

Girraween High School

2015 Year 12 Trial Higher School Certificate

Mathematics (2 unit)

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

- For **Section II: Questions 11 – 16** MUST be returned in clearly marked *separate sections*.
- On each page of your answers, clearly write:
 - the **QUESTION** being answered
 - **YOUR NAME**
 - your **Mathematics TEACHER'S NAME**.
- Start each new question on a **NEW PAGE**.
- You may ask for extra pieces of paper if you need them.

SECTION I**10 Marks****Attempt all of Questions 1 – 10****Allow about 15 minutes for this section.**

Use the multiple-choice answer sheet for Questions 1 – 10.

1. What is the value of $\frac{2\pi^2}{19}$, correct to 3 significant figures?

- (A) 1.03
- (B) 1.04
- (C) 1.038
- (D) 1.039

2. Which of the following is equal to $\frac{1}{3\sqrt{5} + \sqrt{2}}$?

- (A) $\frac{3\sqrt{5} - \sqrt{2}}{13}$
- (B) $\frac{3\sqrt{5} + \sqrt{2}}{13}$
- (C) $\frac{3\sqrt{5} - \sqrt{2}}{43}$
- (D) $\frac{3\sqrt{5} + \sqrt{2}}{43}$

3. Which of the following is equivalent to $\frac{\log_a 8}{\log_a 4}$?

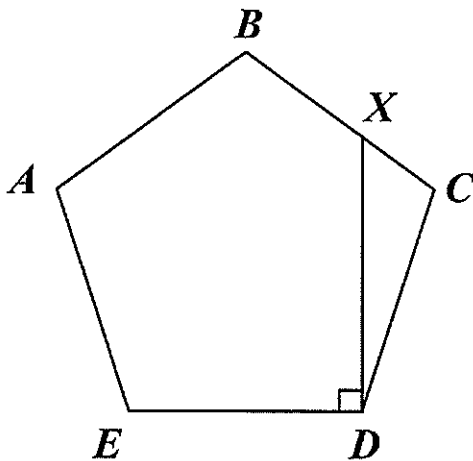
- (A) $1\frac{1}{2}$
- (B) 2
- (C) $\log_a 2$
- (D) $\log_a 4$

4. The quadratic equation $2x^2 - 4x - 3 = 0$ has roots α and β .

What is the value of $(\alpha + \beta) - (\alpha\beta)$?

- (A) $-\frac{1}{2}$
 (B) $-3\frac{1}{2}$
 (C) $\frac{1}{2}$
 (D) $3\frac{1}{2}$

- 5.



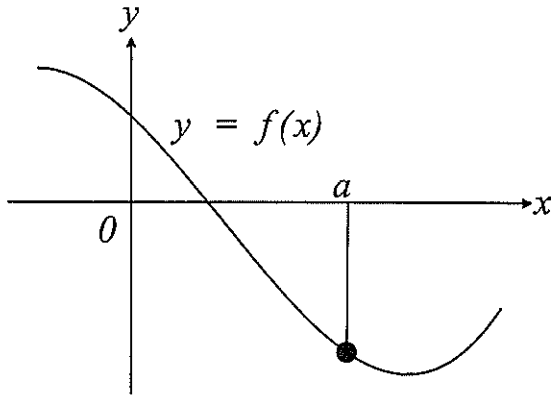
NOT TO SCALE

$ABCDE$ is a regular pentagon and $DX \perp DE$.

The size of $\angle CXD$ is :

- (A) 36°
 (B) 54°
 (C) 64°
 (D) 72°

6. The diagram shows the graph of $y = f(x)$.



Which of the following statements is true?

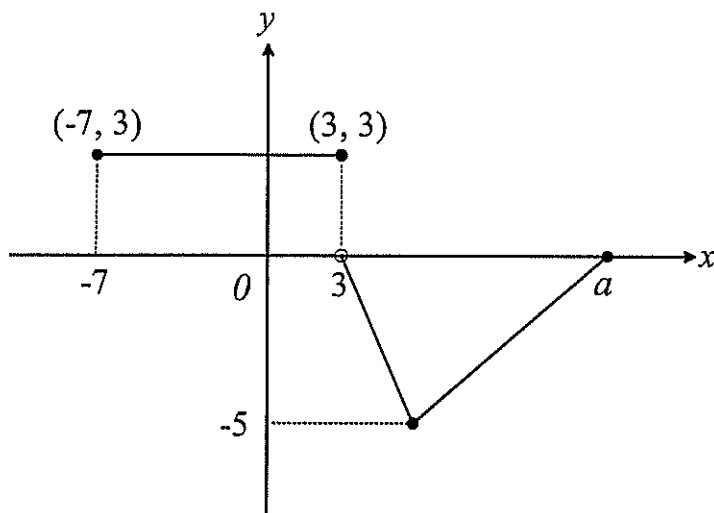
- (A) $f'(a) < 0$ and $f''(a) < 0$
 (B) $f'(a) < 0$ and $f''(a) > 0$
 (C) $f'(a) > 0$ and $f''(a) < 0$
 (D) $f'(a) > 0$ and $f''(a) > 0$
7. A parabola has focus $(0, -4)$ and directrix $y = 2$.

What is the equation of the parabola?

- (A) $x^2 = -12(y + 1)$
 (B) $x^2 = -24(y + 4)$
 (C) $x^2 = 12(y + 1)$
 (D) $x^2 = 24(y + 4)$
8. What is the derivative of $\frac{x}{\sin x}$?

- (A) $\frac{-x \cos x - \sin x}{\sin^2 x}$
 (B) $\frac{x \cos x - \sin x}{\sin^2 x}$
 (C) $\frac{\sin x + x \cos x}{\sin^2 x}$
 (D) $\frac{\sin x - x \cos x}{\sin^2 x}$

9. The diagram shows the graph of $y = f(x)$.

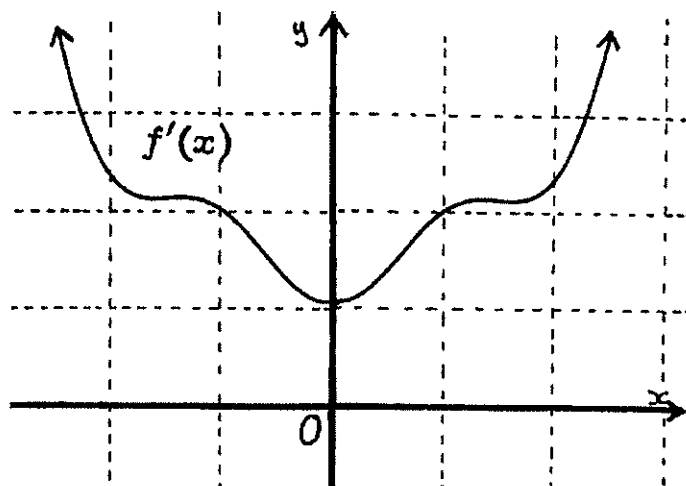


NOT TO SCALE

Use the graph to determine the value of a which satisfies the condition $\int_{-7}^a f(x) dx = 0$.

