln(x^2 + 3)$  1
- ii)  $\tan^2 4x$  2
- f) Given the parabola  $4y = x^2 - 12$ , find the:
- i) focal length. 1
- ii) coordinates of the focus. 1
- g) Use Simpson's Rule and the five function values in the table below to estimate  $\int_2^4 f(x) dx$ . 2

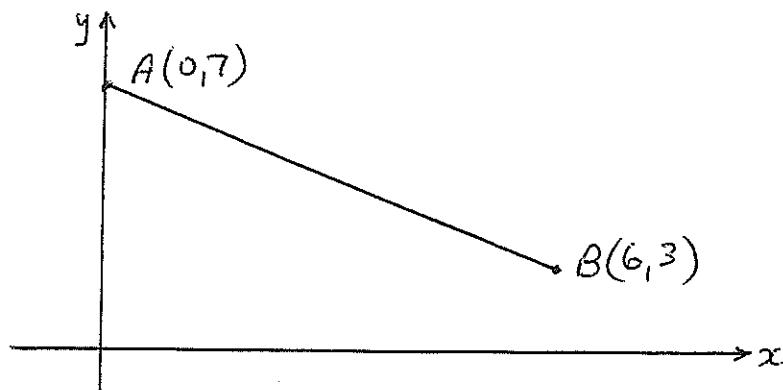
$x$	2	2.5	3	3.5	4
$f(x)$	4	1	-2	3	8

Question 13 (15 marks)

Marks

- a) i) Factorise  $24 + 2m - m^2$  1  
ii) Hence solve  $24 + 2m - m^2 < 0$  1

b)



A(0,7) and B(6,3) are points on the number plane and the equation of AB is  $2x + 3y - 21 = 0$ .

- i) Find the length of AB. 1  
ii) Find the gradient of AB. 1  
iii) Show that the equation of the perpendicular from D(-2,0) to AB is  $3x - 2y + 6 = 0$ . 2  
iv) Find the perpendicular distance from D to AB. 2  
v) Find the coordinates of a point C such that ABCD is a parallelogram. 1
- c) An amount of money doubles in value over a period of  $n$  months. Interest is compounded at the rate of 1% per month. Use the compound interest formula to find the number of months required, correct to the nearest month. 2
- d) i) Find  $\frac{d}{dx}(\operatorname{cosec} x)$  2  
ii) Hence evaluate  $\int_{\pi/3}^{\pi/2} \cot x \operatorname{cosec} x \, dx$ . Give your answer in exact form. 2



**Question 14** (15 marks)

Marks

- a) Find the angle that the line  $3x + 5y + 2 = 0$  makes with the positive direction of the  $x$ -axis. 2
- b) Find: i)  $\int \sin \frac{2x}{3} dx$  2  
 ii)  $\int \frac{x^2 e^{x^2} + 1}{x} dx$  2
- c) Prove that  $\frac{\cos \theta}{1 + \sin \theta} + \frac{\cos \theta}{1 - \sin \theta} = 2 \sec \theta$  2
- d) Solve for  $m$ :  $\log_m 8 + 3 \log_m 4 = 6$ . Leave your answer in exact form. 3

e)

