



2014

TRIAL HSC EXAMINATION

Mathematics Extension 1

GENERAL INSTRUCTIONS

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this booklet
- Show all necessary working in questions 11 – 14

Total Marks – 70

Section 1 10 marks

- Attempt Questions 1 -10
- Allow about 15 minutes for this section.

Section 2 60 marks

- Attempt Questions 11 – 14
- Allow about 1 hour 45 minutes for this section.

NAME: _____

TEACHER: _____

NUMBER: _____

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
TOTAL	/70

Section I

Total marks – 10

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

1 Which of the following is an expression for $\int \cos^2 2x \, dx$?

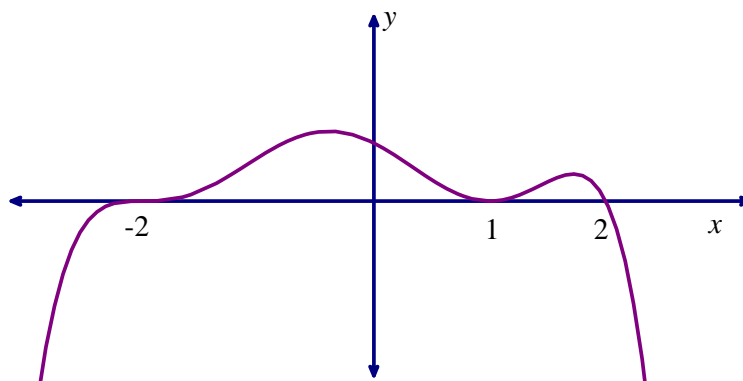
(A) $x - \frac{1}{4} \sin 4x + C$

(B) $x + \frac{1}{4} \sin 4x + C$

(C) $\frac{x}{2} - \frac{1}{8} \sin 4x + C$

(D) $\frac{x}{2} + \frac{1}{8} \sin 4x + C$

2 Below is the graph of the polynomial $y = P(x)$



Which of the following is a possible equation for $P = (x)$?

(A) $P(x) = (x - 2)^3 (1 - x)^2 (x + 2)$

(B) $P(x) = (x - 2)(1 - x)^2 (x + 2)^3$

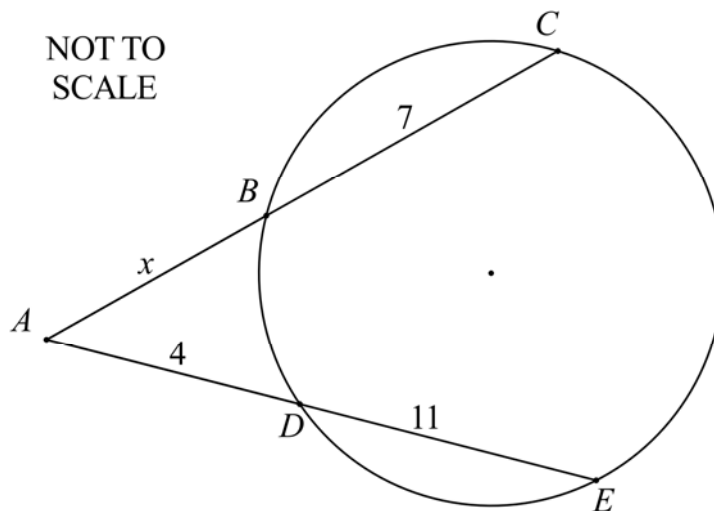
(C) $P(x) = (2 - x)^3 (1 - x)^2 (x + 2)$

(D) $P(x) = (2 - x)(1 - x)^2 (x + 2)^3$

3 What is the maximum value of $\sqrt{5} \cos x - 2 \sin x$?

- (A) $\sqrt{5}$
- (B) 3
- (C) 5
- (D) $\sqrt{29}$

4 In the diagram below, BC and DE are chords of a circle. CB and ED produced meet at A .



What is the value of x ?

- (A) $\frac{11}{28}$
- (B) $\frac{28}{11}$
- (C) 5
- (D) 12

5 What is the domain and range of the equation $y = 4 \cos^{-1} 3x$?

- (A) Domain: $-\frac{1}{3} \leq x \leq \frac{1}{3}$ and Range: $-2\pi \leq y \leq 2\pi$
- (B) Domain: $-3 \leq x \leq 3$ and Range: $-2\pi \leq y \leq 2\pi$
- (C) Domain: $-\frac{1}{3} \leq x \leq \frac{1}{3}$ and Range: $0 \leq y \leq 4\pi$
- (D) Domain: $-3 \leq x \leq 3$ and Range: $0 \leq y \leq 4\pi$