- (A) 9
 (B) 12
 (C) 13
 (D) 15

- 10.



It is known that $y = f(x)$ passes through the origin.

By examining the graph of $y = f'(x)$ shown above, state which of the following statements is true?

- (A) $f(x)$ is an even function with a point of inflexion at $f(0)$.
 (B) $f(x)$ is an even function with a local maximum at $f(0)$.
 (C) $f(x)$ is an odd function with a point of inflexion at $f(0)$.
 (D) $f(x)$ is an odd function with a local maximum at $f(0)$.

The examination continues on the next page.

SECTION II**90 Marks****Attempt all of Questions 11 – 16****Allow about 2 hours and 45 minutes for this section.**

Answer each question on the paper provided.

Start each new question on a **NEW PAGE**.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks)**Marks**Start a **NEW PAGE**.

- (a) Factorise $2x^2 + 11x - 21$. 2
- (b) Solve $|3x - 5| < 4$. 2
- (c) Find the equation of the tangent to the curve $y = x^3$ at the point where $x = 2$. 2
- (d) Differentiate $(e^{3x} - 5)^4$. 2
- (e) Evaluate $\int_1^2 \frac{1}{(3x - 1)^2} dx$ 3
- (f) (i) Find $\frac{d}{dx}(\sin x^3)$. 2
- (ii) Hence find $\int x^2 \cos x^3 dx$. 2

The examination continues on the next page.

Question 12 (15 Marks)**Marks**

Start a NEW PAGE.

- (a) What is the radius of a circle in which an arc of length 15 cm subtends an angle of 60° at the centre? 2

Give your answer correct to the nearest mm.

- (b) Solve $2 \log_e x = \log_e (3x + 10)$. 2

- (c) Differentiate $y = \frac{\sin x}{1 + \cos x}$ 3

and hence show that $\frac{dy}{dx} = \frac{1}{1 + \cos x}$.

- (d) Find $\int \frac{10x}{x^2 - 8} dx$. 2

- (e) Evaluate $\int_0^{\frac{\pi}{4}} \cos(2x) dx$ 2

- (f) Mayomi and Christina each throw a die.

- (i) Find the probability that they throw the same number. 2

- (ii) Find the probability that the number thrown by Christina is greater than the number thrown by Mayomi. 2

The examination continues on the next page.

Question 13 (15 Marks)

Marks

Start a NEW PAGE.

- (a) The first and last terms of an arithmetic series are 10 and 60 respectively, and the sum of the series is 3535.

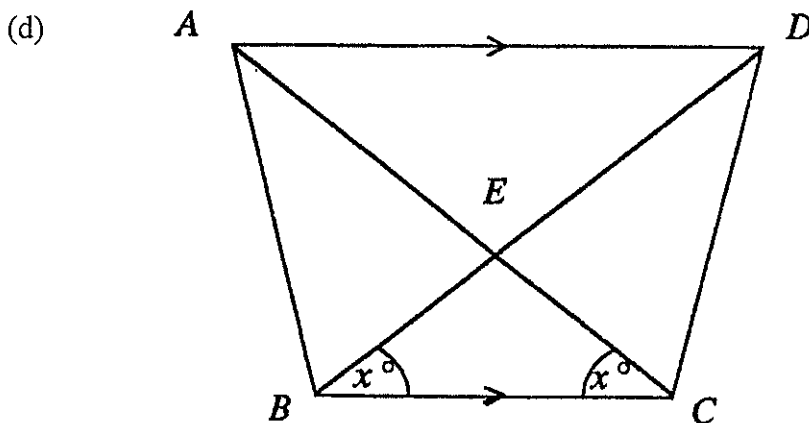
Find:

- (i) the number of terms in the series. 2
- (ii) the common difference. 2
- (b) “Although the number of unemployed is increasing, the Government’s policies to reduce unemployment seem to be taking effect.” 2

Given E is the number of unemployed, what does the above statement imply about

$$\frac{dE}{dt} \text{ and } \frac{d^2 E}{dt^2} ?$$

- (c) What is the volume of the solid of revolution formed by rotating the curve $y = \sec x$ about the x -axis for $0 \leq x \leq \frac{\pi}{3}$? 2



In the diagram AD is parallel to BC and $\angle DBC = \angle ACB = x^\circ$.

- (i) Show that $AE = DE$. 2
- (ii) Prove that the triangles $\triangle ABC$ and $\triangle DCB$ are congruent. 3
- (iii) Deduce that $\angle ABD = \angle DCA$. 2

The examination continues on the next page.

Question 14 (15 Marks)

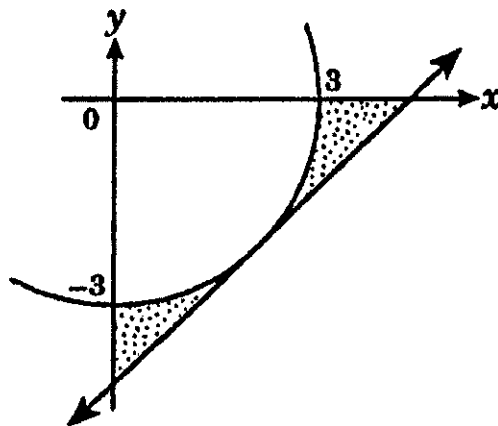
Marks

Start a NEW PAGE.

- (a) Consider the geometric series $1 + (\sqrt{5} - 2) + (\sqrt{5} - 2)^2 + \dots$
- (i) Explain why the geometric series has a limiting sum. 1
- (ii) Find the exact value of the limiting sum. 2
- Write your answer with a rational denominator.