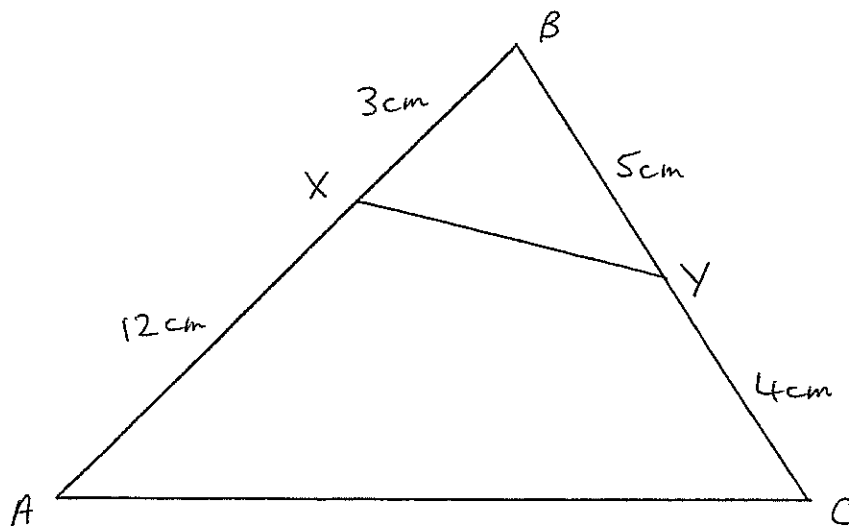


Figure not to scale

- i) Prove that  $\triangle BXY$  is similar to  $\triangle ABC$ . 2
- ii) If angle A is  $35^\circ$ , use the Sine Rule to find the size of angle C, correct to the nearest degree. 2

Question 15 (15 marks)

Marks

a)

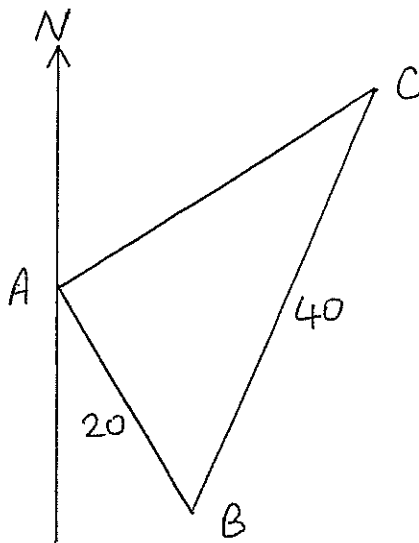
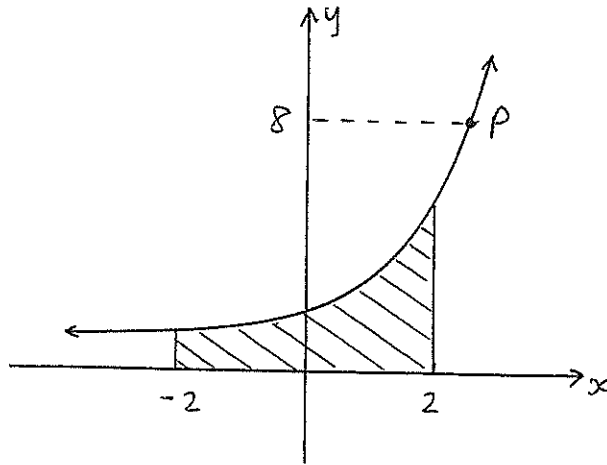


Figure not to scale

Two geologists on a large level area of land drive 20 km from point A on a bearing of  $150^\circ\text{T}$  to a point B. They then drive 40 km on a bearing of  $020^\circ\text{T}$  to point C.

- i) Copy the above diagram into your answer booklet, and find the size of  $\angle ABC$ . 1
  - ii) Use the Cosine Rule to find the distance AC to the nearest kilometre. 2
- b) Consider the curve defined by  $y = 4 - \cos 2x$ .
- i) State the amplitude and period of this curve. 2
  - ii) Sketch the curve for  $0 \leq x \leq \pi$ . Show clear, relevant information on the axes. 2
  - iii) Find the area between the curve and the line  $y = 2$  for  $0 \leq x \leq \pi$ . 3

- c) The diagram shows the curve  $y = e^x$ , a shaded area from  $x = -2$  to  $x = 2$ , and a point P on the curve.



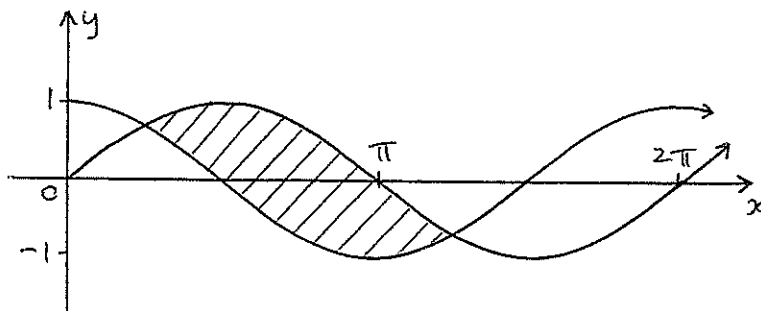
Not to scale

- i) The point P has a y coordinate of 8. Find its x coordinate. 1
- ii) The shaded area is rotated about the x-axis. Find the volume of the generated solid, giving your answer correct to 3 significant figures. 3
- d) Factorise  $x^2 + 2xy + y^2 - 1$  1

**Question 16 (15 marks)**

Marks

a)



The diagram shows the curves  $y = \sin x$  and  $y = \cos x$ .