6 What is the solution of the inequation $3x + 2 < |2x - 1|$?

(A) $x < -\frac{1}{5}$

(B) $-3 < x < \frac{1}{5}$

(C) $x < -\frac{1}{5}$ or $x > 3$

(D) $x < -3$

7 What is the value of $\lim_{x \rightarrow 0} \frac{3x \cos 4x}{\sin 2x}$?

(A) 0

(B) $\frac{3}{2}$

(C) $\frac{2}{3}$

(D) Undefined

8 Which of the following is the derivative x of $\tan^{-1}(e^{-x})$?

(A) $\frac{e^x}{1 + e^{2x}}$

(B) $\frac{-e^{-x}}{1 + e^{2x}}$

(C) $\frac{-e^{-x}}{1 + e^{-2x}}$

(D) $\frac{e^x}{1 + e^{-2x}}$

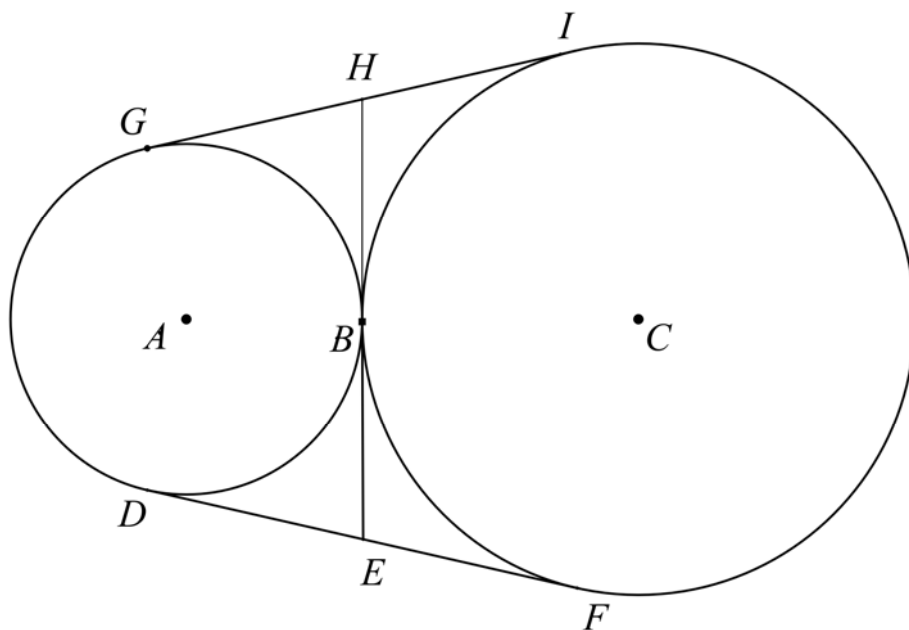
- 9 A particle undergoing simple harmonic motion and has acceleration according to the equation:

$$v^2 = 9(5 - x^2)$$

What is the amplitude A and Period T of this motion?

- (A) $A = \sqrt{5}$ and $T = \frac{2\pi}{9}$
 (B) $A = 5$ and $T = \frac{2\pi}{9}$
 (C) $A = \sqrt{5}$ and $T = \frac{2\pi}{3}$
 (D) $A = 5$ and $T = \frac{2\pi}{3}$

- 10 In the diagram below DF , GI and HE are common tangents to the two unequal circles that have centres at A and C .



Which of the following quadrilaterals are **NOT** cyclic?

- (A) $ABED$
 (B) $HICB$
 (C) $GIFD$
 (D) $ACFD$

Section II**Total marks – 60****Attempt Questions 11–14****Allow about 1 hour and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Factorise the polynomial $P(x) = x^3 - 7x + 6$. **2**

(b) Evaluate $\int_0^1 \frac{1}{\sqrt{4-3x^2}} dx$. **2**

(c) Solve $\frac{2x+3}{x} \geq x$. **3**

(d) The point P divides the interval from $A(-1, 3)$ to $B(7, -4)$ externally in the ratio $2:5$. Find the coordinates of P . **2**

(e) Use the substitution $u = x - 1$ to evaluate: **3**

$$\int_2^4 \frac{3x}{(x-1)^2} dx.$$

(f) The two curves $y = 2 - x^2$ and $y = x^3$ intersect at $(1, 0)$. **3**

Find the acute angle between the two curves at $(1, 0)$.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Given the function $f(x) = \frac{2x+1}{x-1}$

- (i) Write down the equation of any vertical and horizontal asymptotes. 1
- (ii) Given that $f(x)$ is a hyperbola, state the domain of the inverse function $f^{-1}(x)$. 1
- (iii) Sketch the graph of the inverse function $y = f^{-1}(x)$.
Clearly label all important features of the graph. 2

(b) A can of soft drink at temperature T degrees is removed from a fridge and placed in a room that has a constant temperature of A degrees. The rate at which the can of soft drink warms can be expressed using the equation:

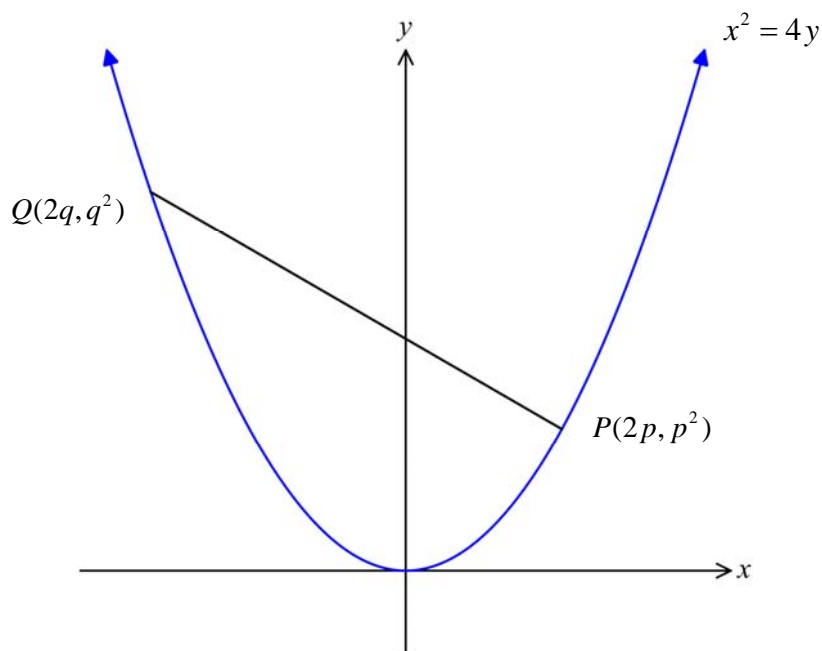
$$\frac{dT}{dt} = -k(T - A)$$

where t is the time in minutes after the can is placed in the room, and k is a positive constant.