- (b) Find the values of k for which the quadratic equation $x^2 + (k - 2)x + 4 = 0$ has no real roots. 2

- (c) (i) Prove that the line $3x - 4y = 15$ is a tangent to the circle $x^2 + y^2 = 9$. 2



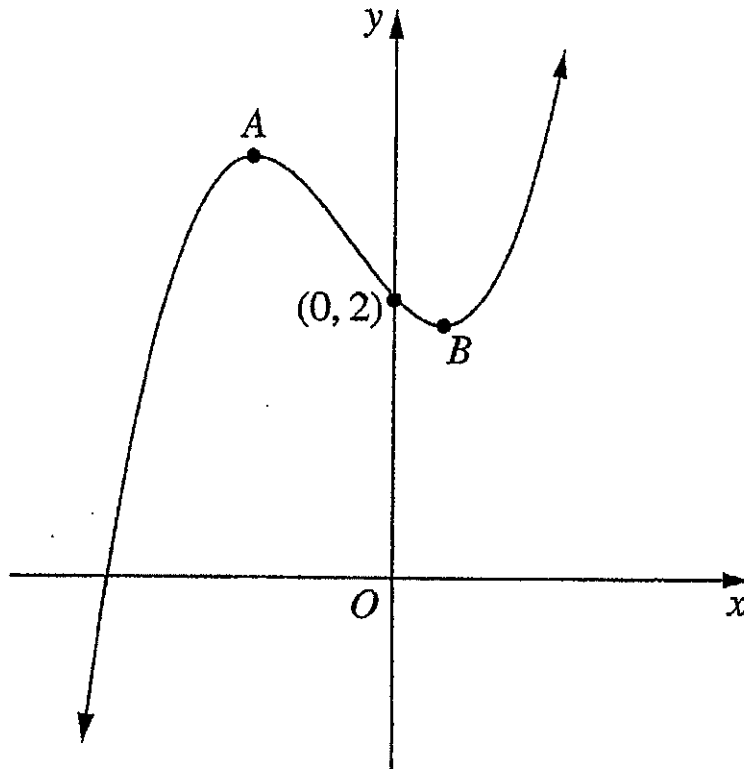
- (ii) Calculate the exact area in the fourth quadrant between the tangent $3x - 4y = 15$ and the circle $x^2 + y^2 = 9$ (as shown by the shaded area in the diagram above). 2

Question 14 continues on the next page.

Question 14 (continued)

(d) The graph of $y = x^3 + x^2 - x + 2$ is sketched below.

The points A and B are the turning points.



NOT TO SCALE

- | | | |
|-------|---|---|
| (i) | Find the coordinates of A and B . | 3 |
| (ii) | For what values of x is the curve concave up?
Give reasons for your answer. | 2 |
| (iii) | For what values of k has the equation $x^3 + x^2 - x + 2 = k$ three real solutions? | 1 |

End of Question 14.

The examination continues on the next page.

Question 15 (15 Marks)**Marks**

Start a NEW PAGE.

- (a) A standard pack of 52 cards consists of four suits (Diamonds, Hearts, Clubs and Spades) with 13 cards in each suit.

- (i) One card is drawn from the pack and kept on the table. A second card is drawn and placed beside it on the table. 1

What is the probability that the second card is from a different suit to the first?

- (ii) The two cards are replaced and the pack shuffled. Four cards are chosen from the pack and placed side by side on the table. 2

What is the probability that these four cards are all from different suits?

- (b) A particle is initially at rest at the origin.

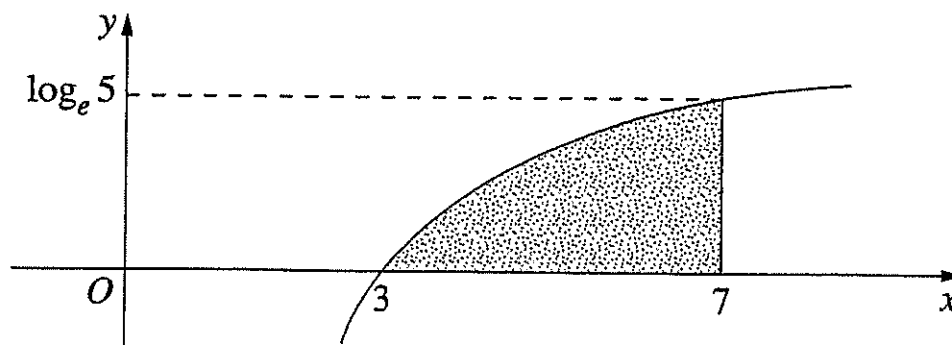
Its acceleration as a function of time t is given by $\ddot{x} = 4 \sin 2t$.

- (i) Show that the velocity of the particle is given by $\dot{x} = 2 - 2 \cos 2t$. 2

- (ii) Sketch the graph of the velocity for $0 \leq t \leq 2\pi$ AND determine the time at which the particle first comes to rest after $t = 0$. 3

- (iii) Find the distance travelled by the particle between $t = 0$ and the time at which the particle first comes to rest after $t = 0$. 2

- (c)



In the diagram, the shaded region is bounded by $y = \log_e(x - 2)$, the x -axis and the line $x = 7$. 5

Find the exact value of the area of the shaded region.

The examination continues on the next page.

Question 16 (15 Marks)

Marks

Start a NEW PAGE.

- (a) A drug is used to control a medical condition. It is known that the quantity Q of drug remaining in the body after t hours satisfies an equation of the form

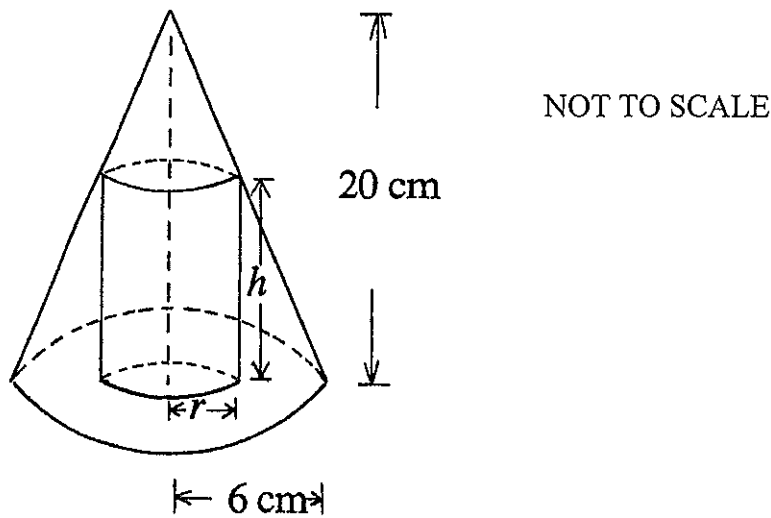
$$Q = Q_0 e^{-kt}$$

where Q_0 and k are constants.