Write an appropriate integral expression to represent the shaded area above.

2

DO NOT EVALUATE THIS INTEGRAL.

- b) Given the curve  $y = x \log x - x$ , for  $x > 0$ .
- i) Find where the curve crosses the  $x$ -axis. 2
  - ii) Find any stationary points and determine their nature. 2
  - iii) Write a statement for the concavity of this curve. 1
  - iv) Find  $y$  when  $x = e^2$ , and sketch the curve for  $0 < x \leq e^2$  2

c) A man has 1 million ( $10^6$ ) dollars in a bank account. The account earns a steady  $\frac{1}{2}\%$  interest per month, compounded monthly.

At the same time, however, a bank employee is stealing a constant amount  $\$M$  per month from this account, immediately after the month's interest is added to the man's account.

Let  $A_n$  be the amount remaining in the man's account at the end of  $n$  months.

- i) Write an expression for  $A_1$ , and show that  $A_2 = 10^6(1.005)^2 - M(1.005 + 1)$  2
- ii) Write a simplified expression for  $A_n$  2
- iii) Determine the value of  $\$M$  that is stolen each month, such that the man will have only  $\$20$  remaining in his account after 10 years. 2

END OF PAPER

## STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left( x + \sqrt{x^2 + a^2} \right)$$

NOTE :  $\ln x = \log_e x, \quad x > 0$

# Solutions.

①  $b^2 - 4ac \geq 0$   
 $36 + 12k \geq 0$   
 $k \geq -3$   
A

② C

③  $\frac{a}{1-r} = 12$   
 $8 = 12 - 12r$   
 $12r = 4$   
 $r = \frac{1}{3}$   
D

④ A

⑤  $x = \frac{-2}{-2} = 1$   
max. value = 7  
C

⑥  $y = 3 \sin \frac{x}{2}$  D

⑦  $y = 2^{-x}$   
 $= (2^{-1})^x$   
 $= (\frac{1}{2})^x$   
C

⑧ C

⑨  $3x + 3y = 180$   
 $x + y = 60$   
B

⑩  $(x-2)(x+1)$   
 $x = 2, \cancel{x}$   
C

⑪ a) i)  $y' = (x \sin 2x + 2 \cos 2x) \cdot x$  ii)  $y' = 4e^{4x} - \frac{1}{x^2}$   
 $= \sin 2x + 2x \cos 2x$

iii)  $y' = \frac{1(3+2x) - 2(x+1)}{(3+2x)^2}$   
 $= \frac{3 + \cancel{2x} - 2x - 2}{(3+2x)^2}$   
 $= \frac{1}{(3+2x)^2}$

b)  $\frac{(4x+2)^7}{28} + c$

c)  $3^{1-x} = 3^{-\frac{3}{2}}$   
 $1-x = -\frac{3}{2}$   
 $x = 2\frac{1}{2}$

d)  $\sin x = -1$  or  $\sin x = \frac{1}{2}$   
 $\therefore x = \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

e)  $\int_{50}^{\infty} = \frac{50}{2} (5 + 103)$   
 $= 2700$

(12) a)  $x+2 = 3x$  or  $-(x+2) = 3x$   
 $x=1$  ✓  $-x-2 = 3x$   
 $4x = -2$   
 $x = -\frac{1}{2}$

only solution is  $x=1$

b)  $\frac{\log 50}{\log 2} \doteq 5.64$

c)  $y' = e^{\sin x} \times \cos x$   
 when  $x=0$ ,  $m_T = e^0 \times \cos 0$   
 $= 1$

d) i)  $\alpha + \beta = -4$ ,  $\alpha\beta = 1$

ii)  $\frac{\alpha^2 + \beta^2}{\alpha\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$   
 $= \frac{16 - 2}{1}$   
 $= 14$