- (i) Show that $T = A + Pe^{-kt}$ satisfies the equation, where P is a constant. 1
- (ii) If the room temperature is 30° and the soft drink warms from 3° to 15° in the first 10 minutes find the exact value of k . 2
- (iii) Find the time taken to the nearest second for temperature of the can to increase by another 12 degrees. 2

Question 12 continues on page 8

- (c) The diagram shows two distinct points $P(2p, p^2)$ and $Q(2q, q^2)$ that lie on the parabola $x^2 = 4y$.



- (i) Show that the line PQ has the equation $y = \frac{p+q}{2}x - pq$. 2
- (ii) The point $R(8, 0)$ lies on the straight line that passes through P and Q .
- (1) Show that $4(p+q) = pq$ 1
- (2) Given that M is the midpoint of PQ , show that the equation of the locus of M , as P and Q vary is 1
- $$2y = x^2 - 8x$$
- (3) State any restrictions on the locus. 2

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

- (a) Use mathematical induction to prove that if n is an integer and $n \geq 1$ and $a > 0$ **3**

$$\frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \dots - \frac{1}{(a+1)^n} = \frac{1}{a(a+1)^n}$$

- (b) A particle moves in simple harmonic motion such that its displacement x centimetres after t seconds is given by $x = 2\sqrt{3} \cos 2t - 2 \sin 2t - 7$.

- (i) Express x in the form $x = 4 \cos(2t + \alpha) - 7$, where $0 \leq \alpha \leq \frac{\pi}{2}$ **2**

- (ii) What is the period of the motion? **1**

- (iii) What is the range of possible values for x ? **1**

- (iv) How much time elapses from when the particle first starts until it is first at its minimum displacement? **2**

- (c) By sketching appropriate graphs or otherwise solve: **3**

$$\sin^{-1}(3x+1) = \cos^{-1} x$$

- (d) The polynomial $P(x) = x^3 + ax^2 - 6x - 4$, where $a > 0$, has three zeroes. **3**
One zero is the product of the other two.

Find the value of a .

Question 14 (15 marks) Use a SEPARATE writing booklet.

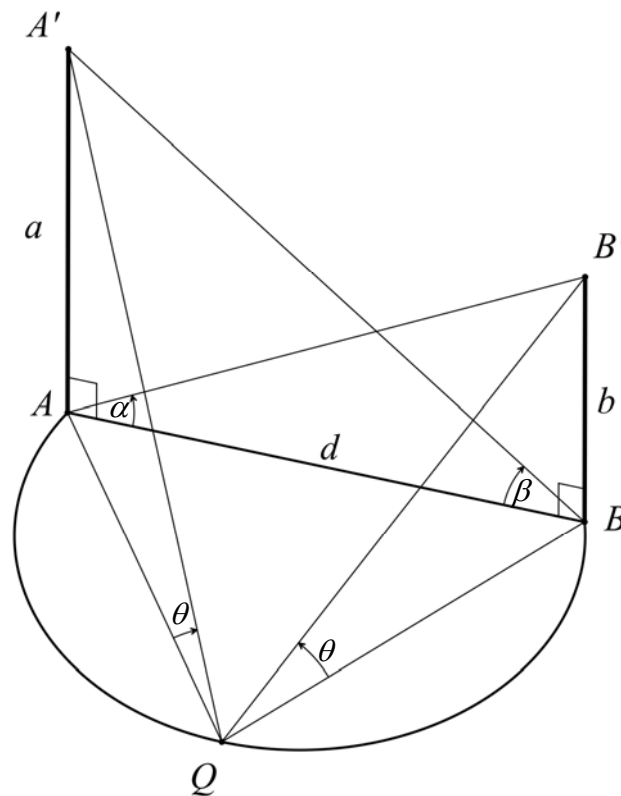
- (a) (i) An object is initially 2 m to the right of the origin travelling with velocity 6 m/s. The acceleration \ddot{x} ms^{-2} of the object is given by: 2

$$\ddot{x} = 2x^3 + 4x$$

Find an expression for v^2 in terms of x .

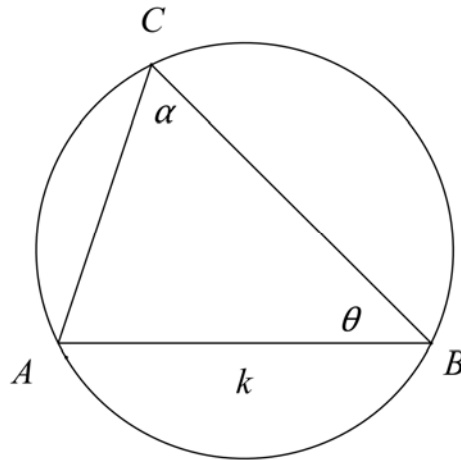
- (ii) Find the minimum speed of the object giving a reason. 2

- (b) AQB is a semi-circle in the horizontal plane with diameter AB of length d metres. There are two vertical posts AA' and BB' of heights a and b respectively. From Q , the angle of elevation to the tops of both posts A' and B' is θ . From A the angle of elevation to B' is α and from B the angle of elevation to A' is β .



- (i) Prove that $d^2 = \frac{a^2 + b^2}{\tan^2 \theta}$. 3
- (ii) Show that $\tan^2 \alpha + \tan^2 \beta = \tan^2 \theta$. 1

- (c) The points A , B and C all lie on a circle. A and B are fixed, AB has a constant length k cm and C lies on the major arc. $\angle ABC = \theta$ and $\angle ACB = \alpha$ as shown in the diagram below. C moves on the major arc so that θ increases at 0.1 radians per second.