The initial dose is 6 milligrams and after 15 hours the amount remaining in the body is half the initial dose.

- (i) Find the values of Q_0 and k , correct to 3 decimal places where necessary. 2
- (ii) When will one-eighth of the initial dose remain? 2

- (b)



A cylinder of radius r cm and height h cm is inscribed in a cone with base radius 6 cm and height 20 cm as in the diagram.

- (i) Show that the volume V of the cylinder is given by 2

$$V = \frac{10 \pi r^2 (6 - r)}{3}$$

- (ii) Hence find the values of r and h for the cylinder which has maximum volume. 2

Question 16 continues on the next page.

Question 16 (continued)

- (c) A farmer borrows \$80 000 to purchase new machinery. The interest is calculated monthly at the rate of 2% per month, and is compounded each month.

The farmer intends to repay the loan with interest in two equal annual instalments of \$ M at the end of the first and second years.

- (i) How much does the farmer owe at the end of the first month? 1
- (ii) Write an expression involving M for the total amount owed by the farmer after 12 months, just after the first instalment of \$ M has been paid. 2
- (iii) Find an expression for the amount owed at the end of the second year and deduce that 2

$$M = \frac{80\,000 \times (1.02)^{24}}{(1.02)^{12} + 1}$$

- (iv) What is the total interest over the two year period? 2

End of examination.

Question 11 (15 Marks)

$$\begin{aligned} \text{(a)} \quad & 2x^2 + 11x - 21 \quad \begin{array}{l} -42 \\ -42 \\ -11 \end{array} \begin{array}{l} \times \\ \times \\ \times \end{array} \begin{array}{l} 14 \\ 14 \\ 3 \end{array} \\ & = 2x^2 + 14x - 3x - 21 \\ & = 2x(x+7) - 3(x+7) \\ & = (2x-3)(x+7) \quad (2m) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & |3x-5| < 4 \\ & \Rightarrow (3x-5) < 4 \\ & +(3x-5) < 4 \quad \text{or} \quad -(3x-5) < 4 \\ & 3x < 9 \qquad \qquad \qquad 3x-5 > -4 \\ & x < 3 \qquad \qquad \qquad 3x > 1 \\ & \qquad \qquad \qquad \qquad \qquad \qquad x > \frac{1}{3} \\ & \therefore \frac{1}{3} < x < 3 \quad (2m) \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & y = x^3 \\ & \frac{dy}{dx} = 3x^2 \\ \text{when } x=2: & \frac{dy}{dx} = 3 \times 4 \\ & \frac{dy}{dx} = 12 \\ \text{when } x=2: & y = 2^3 \\ & y = 8 \\ \text{Require tangent } m=12, (2, 8): \\ & y - y_1 = m(x - x_1) \\ & y - 8 = 12(x - 2) \\ & y - 8 = 12x - 24 \\ & y = 12x - 16 \\ \text{or } 12x - y - 16 = 0 \quad (2m) \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad & y = (e^{3x} - 5)^4 \\ & \frac{dy}{dx} = 4(e^{3x} - 5)^3 \cdot (3e^{3x}) \\ & = 12e^{3x} (e^{3x} - 5)^4 \quad (2m) \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad & \int_1^2 \frac{1}{(3x-1)^2} dx = \int_1^2 (3x-1)^{-2} dx \\ & = \left[\frac{(3x-1)^{-1}}{3 \cdot (-1)} \right]_1^2 \\ & = \left[\frac{-1}{3(3x-1)} \right]_1^2 \\ & = \frac{-1}{15} - \left(\frac{-1}{6} \right) \\ & = \frac{-1}{15} + \frac{1}{6} \\ & = \frac{1}{10} \quad (3m) \end{aligned}$$

Question 11 continued:

$$\begin{aligned} \text{1) (f) (i)} \quad & y = \sin x^3 \\ & y = \sin u \qquad u = x^3 \\ & \frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 3x^2 \\ & \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \\ & = \cos u \times 3x^2 \\ & = 3x^2 \cos x^3 \quad (2m) \end{aligned}$$

$$\begin{aligned} \text{f) (ii)} \quad & \text{If } \frac{d}{dx} (\sin x^3) = 3x^2 \cos x^3 \\ & \text{then } \int 3x^2 \cos x^3 dx = \sin x^3 + C_1 \\ & \therefore \int x^2 \cos x^3 dx = \frac{1}{3} \sin x^3 + C \quad (2m) \end{aligned}$$

Question 12 (15 Marks)

$$\begin{aligned} \text{(a)} \quad & L = 15 \quad \theta = 60^\circ \\ & \qquad \qquad \qquad = \frac{\pi}{3} \\ & L = r\theta \\ & 15 = r \cdot \frac{\pi}{3} \\ & r = \frac{45}{\pi} \\ & r = 14.32394488 \\ & \therefore r = 14.3 \text{ cm, or } r = 143 \text{ mm} \\ & \text{(correct to nearest mm.)} \quad (2m) \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & 2 \log_e x = \log_e (3x+10) \\ & \log_e x^2 = \log_e (3x+10) \\ & \therefore x^2 = 3x+10 \\ & x^2 - 3x - 10 = 0 \\ & (x+2)(x-5) = 0 \\ & x = -2, \text{ or } x = 5 \\ & x \neq -2 \text{ as for } y = \log_e x, x > 0. \\ & \therefore \text{only solution is } x = 5. \quad (2m) \end{aligned}$$

Question 12 continued:

$$12)(c) \quad y = \frac{\sin x}{1 + \cos x}$$

$$u = \sin x \quad v = 1 + \cos x$$

$$\frac{du}{dx} = \cos x \quad \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$= \frac{(1 + \cos x)(\cos x) - (\sin x)(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$$

$$= \frac{\cos x + 1}{(1 + \cos x)^2} \quad \text{since } \sin^2 x + \cos^2 x = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + \cos x}$$