e) i)  $y' = \frac{2x}{x^2 + 3}$

ii)  $y' = 2 \tan 4x \times \sec^2 4x \times 4$   
 $= 8 \tan 4x \sec^2 4x$

f)  $x^2 = 4y + 12$   
 $= 4(y + 3)$

i) focal length  $a = 1$

ii) vertex at  $(0, -3)$   $\therefore$  focus at  $(0, -2)$

$$\begin{aligned} \textcircled{12} \text{ g) } \int_2^4 f(x) dx &\doteq \frac{0.5}{3} (4 + 4x + 2x(-2) + 4x^3 + 8) \\ &= \frac{1}{6} (4 + 4x - 4 + 12 + 8) \\ &= \frac{1}{6} \times 24 \\ &= 4 \end{aligned}$$

$$\textcircled{13} \text{ a) i) } (6-m)(4+m)$$

$$\text{ii) } \frac{-4}{6} \quad \therefore m < -4 \text{ or } m > 6$$

$$\text{b) i) } d_{AB} = \sqrt{36+16} \\ = \sqrt{52} \text{ or } 2\sqrt{13}$$

$$\text{ii) } M_{AB} = \frac{-4}{6} \\ = \frac{-2}{3}$$

$$\text{iii) } \text{UR } M_{\perp} = \frac{3}{2}$$

$$\therefore y-0 = \frac{3}{2}(x+2)$$

$$2y = 3x+6$$

$$3x-2y+6=0 \text{ as reqd.}$$

$$\text{iv) } p.d. = \frac{|-4+0-2||}{\sqrt{2^2+3^2}} \\ = \frac{25}{\sqrt{13}}$$

$$\text{v) } C \text{ is } (4, -4)$$

$$\text{e) } 2P = P(1+r)^n \\ 2 = 1.01^n$$

$$\log 2 = n \log 1.01$$

$$n = \frac{\log 2}{\log 1.01}$$

$$\doteq 70 \text{ months}$$

$$\text{d) } \frac{d}{dx} [(\sin x)^{-1}] \\ = -(\sin x)^{-2} \times \cos x \\ = \frac{-\cos x}{\sin^2 x} \\ = -\cot x \operatorname{cosec} x$$

$$\text{ii) } \left[ -\operatorname{cosec} x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ = \frac{-1}{\sin \frac{\pi}{2}} - \left( \frac{-1}{\sin \frac{\pi}{3}} \right) \\ = \frac{-1}{1} + \frac{1}{\frac{\sqrt{3}}{2}} \\ = -1 + \frac{2}{\sqrt{3}}$$



$$(14) a) 5y = -3x - 2$$

$$y = -\frac{3}{5}x - \frac{2}{5}$$

$$\therefore \text{grad.} = -\frac{3}{5}$$

$$\tan \theta = -\frac{3}{5}$$

$$\therefore \theta = 149^\circ$$

$$b) i) -\cos \frac{2\pi}{3} \times \frac{3}{2} + c$$

$$= -\frac{3}{2} \cos \frac{2\pi}{3} + c$$

$$ii) \int (xe^{x^2} + \frac{1}{x}) dx$$

$$= \frac{e^{x^2}}{2} + \log x + c$$

$$c) \text{LHS} = \frac{\cos \theta (1 - \sin \theta) + \cos \theta (1 + \sin \theta)}{1 - \sin^2 \theta}$$