- (i) If P is the perimeter of the triangle ABC then show that: 2

$$P = \frac{k[\sin \alpha + \sin \theta + \sin(\theta + \alpha)]}{\sin \alpha}$$

- (ii) Find the value of $\frac{dP}{dt}$ in terms of k and α when $\theta = \alpha$. 2

- (iii) Show that $\frac{dP}{dt} = 0$ when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$. 1

- (iv) Find the value of θ that will maximise P , given $\alpha = \frac{3\pi}{8}$. 2

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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1 Which of the following is an expression for $\int \cos^2 2x \, dx$?

Since

$$\cos 2\theta = 2\cos^2 \theta - 1$$

$$\cos^2 \theta = \frac{\cos 2\theta + 1}{2}$$

$$\cos^2 2\theta = \frac{\cos 4\theta + 1}{2}$$

$$\begin{aligned} \int \cos^2 2x \, dx &= \int \frac{1 + \cos 4x}{2} \, dx \\ &= \frac{x}{2} + \frac{\sin 4x}{8} + C \end{aligned}$$

ANSWER is D

2 ANSWER is D

$$P(x) = (2-x)(1-x)^2(x+2)^3$$

3 What is the maximum value of $\sqrt{5} \cos x - 2 \sin x$?

$$\sqrt{5} \cos x - 2 \sin x = R \cos(x + \alpha) \text{ so thus}$$

the maximum value is when $\cos(x + \alpha) = 1$.

$$R = \sqrt{(\sqrt{5})^2 + 2^2}$$

$$R = 3$$

ANSWER is (C) 3

4

$$\frac{AB}{AE} = \frac{AD}{AC}$$

$$\frac{x}{15} = \frac{4}{x+7}$$

$$x^2 + 7x = 60$$

$$x^2 + 7x - 60 = 0$$

$$(x+12)(x-5) = 0$$

Since $x > 0$

$$x = 5$$

ANSWER is C

5 What is the domain and range of the equation $y = 4 \cos^{-1} 3x$?

Domain

$$-1 \leq 3x \leq 1$$

$$-\frac{1}{3} \leq x \leq \frac{1}{3}$$

Range

$$0 \leq \cos^{-1} 3x \leq \pi$$

$$0 \leq 4 \cos^{-1} 3x \leq 4\pi$$

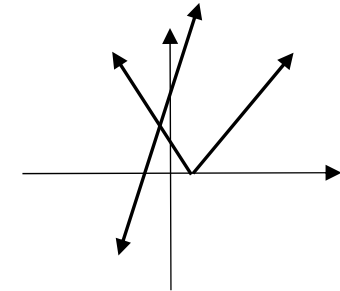
ANSWER is (C)

$$\text{Domain: } -\frac{1}{3} \leq x \leq \frac{1}{3} \text{ and Range: } 0 \leq y \leq 4\pi$$

6 What is the solution of the inequation $3x + 2 < |2x - 1|$?

Best method is to sketch the two functions

$$y = 3x + 2 \text{ and } y = |2x - 1|$$



Only one intersection where

$$3x + 2 = -[2x - 1]$$

$$5x = -1$$

$$x = -\frac{1}{5}$$

ANSWER is (A) $x < -\frac{1}{5}$

7 What is the value of $\lim_{x \rightarrow 0} \frac{3x \cos 4x}{\sin 2x}$?

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3x \cos 4x}{\sin 2x} &= \lim_{x \rightarrow 0} \frac{3x}{\sin 2x} \times \lim_{x \rightarrow 0} \cos 4x \\ &= \frac{3}{2} \lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \times \lim_{x \rightarrow 0} \cos 4x \\ &= \frac{3}{2} \times 1 \times \cos 0 = \frac{3}{2} \end{aligned}$$

ANSWER is (B) $\frac{3}{2}$

8 Which of the following is the derivative of $\tan^{-1}(e^{-x})$?

$$\begin{aligned} \frac{d}{dx} \tan e^{-x} &= \frac{1}{1+(e^{-x})^2} \times \frac{d}{dx}(e^{-x}) \\ &= \frac{-e^{-x}}{1+e^{-2x}} \end{aligned}$$

ANSWER is (C) $\frac{-e^{-x}}{1+e^{-2x}}$

9

$$\begin{aligned} \ddot{x} &= -n^2x \\ \frac{d}{dx} \left(\frac{1}{2} v^2 \right) &= -n^2(x - x_0) \\ \left(\frac{1}{2} v^2 \right) &= -\frac{n^2 x^2}{2} + C \\ v^2 &= -\frac{n^2 x^2}{2} + D \end{aligned}$$

When $v = 0$ $x = A$

$$\begin{aligned} 0 &= -n^2 A^2 + D \\ D &= n^2 A^2 \end{aligned}$$

$$\begin{aligned} v^2 &= -n^2 x^2 + n^2 A^2 \\ v^2 &= n^2 (A^2 - x^2) \end{aligned}$$

Thus $n = 3$ and $A = \sqrt{5}$

$$T = \frac{2\pi}{3}$$

ANSWER is (C) $A = \sqrt{5}$ and $T = \frac{2\pi}{3}$

10

$\angle ADF = 90^\circ$ Angle between a radius and tangent

$\angle CFD = 90^\circ$ Angle between a radius and tangent

If $ACFD$ is cyclic then $\angle CADF = 90^\circ$ and $\angle ACF = 90^\circ$ opposite angle are supplementary.

So $ACFD$ must be a rectangle if it is cyclic.

But the circles are not equal and so AD and CF are not equal.