(3m)

$$12)(d) \quad \int \frac{10x}{x^2-8} dx = 5 \int \frac{2x}{x^2-8} dx$$

$$= 5 \log_e(x^2-8) + C$$

$$\text{since } \int \frac{f'(x)}{f(x)} dx = \log_e f(x) + C$$

(2m)

$$12)(e) \quad \int_0^{\frac{\pi}{4}} \cos(2x) dx$$

$$= \frac{1}{2} \left[\sin(2x) \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2} \left[\sin \frac{\pi}{2} - \sin 0 \right]$$

$$= \frac{1}{2} (1 - 0)$$

$$= \frac{1}{2}$$

12)(f)(i) P(throw same number)

$$= P(1,1) \text{ or } P(2,2) \text{ or } P(3,3) \text{ or } P(4,4) \\ \text{or } P(5,5) \text{ or } P(6,6)$$

$$= 6 \times \frac{1}{36}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

(2m)

Question 12(f) continued:

12)(f)(ii) Let Mayomi be first number, then Christina throw second number.
Possible outcomes:

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
 (2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
 (3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
 (4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
 (5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
 (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

15 possible outcomes where Christina throws number greater than Mayomi.

$\therefore P(\text{Christina's number} > \text{Mayomi's number})$

$$= \frac{15}{36}$$

$$= \frac{5}{12}$$

(2m)

12)(f)(ii) ALTERNATIVE METHOD

from f(i), $P(\text{throw same number}) = \frac{1}{6}$

$\therefore P(\text{do NOT throw same number}) = \frac{5}{6}$

of this, $\frac{1}{2} \times \frac{5}{6} = P(\text{Christina} > \text{Mayomi})$
 $= P(\text{Mayomi} > \text{Christina})$

$\therefore P(\text{Christina's number} > \text{Mayomi's number})$

$$= \frac{1}{2} \times \frac{5}{6}$$

$$= \frac{5}{12}$$

(2m)

Question 13 (15 Marks)

(a)(i) Arithmetic series: $a = 10$, $l = 60$
 $S_n = 3535$

$$S_n = \frac{n}{2} (a + l)$$

$$3535 = \frac{n}{2} (10 + 60)$$

$$35n = 3535$$

$$n = \frac{3535}{35}$$

$$\therefore n = 101$$

(2m)

(b)(a)(ii) $l = a + (n-1)d$

$$60 = 10 + 100d$$

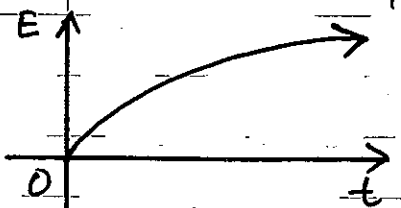
$$100d = 50$$

$$\therefore d = \frac{1}{2}$$

(2m)

Question 13 continued:

13)(b) Possible graph of unemployment ^(E) over time may be:

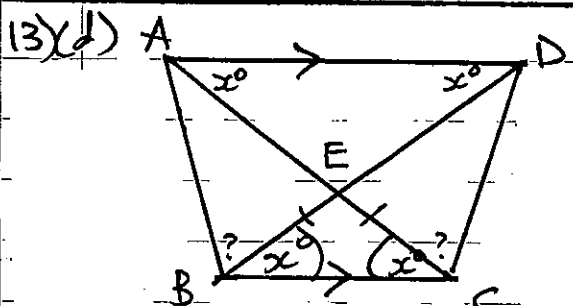


Unemployment is increasing: so $\frac{dE}{dt} > 0$.
 Since policy to reduce unemployment is taking effect, then even though the number of unemployed is still increasing, rate of increase is slowing down, i.e. curve is concave down.

i.e. $\frac{d^2E}{dt^2} < 0$

so in this case: $\frac{dE}{dt} > 0$, $\frac{d^2E}{dt^2} < 0$. (2m)

13)(c) volume about x-axis = $\pi \int y^2 dx$
 $= \pi \int_0^{\frac{\pi}{3}} \sec^2 x dx$
 $= \pi [\tan x]_0^{\frac{\pi}{3}}$
 $= \pi [\tan \frac{\pi}{3} - \tan 0]_0^{\frac{\pi}{3}}$
 $= \pi \sqrt{3}$ cubic units (2m)



(i) $\angle ADE = \angle CBE$ (alternate angles equal)
 $AD \parallel BC$
 and $\angle DAE = \angle ACB$ (alternate angles equal)
 $AD \parallel BC$
 $\therefore \triangle ADE$ is isosceles triangle
 $\therefore AE = DE$ (sides opposite equal angles are equal) (2m)

Question 13(d) continued:

13)(d)(ii) $\triangle BEC$ is isosceles
 (since given $\angle DBC = x^\circ = \angle ACB$)
 $\therefore EB = EC$ (sides opposite equal angles are equal)

But $AC = AE + EC$

and $DB = DE + EB$

but $AE = DE$, and $EB = EC$

$\therefore AC = DB$.

Now in $\triangle ABC$, $\triangle DCB$:

$AC = DB$ (as shown) S

$\angle ACB = \angle DBC$ (given) A

BC is common S

$\therefore \triangle ABC \cong \triangle DCB$ (SAS) (3m)

13)(d)(iii) $\angle ABC = \angle DCB$

(matching angles in congruent triangles are equal)

i.e. $\angle ABD + \angle DBC = \angle DCA + \angle ACB$

But $\angle DBC = x^\circ = \angle ACB$ (given)

$\therefore \angle ABD = \angle DCA$ (2m)

Question 14 (15 Marks)

14)(a)(i) $1 + (\sqrt{5}-2) + (\sqrt{5}-2)^2 + \dots$

Totd this is geometric series:

have $a=1$, $r = \sqrt{5}-2$

Limiting sum exists if $|r| < 1$

in this case $r = \sqrt{5}-2 = 0.236\dots$

i.e. $r < 1$

\therefore limiting sum exists. (1m)

14)(a)(ii) $S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 - (\sqrt{5}-2)}$

$= \frac{1}{3-\sqrt{5}}$

$= \frac{1}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}$

$= \frac{3+\sqrt{5}}{9-5}$

$\therefore S_\infty = \frac{3+\sqrt{5}}{4}$

Question 14 continued:

(14)(b) $x^2 + (k-2)x + 4 = 0$

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (k-2)^2 - 4 \times 1 \times 4 \\ &= k^2 - 4k + 4 - 16 \\ \Delta &= k^2 - 4k - 12 \\ \Delta &= (k-6)(k+2) \end{aligned}$$

for no real roots,

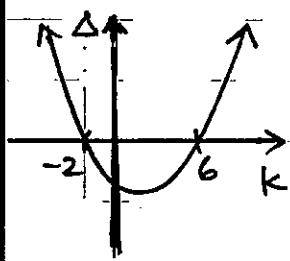
$$\Delta < 0$$

from sketch,

$$\Delta < 0 \text{ when}$$

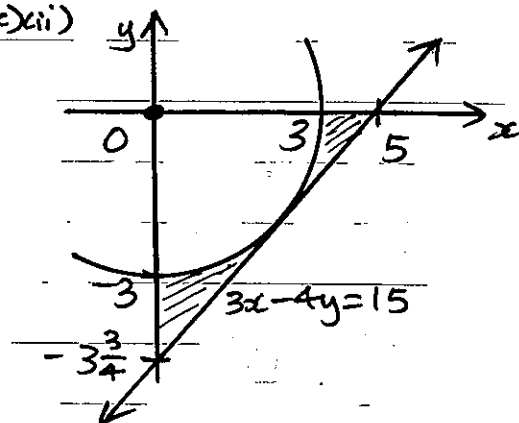
$$-2 < k < 6$$

(2m)



Question 14(c) (ii) continued:

(14)(c)(ii)



Shaded Area

$$\begin{aligned} &= (\text{Area triangle}) - (\text{Area Quadrant}) \\ &= \left(\frac{1}{2} \times 5 \times \frac{15}{4}\right) - \left(\frac{1}{4} \times \pi \times 3^2\right) \\ &= \frac{75}{8} - \frac{9\pi}{4} \\ &= \frac{75 - 18\pi}{8} \\ &= \frac{3(25 - 6\pi)}{8} \text{ square units} \end{aligned}$$

(2m)

(14)(c)(i) $x^2 + y^2 = 9$ is a circle centre (0,0) radius 3.

For $3x - 4y = 15$ to be a tangent to circle, \perp distance from (0,0) to $3x - 4y - 15 = 0$ must be 3 units.

$$\begin{aligned} d &= \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right| \\ &= \left| \frac{3 \times 0 - 4 \times 0 - 15}{\sqrt{3^2 + 4^2}} \right| \\ &= \left| \frac{-15}{\sqrt{25}} \right| \\ &= \frac{15}{5} \end{aligned}$$

$\therefore d = 3$

(2m)

So perpendicular distance is 3 units

$\therefore 3x - 4y = 15$ is tangent to circle $x^2 + y^2 = 9$

(14)(c)(ii) $3x - 4y = 15$ crosses:

x-axis: when $y=0$: $3x = 15$
 $x = 5$

x-intercept is (5,0)

y-axis: when $x=0$: $-4y = 15$

$$y = -\frac{15}{4}$$

$$y = -3\frac{3}{4}$$

(14)(d) $y = x^3 + x^2 - x + 2$

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$

A and B are stationary points,

So need to solve $\frac{dy}{dx} = 0$:

$$3x^2 + 2x - 1 = 0$$

$$3x^2 + 3x - x - 1 = 0$$

$$3x(x+1) - 1(x+1) = 0$$

$$(3x-1)(x+1) = 0$$

$$x = \frac{1}{3} \text{ or } x = -1$$

when $x = -1$: $y = (-1)^3 + (-1)^2 - (-1) + 2$
 $= -1 + 1 + 1 + 2$

$$y = 3$$

when $x = \frac{1}{3}$: $y = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + 2$

$$= \frac{49}{27}$$

$$y = 1\frac{22}{27}$$

$\therefore A(-1, 3), B\left(\frac{1}{3}, 1\frac{22}{27}\right)$

(3m)

Question 14 (d) continued:

14)(d)(ii) $\frac{dy}{dx} = 3x^2 + 2x - 1$

$\frac{d^2y}{dx^2} = 6x + 2$

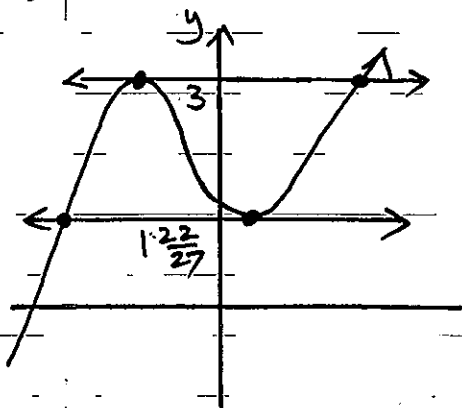
concave up when $\frac{d^2y}{dx^2} > 0$

i.e. when $6x + 2 > 0$

$6x > -2$

\therefore concave up when $x > -\frac{1}{3}$ (2m)

14)(d)(iii)



$x^3 + x^2 - x + 2 = k$ represents the intersection of $y = x^3 + x^2 - x + 2$ and $y = k$.

There will be 3 real solutions for

$1 \frac{22}{27} < k < 3$ (1m)

(Note: only 2 roots if $k = 1 \frac{22}{27}$ or $k = 3$)

Question 15 (15 Marks)

a) (i) After first card is drawn, 51 cards remain of which 3x13 are of a different suit

\therefore P (2nd card different suit)

$= \frac{39}{51}$

$= \frac{13}{17}$

(1m)

Question 15 (a) continued:

15)(a)(ii)

P (all 4 different suits)

$= P(\text{card 1 any suit}) \times P(\text{card 2 any of 3 suits}) \times P(\text{card 3 any of 2 suits}) \times P(\text{card 4 last remaining suit})$

$= \left(\frac{52}{52}\right) \times \left(\frac{39}{51}\right) \times \left(\frac{26}{50}\right) \times \left(\frac{13}{49}\right)$

$= \frac{685464}{6497400}$

$= \frac{2197}{20825}$

$= 0.1054981993...$

$\doteq 0.105$ (3 decimal places) (2m)

15)(b)(i) $\ddot{x} = 4 \sin 2t$

$\dot{x} = \int \ddot{x} dt$

$= \int 4 \sin 2t$

$\dot{x} = -2 \cos 2t + C_1$

when $t=0, \dot{x}=0: 0 = -2 \cos 2(0) + C_1$

$0 = -2 + C_1$

$\therefore C_1 = 2$

$\therefore \dot{x} = -2 \cos 2t + 2$ (2m)