$$= \frac{2 \cos \theta}{\cos^2 \theta}$$

$$= 2 \sec \theta$$

$$= \text{RHS}$$

$$d) \log_m 8 + \log_m 64 = 6$$

$$\log_m 512 = 6$$

$$m^6 = 512$$

$$\therefore m = \sqrt[6]{512}$$

$$(\text{or } 2\sqrt{2})$$

$$e) \frac{BX}{BC} = \frac{3}{9} = \frac{1}{3}$$

$$\frac{BY}{BA} = \frac{5}{15} = \frac{1}{3}$$

and  $\angle B$  is common

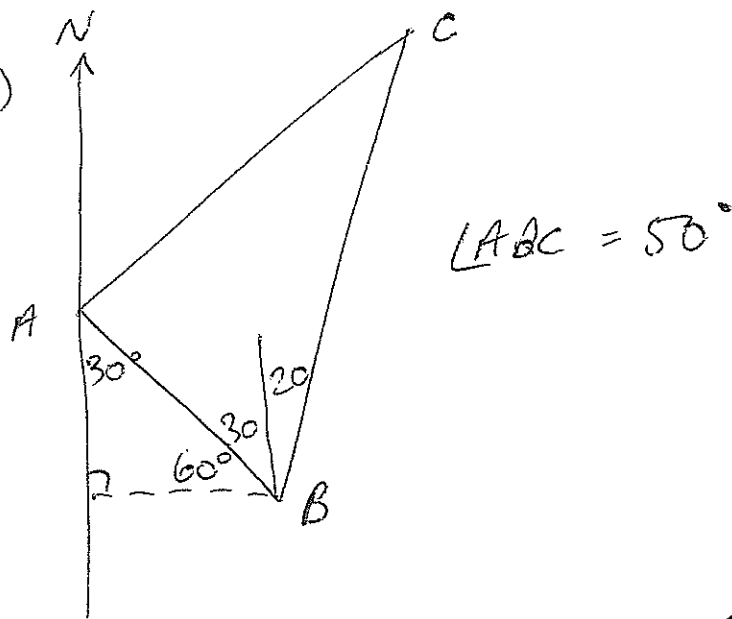
$\therefore \triangle BXY \parallel \triangle ABC$  (equal ratios of sides about equal incl. ang.)

$$ii) \frac{9}{\sin 35^\circ} = \frac{15}{\sin C}$$

$$\sin C = \frac{15 \sin 35^\circ}{9}$$

$$\therefore C \doteq 73^\circ$$

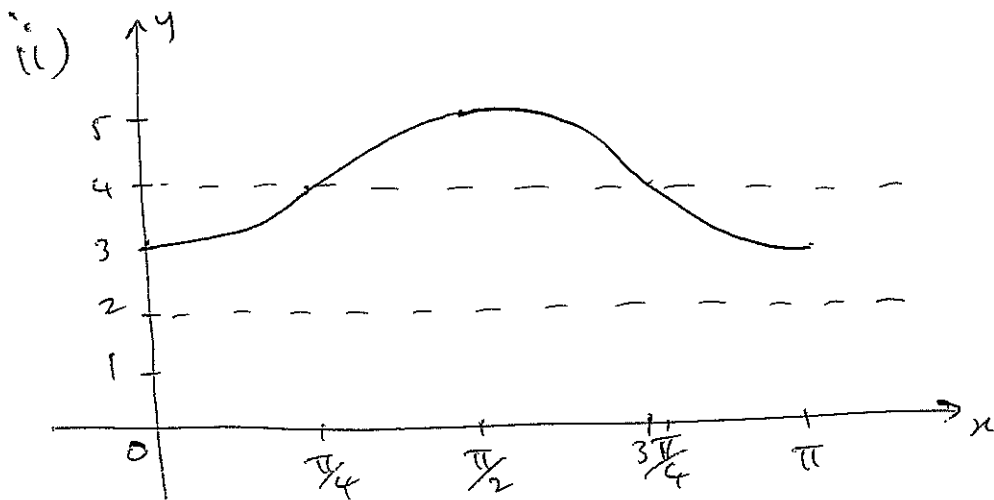
(15) a) i)



$$\begin{aligned} \text{ii) } AC^2 &= 20^2 + 40^2 - 2 \times 20 \times 40 \times \cos 50^\circ \\ &= 400 + 1600 - 1600 \cos 50^\circ \\ &= 971.54 \end{aligned}$$

$$\therefore AC \doteq 31 \text{ km}$$

b) i) amp. = 1, period =  $\frac{2\pi}{2}$   
=  $\pi$  units



$$\begin{aligned} \text{iii) Area} &= \int_0^\pi (4 - \cos 2x) dx - 2\pi \\ &= \left[ 4x - \frac{\sin 2x}{2} \right]_0^\pi - 2\pi \\ &= (4\pi - 0) - (0 - 0) - 2\pi \\ &= 2\pi \text{ u}^2 \end{aligned}$$