$ACFD$ is not cyclic

ANSWER is (D) $ACFD$

Question 11

(a) Factorise the polynomial

$$P(x) = x^3 - 7x + 6.$$

$$P(x) = x^3 - 7x + 6$$

$$P(1) = 1^3 - 7(1) + 6$$

$$= 0$$

$\therefore (x-1)$ is a factor of $P(x)$

$$P(x) = x^3 - 7x + 6$$

$$= (x-1)(x^2 + bx - 6)$$

x term is

$$-7x = -bx - 6x$$

$$7 = b + 6$$

$$b = 1$$

$$P(x) = (x-1)(x^2 + x - 6)$$

$$= (x-1)(x-2)(x+3)$$

(b) Evaluate $\int_0^1 \frac{1}{\sqrt{4-3x^2}} dx$. 2

$$\int_0^1 \frac{1}{\sqrt{4-3x^2}} dx = \int_0^1 \frac{1}{\sqrt{4-(x)^2}}$$

$$= \int_0^1 \frac{1}{\sqrt{3}\sqrt{\frac{4}{3}-(x)^2}} dx$$

$$= \frac{1}{\sqrt{3}} \sin^{-1} \frac{x\sqrt{3}}{2} dx$$

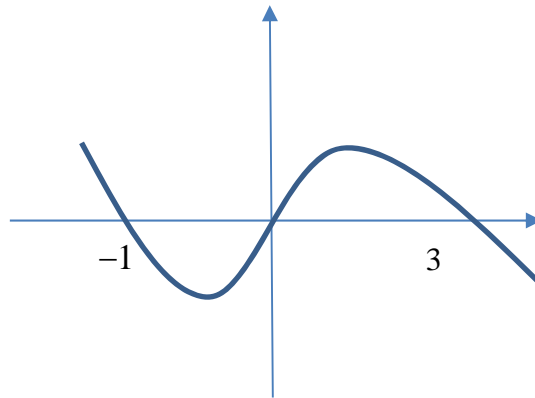
(c)

$$x(2x+3) \geq x^3 \quad x \neq 0$$

$$x(2x+3) - x^3 \geq 0$$

$$x(2x+3-x^2) \geq 0$$

$$x(x+1)(3-x) \geq 0$$



From the graph of $y = x(x+1)(3-x)$

$x \leq -1$ or $0 < x \leq 3$ since $x \neq 0$

(d)

$A(-1,3)$ to $B(7,-4)$

$-2:5$

$$x = \frac{(5)(-1) + (-2)(7)}{5-2} \quad y = \frac{(5)(3) + (-2)(-4)}{5-2}$$

$$= \frac{-5-14}{3} \quad = \frac{15+8}{3}$$

$$= \frac{-19}{3} \quad = \frac{23}{3}$$

P has Coordinates $\left(\frac{-19}{3}, \frac{23}{3}\right)$

(e) Evaluate: $\int_2^4 \frac{3x}{(x-1)^2} dx$.

$$u = x - 1$$

$$x = u + 1$$

$$x = 4$$

$$u = 4 - 1 = 3$$

When

$$\frac{du}{dx} = 1$$

$$x = 2$$

$$u = 2 - 1 = 1$$

$$du = dx$$

$$\int_2^4 \frac{3x}{(x-1)^2} dx = \int_1^3 \frac{3(u+1)}{(u)^2} du$$

$$= \int_1^3 \frac{3u}{u^2} + \frac{3}{u^2} du$$

$$= \int_1^3 \frac{3}{u} + \frac{3}{u^2} du$$

$$= [3 \ln u - 3u^{-1}]_1^3$$

$$= \left(3 \ln 3 - \frac{3}{3}\right) - \left(3 \ln 1 - \frac{3}{1}\right)$$

$$= 3 \ln 3 + 2$$

(f) The two curves $y = 2 - x^2$ and $y = x^3$ intersect at $(1,0)$. Find the acute angle between the two curves at $(1,0)$.

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \quad \text{where } m_1 \text{ and } m_2 \text{ are the}$$

gradients of the two functions at $x = 1$

$$y = 2 - x^2 \quad y = x^3$$

$$\frac{dy}{dx} = -2x \quad \frac{dy}{dx} = 3x^2$$

at $x = 1$ at $x = 1$

$$m_1 = -2 \quad m_2 = 3$$

$$\tan \theta = \left| \frac{-2-3}{1-(-2)(3)} \right|$$

$$\tan \theta = \left| \frac{-5}{-5} \right|$$

$$\tan \theta = 1$$

$$\theta = 45^\circ$$

Question 12

(a) (i)

Vertical Asymptote: $x = 1$

Horizontal Asymptote: $y = 2$

(ii) $y \neq 2$

(iii)

x intercept of $f(x)$ is when

$$f(x) = 0$$

$$0 = \frac{2x+1}{x-1}$$

$$0 = 2x+1$$

$$x = -\frac{1}{2}$$

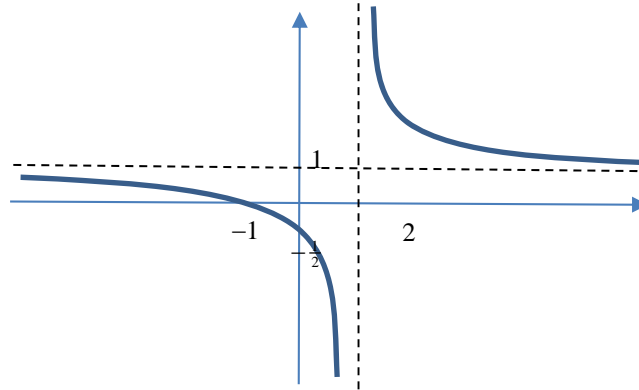
y intercept of $f(x)$ is

$$y = \frac{2(0)+1}{(0)-1}$$

$$= -1$$

So the x intercept of $f^{-1}(x)$ is $x = -1$

So the y intercept of $f^{-1}(x)$ is $y = -\frac{1}{2}$



(b) (i) Show that $T = A + Pe^{-kt}$ satisfies the equation, where P is a constant.