15)(b)(ii) $\dot{x} = 2 - 2 \cos 2t$

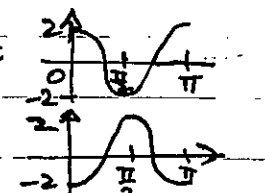
Period = $\frac{2\pi}{2}$

Amplitude = 2

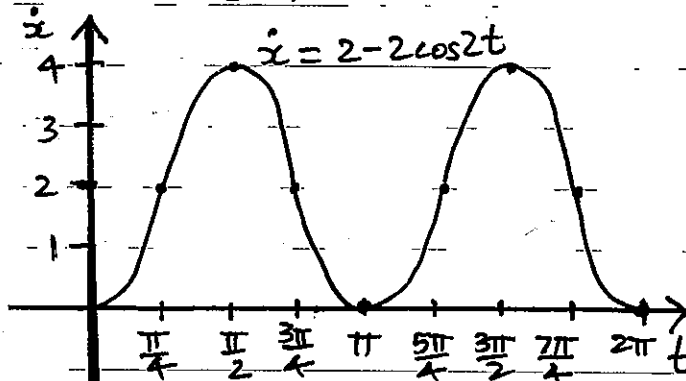
$= \pi$

$\dot{x} = 2 \cos 2t$

$\dot{x} = -2 \cos 2t$



then translate UP 2 units to get final sketch of graph:



Particle first comes to rest (after $t=0$)

at $t = \pi$ (3m)

Question 15(b) continued:

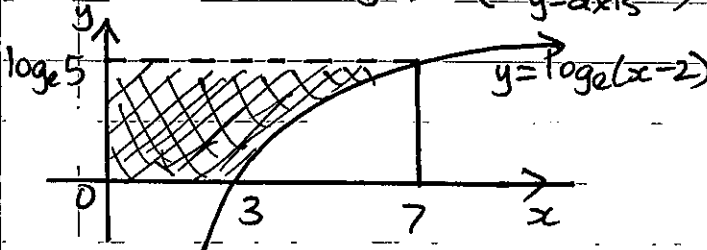
15)(b)(ciii) Distance travelled between $t=0$ and $t=\pi$

$$\begin{aligned} x &= \int_0^{\pi} (2 - 2\cos 2t) dt \\ &= [2t - \sin 2t]_0^{\pi} \\ &= (2\pi - \sin 2\pi) - (0 - \sin 0) \\ &= (2\pi - 0) - (0 - 0) \end{aligned}$$

Distance = 2π
 \therefore distance travelled is 2π units 2m

15)(c) Required area needs to be found by subtraction

Area = (area of rectangle) - (area between curve and y-axis)



Shaded area about y-axis needed:

$$y = \log_e(x-2)$$

From definition of logarithm:

$$e^y = x - 2$$

$$\therefore x = 2 + e^y$$

$$\begin{aligned} \text{Area} &= \int_0^{\log_e 5} (2 + e^y) dy \\ &= [2y + e^y]_0^{\log_e 5} \\ &= (2\log_e 5 + e^{\log_e 5}) - (2 \times 0 + e^0) \end{aligned}$$

$$= 2\log_e 5 + 5 - 1$$

$$\text{Area} = 2\log_e 5 + 4$$

Now Area Rectangle = $7 \times \log_e 5$
 $= 7\log_e 5$

So required Area

$$\begin{aligned} &= (\text{Area rectangle}) - (\text{area between curve \& y-axis}) \\ &= 7\log_e 5 - (2\log_e 5 + 4) \\ &= 5\log_e 5 - 4 \text{ square units} \end{aligned}$$

5m

Question 16 (15 Marks)

16)(a)(i) $Q = Q_0 e^{-kt}$

when $t=0$, $Q=6$: $6 = Q_0 e^0$

$$\therefore Q_0 = 6$$

So $Q = 6e^{-kt}$

when $t=15$, $Q = \frac{Q_0}{2}$

$$\therefore \frac{Q_0}{2} = Q_0 e^{-15k}$$

$$\frac{1}{2} = e^{-15k}$$

Taking logarithms base e of both sides:

$$\log_e\left(\frac{1}{2}\right) = -15k$$

$$k = -\frac{1}{15} \log_e\left(\frac{1}{2}\right)$$

$$= 0.04620981204\dots$$

$$\therefore k \approx 0.046 \text{ (3 decimal places)}$$

$\therefore Q_0 = 6$, $k \approx 0.046$ 2m

16)(a)(ii) Require $\frac{1}{8}$ of initial dose Q_0 :

i.e. require $\frac{Q_0}{8}$

Solving $\frac{Q_0}{8} = Q_0 e^{-kt}$

$$\frac{1}{8} = e^{-kt}$$

Taking logarithms base e of both sides:

$$\log_e\left(\frac{1}{8}\right) = -kt$$

$$t = \frac{\log_e\left(\frac{1}{8}\right)}{-k}$$

$$= \frac{\log_e\left(\frac{1}{8}\right)}{-0.04620981204}$$

$$= 45$$

\therefore After 45 hours, there will be $\frac{1}{8}$ of initial dose remaining

2m

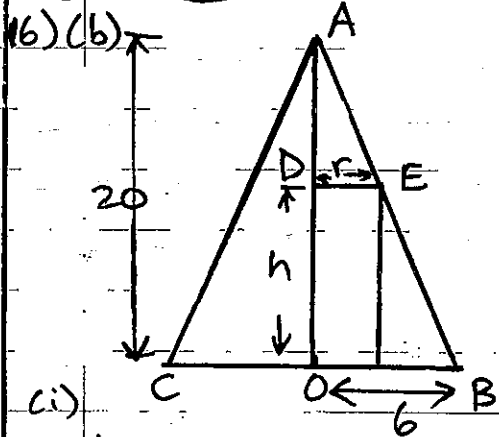
ALTERNATIVE METHOD 16(a)(ii)

15 hours \rightarrow half life

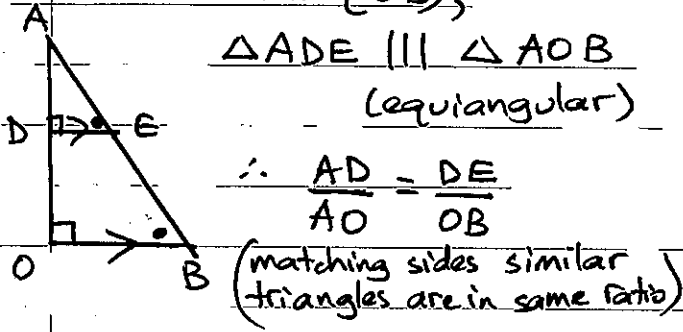
$$\frac{1}{8} \rightarrow 3 \text{ "half lives"} \rightarrow \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