$$(15) \text{ c) i) } 8 = e^x \Rightarrow x = \log_2 8 \text{ or } \ln 8 \text{ or } 2.079$$

$$\text{ii) } V = \pi \int_{-2}^2 (e^x)^2 dx$$

$$= \pi \int_{-2}^2 e^{2x} dx$$

$$= \pi \left[ \frac{e^{2x}}{2} \right]_{-2}^2$$

$$= \frac{\pi}{2} (e^4 - e^{-4})$$

$$\doteq 85.7 \text{ u}^3$$

$$\text{d) } (x+y)^2 - 1 = (x+y+1)(x+y-1)$$

$$(16) \text{ a) Area} = \left( \int_{\pi/4}^{\pi} \sin x dx - \int_{\pi/4}^{\pi/2} \cos x dx \right) \times 2$$

$$\text{b) i) } y=0 \Rightarrow x \log x - x = 0$$

$$x(\log x - 1) = 0$$

$$x=0 \text{ (no sol.) or } \log x - 1 = 0$$

$$\log x = 1$$

$$\therefore \underline{x=e} \text{ only}$$

$$\text{ii) S.P.'s when } y' = \log x + \frac{1}{x} \times x - 1 = 0$$

$$\therefore \log x + 1 - 1 = 0$$

$$\log x = 0$$

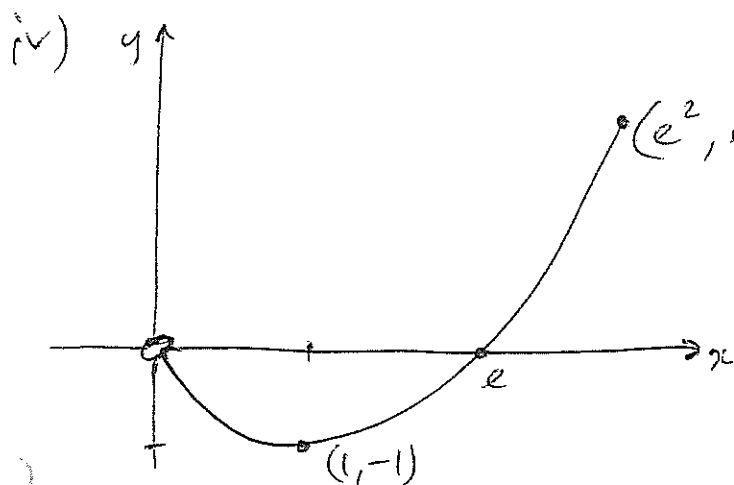
$$\therefore x = 1$$

$$y'' = \frac{1}{x}$$

When  $x=1$ ,  $y'' > 0 \Rightarrow$  minimum turning pt at  $(1, -1)$

iii) For all  $x > 0$ ,  $y'' > 0$

$\therefore$  curve is always concave up.



$$\begin{aligned} x &= e^2 \\ \therefore y &= e^2 \log(e^2) - e^2 \\ &= e^2 \times 2 - e^2 \\ &= e^2 \end{aligned}$$

e) i)  $A_1 = 10^6(1.005) - M$

$$A_2 = [10^6(1.005) - M] \times 1.005 - M$$

$$= 10^6(1.005)^2 - 1.005M - M$$

$$= 10^6(1.005)^2 - M(1.005 + 1) \text{ as reqd.}$$

ii)  $A_n = 10^6(1.005)^n - M(1.005^{n-1} + 1.005^{n-2} + \dots + 1)$

$$= 10^6(1.005)^n - M \times \frac{(1.005^n - 1)}{1.005 - 1}$$

$$= 10^6(1.005)^n - M \frac{(1.005^n - 1)}{0.005}$$

iii)  $A_{120} = 20$

$$\therefore 20 = 10^6(1.005)^{120} - 200M(1.005^{120} - 1)$$

$$\therefore M = \frac{10^6(1.005)^{120} - 20}{200(1.005)^{120} - 1}$$

$$= \$11,101.93 \text{ (accept 101 or 102)}$$