$$T = A + Pe^{-kt}$$

$$\frac{dT}{dt} = -kPe^{-kt}$$

$$= -k(A + Pe^{-kt} - A)$$

$$\frac{dT}{dt} = -k(T - A)$$

(ii) When $t = 0$ $T = 3$ and When $t = 10$

$$T = 15$$

$$3 = 30 + Pe^0$$

$$P = -27$$

$$15 = 30 - 27e^{-10k}$$

$$15 = 27e^{-10k}$$

$$\frac{15}{27} = e^{-10k}$$

$$\ln\left(\frac{15}{27}\right) = -10k$$

$$-\frac{1}{10}\ln\left(\frac{15}{27}\right) = k$$

OR

$$\frac{1}{10}\ln\left(\frac{27}{15}\right) = k$$

(iii)

$$27 = 30 - 27e^{-kt}$$

$$3 = 27e^{-kt}$$

$$\frac{3}{27} = e^{-kt}$$

$$\ln\left(\frac{1}{9}\right) = -kt$$

$$-\frac{\ln\left(\frac{1}{9}\right)}{k} = t$$

$$\frac{\ln\left(\frac{1}{9}\right)}{k} = t$$

$$\frac{1}{10}\ln\left(\frac{15}{27}\right) = t$$

$$t = 37.38132742$$

Time Taken is
 $37.38132742 - 10 = 27.38132742$ minutes
 $= 27^{\circ}23''$

c) (i) Show that the line PQ has the equation $y = \frac{p+q}{2}x - pq$.

$$m_{\text{CHORD}} = \frac{p^2 - q^2}{2p - 2q}$$

$$m_{\text{CHORD}} = \frac{p+q}{2}$$

Equation

$$y - p^2 = \frac{p+q}{2}(x - 2p)$$

$$y - p^2 = \frac{p+q}{2}x - \frac{2p^2 - 2pq}{2}$$

$$y = \frac{p+q}{2}x - p^2 + 2pq + p^2$$

$$y = \frac{p+q}{2}x - p^2 + 2pq + p^2$$

$$y = \frac{p+q}{2}x - pq$$

(ii) The point $R(8,0)$ lies on the straight line that passes through P and Q .

(1) Show that $4(p+q) = pq$.

$$y = \frac{p+q}{2}x - pq$$

$$0 = \frac{p+q}{2} \times 8 - pq$$

$$0 = 4(p+q) - pq$$

$$4(p+q) = pq$$

(2) Given that M is the midpoint of PQ , show that as P and Q vary the

$$2y = x^2 - 8x$$

Midpoint

$$x = \frac{2p+2q}{2} \quad y = \frac{p^2+q^2}{2}$$

$$= p+q$$

Locus of M

$$x^2 - 8x = (p+q)^2 - 8x$$

$$= (p^2 + 2pq + q^2) - 8(p+q)$$

From part (ii) (1) $4(p+q) = pq$

$$= (p^2 + 2pq + q^2) - 2 \times 4(p+q)$$

$$= (p^2 + 2pq + q^2) - 2 \times pq$$

$$= p^2 + q^2$$

$$= 2y$$

(3) State any restrictions on the locus.

The y -value of M must always be greater than or equal to the y -value of the parabola $x^2 = 4y$.

$$\frac{x^2 - 8x}{2} \geq \frac{x^2}{4}$$

$$2x^2 - 16x \geq x^2$$

$$x^2 - 16x \geq 0$$

$$x(x - 16) \geq 0$$

$$x \leq 0 \quad x \geq 16$$

Question 13

(a) Use mathematical induction to prove that if n is an integer and $n \geq 1$ and $a > 0$

$$\frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \dots - \frac{1}{(a+1)^n} = \frac{1}{a(a+1)^n}$$

When $n = 1$

$$\begin{aligned} LHS &= \frac{1}{a} - \frac{1}{a+1} & RHS &= \frac{1}{a(a+1)^1} \\ &= \frac{a+1-a}{a(a+1)} & &= \frac{1}{a(a+1)} \\ &= \frac{1}{a(a+1)} \end{aligned}$$

True for $n = 1$

Assume true when $n = k$

$$\frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \dots - \frac{1}{(a+1)^k} = \frac{1}{a(a+1)^k}$$

Prove true when $n = k+1$ if true for $n = k$

Aim : Prove that

$$\frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \dots - \frac{1}{(a+1)^{k+1}} = \frac{1}{a(a+1)^{k+1}}$$

$$\begin{aligned} LHS &= \frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \dots - \frac{1}{(a+1)^{k+1}} \\ &= \frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \dots - \frac{1}{(a+1)^k} - \frac{1}{(a+1)^{k+1}} \\ &= \frac{1}{a(a+1)^k} - \frac{1}{(a+1)^{k+1}} && \text{By Assumption} \\ &= \frac{a+1-a}{a(a+1)^{k+1}} \\ &= \frac{1}{a(a+1)^{k+1}} = RHS \end{aligned}$$

By Mathematical Induction

$$\frac{1}{a} - \frac{1}{a+1} - \frac{1}{(a+1)^2} - \dots - \frac{1}{(a+1)^n} = \frac{1}{a(a+1)^n} \text{ is}$$

true for all integers $n \geq 1$

(b) (i) Express x in the form

$$x = 4 \cos(2t + \alpha) - 7, \text{ where } 0 \leq \alpha \leq \frac{\pi}{2}$$

$$2\sqrt{3} \cos 2t - 2 \sin 2t = 4 \cos(2t + \alpha)$$

$$= 4 \cos 2t \cos \alpha - 4 \sin 2t \sin \alpha$$

$$2\sqrt{3} \cos 2t - 2 \sin 2t = 4 \cos(2t + \alpha)$$

$$2\sqrt{3} = 4 \cos \alpha \quad -2 = -4 \sin \alpha$$

$$\frac{\sqrt{3}}{2} = \cos \alpha \quad \frac{1}{2} = \sin \alpha$$

$$\therefore \tan \alpha = \frac{1}{\sqrt{3}} \text{ where } 0 \leq \alpha \leq \frac{\pi}{2}$$

$$\alpha = \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

(ii) What is the period of the motion?

$$T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi \text{ seconds}$$

(iii) What is the range of possible values for x ?