So $\frac{1}{8}$ of initial dose = 3×15
 $= 45$ hours

Question 16 continued:



(i) Take cross-section of cone.
 Since top of cylinder (DE) is parallel to base of cone (OB),



$\triangle ADE \parallel \triangle AOB$
 (equiangular)

$$\therefore \frac{AD}{AO} = \frac{DE}{OB}$$

(matching sides similar triangles are in same ratio)

i.e. $\frac{20-h}{20} = \frac{r}{6}$

$$6(20-h) = 20r$$

$$120 - 6h = 20r$$

$$6h = 120 - 20r$$

$$h = \frac{120 - 20r}{6}$$

$$h = \frac{20(6-r)}{6}$$

$$\therefore h = \frac{10(6-r)}{3}$$

\therefore Volume of cylinder
 $= \pi r^2 h$

$$= \pi r^2 \cdot \frac{10(6-r)}{3}$$

$$\therefore V = \frac{10\pi r^2 (6-r)}{3} \quad (2M)$$

Question 16(b) continued:

16)(b) (ii) $V = \frac{10\pi r^2 (6-r)}{3}$

i.e. $V = \frac{10\pi \cdot 6r^2}{3} - \frac{10\pi r^3}{3}$

$$V = 20\pi r^2 - \frac{10\pi r^3}{3}$$

$$\therefore \frac{dV}{dr} = 40\pi r - \frac{3 \cdot 10\pi r^2}{3}$$

$$\frac{dV}{dr} = 40\pi r - 10\pi r^2$$

For stationary points, solve $\frac{dV}{dr} = 0$

$$40\pi r - 10\pi r^2 = 0$$

$$10\pi r(4 - r) = 0$$

$$\therefore r = 0 \text{ or } r = 4$$

Require maximum volume, so $r \neq 0$

Need to check $r = 4$ is a maximum:

$$\frac{dV}{dr} = 40\pi r - 10\pi r^2$$

$$\frac{d^2V}{dr^2} = 40\pi - 20\pi r$$

when $r = 4$: $\frac{d^2V}{dr^2} = 40\pi - 80\pi = -40\pi$

i.e. $\frac{d^2V}{dr^2} < 0$ when $r = 4$
 i.e. concave down \curvearrowright

\therefore maximum when $r = 4$.

from (i): $h = 10 \frac{(6-r)}{3}$

$$= \frac{10 \times 2}{3}$$

$$h = \frac{20}{3} \text{ or } 6\frac{2}{3}$$

\therefore Maximum volume when

$$r = 4 \text{ cm, } h = \frac{20}{3} \text{ cm or } 6\frac{2}{3} \text{ cm}$$

(2M)

Question 16 continued:

16) cc) A_n is amount owed at end of n months

P is principal, amount borrowed.

in this case $P = 80\,000$.

$$\begin{aligned} \text{ci)} \quad A &= P \left(1 + \frac{r}{100}\right)^n \\ A_1 &= 80\,000 \times \left(1 + \frac{2}{100}\right)^1 \end{aligned}$$

$$\begin{aligned} \text{ie.} \quad A_1 &= 80\,000 \times 1.02 \\ &= \$81\,600 \quad (1\text{m}) \end{aligned}$$

$$\begin{aligned} \text{cii)} \quad A_2 &= A_1 \left(1 + \frac{2}{100}\right) \\ &= A_1 \times 1.02 \\ &= (80\,000 \times 1.02) \times 1.02 \\ A_2 &= 80\,000 \times 1.02^2 \end{aligned}$$

$$\begin{aligned} A_3 &= A_2 \times 1.02 \\ &= (80\,000 \times 1.02^2) \times 1.02 \end{aligned}$$

$$A_3 = 80\,000 \times 1.02^3$$

Following this pattern for first 12 months, then:

$$A_{12} = 80\,000 \times 1.02^{12}$$

So after 1st instalment of $\$M$,

$$\begin{aligned} \text{Amount owing after 12 months} \\ &= 80\,000 \times 1.02^{12} - M \quad (2\text{m}) \end{aligned}$$

ciii) 1st repayment made after A_{12} .

But 2nd repayment not made until after A_{24} .

$$\begin{aligned} A_{13} &= A_{12} \left(1 + \frac{2}{100}\right) \\ &= A_{12} \times 1.02 \end{aligned}$$

$$A_{13} = [80\,000 \times 1.02^{12} - M] \times 1.02$$

$$A_{14} = [80\,000 \times 1.02^{12} - M] \times 1.02^2$$

$$A_{15} = [80\,000 \times 1.02^{12} - M] \times 1.02^3$$

Following this pattern:

$$A_{24} = [80\,000 \times 1.02^{12} - M] \times 1.02^{12}$$

Question 16(c)ciii) continued:

2nd (final) repayment of $\$M$ is paid after A_{24} :

$$\text{ie.} \quad [80\,000 \times 1.02^{12} - M] \times 1.02^{12} - M$$

But after 2nd/final repayment there is no money owing.

$$\therefore [80\,000 \times 1.02^{12} - M] \times 1.02^{12} - M = 0$$

$$80\,000 \times 1.02^{24} - 1.02^{12} M - M = 0$$

$$80\,000 \times 1.02^{24} - M(1.02^{12} + 1) = 0$$

$$\text{ie.} \quad M(1.02^{12} + 1) = 80\,000 \times 1.02^{24}$$

$$\therefore M = \frac{80\,000 \times 1.02^{24}}{(1.02^{12} + 1)}$$

$$\text{ie.} \quad M = \frac{80\,000 \times (1.02)^{24}}{(1.02^{12}) + 1} \quad (2\text{m})$$

(iv) From calculator

$$M = \frac{80\,000 \times 1.02^{24}}{1.02^{12} + 1} = 56\,728.95203$$

ie. each instalment $M = \$56\,728.95$

\therefore Total repaid in the 2 instalments

$$= 2 \times \$56\,728.95$$

$$= \$113\,457.90$$

$$\therefore \text{Interest} = (\text{Total repaid}) - (\text{original loan of } \$80\,000)$$

$$= \$113\,457.90 - \$80\,000$$

$$= \$33\,457.90$$