Amplitude is 4 and centre of motion is at $x = -7$. Range of possible x - values is $-11 \leq x \leq -3$

(iv) How much time elapses from when the particle first starts until it is first at its minimum displacement?

When $x = -11$

$$-11 = 4 \cos\left(2t + \frac{\pi}{6}\right) - 7$$

$$-4 = 4 \cos\left(2t + \frac{\pi}{6}\right)$$

$$-1 = \cos\left(2t + \frac{\pi}{6}\right)$$

$$-1 = \cos\left(2t + \frac{\pi}{6}\right)$$

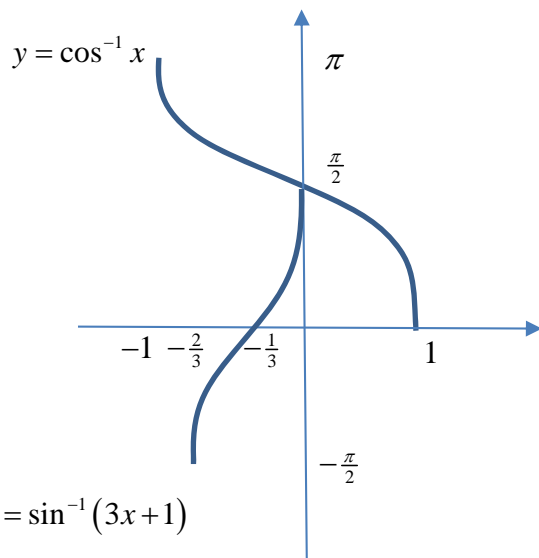
$$2t + \frac{\pi}{6} = \cos^{-1}(-1)$$

$$2t + \frac{\pi}{6} = \pi$$

$$2t = \frac{5\pi}{6}$$

$$t = \frac{5\pi}{12}$$

(c) Sketch both functions $y = \cos^{-1} x$ and $y = \sin^{-1}(3x+1)$ on the same plane.



From the graph, there is only one solution, $x = 0$

ALTERNATIVE :

Solve algebraically:

Let $\alpha = \sin^{-1}(3x+1)$ and thus $\alpha = \cos^{-1} x$

Let $\sin \alpha = 3x+1$ and $\cos \alpha = x$

By Pythagoras using $\cos \alpha = x$ then $\sin \alpha = \sqrt{1-x^2}$

$$(3x+1) = \sqrt{1-x^2}$$

$$(3x+1)^2 = ()$$

$$9x^4 + 6x^3 + 1 = 1 - x^2$$

$$10x^2 + 6x = 0$$

$$2x(5x+3) = 0$$

$$x = 0$$

OR

$$x = \frac{-3}{5}$$

Check solutions in original equation.

$$x = 0 \quad \sin^{-1}(1) = \cos^{-1}(0)$$

$$x = \frac{-3}{5} \quad \sin^{-1}\left(-\frac{9}{5} + 1\right) \neq \cos^{-1}\left(\frac{-3}{5}\right)$$

Solution is $x = 0$

(d) Let the zeroes be α , β and $\alpha\beta$

Product of Roots

$$(\alpha)(\beta)(\alpha\beta) = 4$$

$$\alpha^2 \beta^2 = 4$$

$$\alpha\beta = \pm 2$$

Sum in Pairs

$$\alpha\beta + \alpha^2\beta + \alpha\beta^2 = -6$$

$$\alpha\beta(1 + \alpha + \beta) = -6 \quad I$$

Sum of Roots

$$\alpha + \beta + \alpha\beta = -a \quad II$$

If $\alpha\beta = 2$ then

In I

$$\alpha + \beta + 2 = -a$$

$$\alpha + \beta + 1 = -a - 1$$

And in II

$$\alpha\beta(1 + \alpha + \beta) = -6$$

$$(2)(-a-1) = -6$$

$$-a-1 = -3$$

$$a = 2$$

If $\alpha\beta = -2$ then

In I

$$\alpha + \beta - 2 = -a$$

$$\alpha + \beta + 1 = -a + 3$$

And in II

$$\alpha\beta(1 + \alpha + \beta) = -6$$

$$(-2)(-a+3) = -6$$

$$-a+3 = 3$$

$$-a = 0$$

$$a = 0$$

Since $a > 0$ $a = 2$

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i)

$$\ddot{x} = 2x^3 + 4x$$

$$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = 2x^3 + 4x$$

$$\frac{1}{2} v^2 = \frac{2x^4}{4} + \frac{4x^2}{2} + C$$

$$v^2 = x^4 + 4x^2 + D$$

When $x = 2$ $v = 6$

$$(6)^2 = (2)^4 + 4(2)^2 + D$$

$$36 = 16 + 16 + D$$

$$4 = D$$

$$v^2 = x^4 + 4x^2 + 4$$

$$v^2 = (x^2 + 2)^2$$

(ii) Acceleration $\ddot{x} = 2x^3 + 4x$ is positive when x is positive.

Since the particle is initially at $x = 2$ with positive velocity the velocity of the particle will always be increasing and so the minimum velocity is its initial velocity of 6m/s

(b) (i) Prove that $d^2 = \frac{a^2 + b^2}{\tan^2 \theta}$.

$$\angle AQB = 90^\circ \quad (\text{Angle in a semi circle})$$

Let AQ and BQ be x and y respectively.

In $\triangle AQB$

$$d^2 = x^2 + y^2 \quad \text{By Pythagoras}$$

In $\triangle AQA'$

$$\tan \theta = \frac{a}{x}$$

$$x = \frac{a}{\tan \theta}$$

In $\triangle BQB'$

$$\tan \theta = \frac{b}{y}$$

$$y = \frac{b}{\tan \theta}$$

$$d^2 = \left(\frac{a}{\tan \theta} \right)^2 + \left(\frac{b}{\tan \theta} \right)^2$$

$$d^2 = \frac{a^2 + b^2}{\tan^2 \theta}$$

(ii) Show that $\tan^2 \alpha + \tan^2 \beta = \tan^2 \theta$.

In $\triangle ABA'$

$$\tan \beta = \frac{a}{d}$$

In $\triangle BAB'$

$$\tan \alpha = \frac{b}{d}$$

$$\text{LHS} = \tan^2 \alpha + \tan^2 \beta$$

$$= \left(\frac{a}{d} \right)^2 + \left(\frac{b}{d} \right)^2$$

$$= \frac{a^2 + b^2}{d^2}$$

$$= (a^2 + b^2) \div d^2$$

$$= (a^2 + b^2) \div \frac{a^2 + b^2}{\tan^2 \theta}$$

$$= \tan^2 \theta$$

$$= \text{RHS}$$

(c) (i) If P is the perimeter of the triangle ABC then show that:

$$P = \frac{k [\sin \alpha + \sin \theta + \sin(\theta + \alpha)]}{\sin \alpha}$$

$$P = AB + AC + BC$$

$$AB = k$$

$$\frac{AC}{\sin \theta} = \frac{AB}{\sin \alpha}$$

$$AC = \frac{k \sin \theta}{\sin \alpha}$$

$$\frac{BC}{\sin(\pi - (\theta + \alpha))} = \frac{AB}{\sin \alpha}$$

$$AC = \frac{k \sin(\pi - (\theta + \alpha))}{\sin \alpha}$$

$$AC = \frac{k \sin(\theta + \alpha)}{\sin \alpha}$$

$$P = k + \frac{k \sin(\theta)}{\sin \alpha} + \frac{k \sin(\theta + \alpha)}{\sin \alpha}$$

$$P = \frac{k [\sin \alpha + \sin \theta + \sin(\theta + \alpha)]}{\sin \alpha}$$

(ii) Find the value of $\frac{dP}{dt}$ in terms of k

and α when $\theta = \alpha$.

Since $\angle ACB$ is an angle subtended by AB in the major segment α is a constant.

$$\frac{dP}{dt} = \frac{dP}{d\theta} \times \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = 0.1$$

$$P = \frac{k [\sin \alpha + \sin \theta + \sin(\theta + \alpha)]}{\sin \alpha}$$

$$\frac{dP}{d\theta} = \frac{k [\cos \theta + \cos(\theta + \alpha)]}{\sin \alpha}$$

$$\begin{aligned} \frac{dP}{dt} &= \frac{dP}{d\theta} \times \frac{d\theta}{dt} \\ &= \frac{k [\cos \theta + \cos(\theta + \alpha)]}{\sin \alpha} \times \frac{1}{10} \end{aligned}$$

When $\theta = \alpha$

$$= \frac{k [\cos \alpha + \cos(2\alpha)]}{10 \sin \alpha}$$

(iii) Show that $\frac{dP}{dt} = 0$ when

$$\theta = \frac{\pi}{2} - \frac{\alpha}{2}$$

$$\frac{dP}{dt} = 0 \text{ when } \frac{dP}{d\theta} = 0$$

$$\text{Let } \theta = \frac{\pi}{2} - \frac{\alpha}{2}$$

$$\frac{dP}{d\theta} = \frac{k \left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \cos\left(\frac{\pi}{2} - \frac{\alpha}{2} + \alpha\right) \right]}{\sin \alpha}$$

$$\frac{dP}{d\theta} = \frac{k \left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \cos\left(\frac{\pi}{2} + \frac{\alpha}{2}\right) \right]}{\sin \alpha}$$

NOTE:

$$\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = -\cos\left(\frac{\pi}{2} + \frac{\alpha}{2}\right)$$

$$\frac{dP}{d\theta} = \frac{k \left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) - \cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \right]}{\sin \alpha} = 0$$

(iv) Find the value of θ that will maximise P , given $\alpha = \frac{3\pi}{8}$.

Since $\frac{dP}{d\theta} = 0$ when $\theta = \frac{\pi}{2} - \frac{\alpha}{2}$ then

$\theta = \frac{\pi}{2} - \frac{3\pi}{16}$ is a possible value that can

maximise P

Find the second derivative and test $\theta = \frac{5\pi}{16}$

$$\frac{d^2P}{d\theta^2} = \frac{k [-\sin \theta - \sin(\theta + \alpha)]}{\sin \alpha}$$

$$\text{Let } \theta = \frac{5\pi}{16}$$

$$\frac{d^2P}{d\theta^2} = \frac{-k \left[\sin \frac{5\pi}{16} + \sin\left(\frac{5\pi}{16} + \alpha\right) \right]}{\sin \alpha}$$

Since $0 \leq \alpha \leq \frac{\pi}{2}$

$\sin \frac{5\pi}{16} + \sin\left(\frac{5\pi}{16} + \alpha\right) > 0$ and $\sin \alpha > 0$

Therefore $\frac{d^2P}{d\theta^2} < 0$ when $\theta = \frac{5\pi}{16}$ thus P is a maximum when $\theta = \frac{5\pi}{16}$

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a <$$