
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2015 Trial Examination

FORM VI

MATHEMATICS EXTENSION II

Friday 31st July 2015

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hour
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Candidature — 73 boys

Examiner

BDD

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The roots of the quadratic equation $x^2 - 8ix - 20 = 0$ are:

(A) $4i \pm 2$ (B) $4 \pm 2i$

(C) $-4i \pm 2$ (D) $-4 \pm 2i$

QUESTION TWO

The value of $\displaystyle\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx$ is:

(A) $-\dfrac{1}{2}$ (B) $\dfrac{1}{4}$

(C) $\dfrac{1}{2}$ (D) 1

QUESTION THREE

The gradient of the tangent to the parametric curve $x = 2\sec\theta$, $y = 3\tan\theta$ at θ is:

(A) $\dfrac{2}{3}\sin\theta$ (B) $\dfrac{2}{3}\operatorname{cosec}\theta$

(C) $\dfrac{3}{2}\sin\theta$ (D) $\dfrac{3}{2}\operatorname{cosec}\theta$

QUESTION FOUR

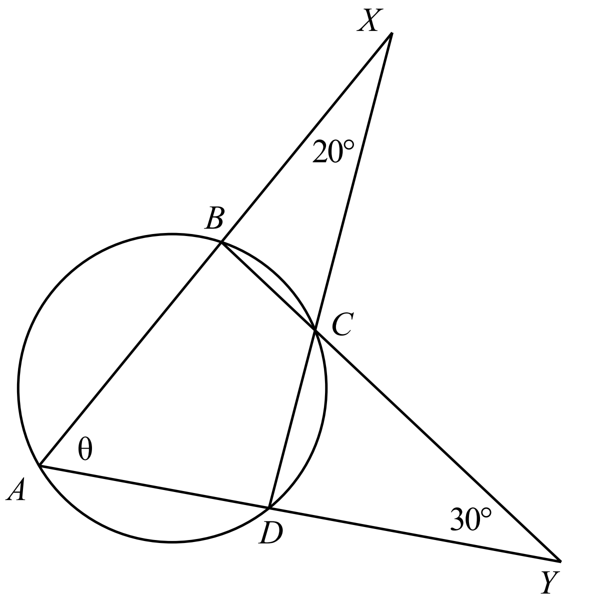
Which of the following functions is odd?

(A) $y = x\sin x$ (B) $y = \sin(\sin(x))$

(C) $y = \ln|x|$ (D) $y = \sin^2(x)$

Exam continues next page ...

QUESTION FIVE

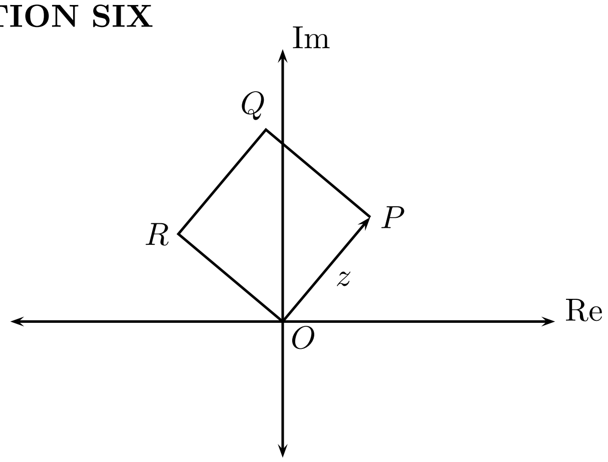


The size of angle θ in the diagram above is:

(A) $50°$ (B) $55°$

(C) $60°$ (D) $65°$

QUESTION SIX



The point P in quadrant one represents complex number z. The points O, P, Q, R are the vertices of a square, as in the diagram.

Which statement is NOT true about the square:

(A) side OR is represented by iz

(B) the centre of the square is represented by $\frac{1}{2}(1-i)z$

(C) diagonal RP is represented by $(1-i)z$

(D) vertex Q is represented by $(1+i)z$

Exam continues overleaf ...

QUESTION SEVEN

A pupil makes the following claims about the roots of the equation $z^6 = 1$:

(I) The roots lie on the vertices of a hexagon in the complex plane

(II) The roots lie on the unit circle in the complex plane

(III) If ω is a root, then so is $\dfrac{1}{\omega}$

(IV) If ω is a root, then so is $\overline{\omega}$

Which of these statements are TRUE?

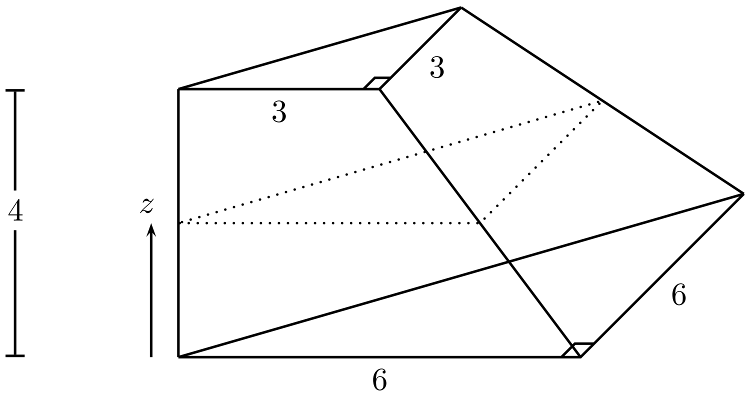
(A) I and IV

(B) II and III

(C) I, II and III

(D) I, II, III and IV

QUESTION EIGHT



The base and top of the solid depicted are right angled isosceles triangles. A pupil is required to determine the volume by slicing parallel to the base. A typical slice parallel to the base at height z from the base is marked.

The cross-sectional area of the slice is:

(A) $\dfrac{1}{2}(6 - \tfrac{3}{4}z)^2$

(B) $\dfrac{1}{4}(6 - \tfrac{1}{4}z)^2$

(C) $\dfrac{1}{2}(7 - z)^2$

(D) $\dfrac{1}{2}(36 - \tfrac{27}{4}z)$

QUESTION NINE

The point defined by the complex number z moves in the complex plane subject to the constraint $\left|z - 3i\right| + \left|z + 3i\right| = 12$.
The locus of z is a conic with eccentricity:

(A) $\dfrac{1}{4}$.

(B) $\dfrac{1}{2}$.

(C) 1.

(D) 2.

QUESTION TEN

A polynomial $P(x)$ of fourth degree with real coefficients has the following properties:

$P(1) = 0, P'(1) \neq 0$

$P(2) \neq 0, P'(2) = P''(2) = 0$

What is the greatest number of complex non-real roots the polynomial could have?

(A) 0 (B) 1

(C) 2 (D) 3

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Given $w = 1 - 2i$, $z = 3 + 4i$, express the following in the form $a + ib$ for real a, b:

(i) w^2 **1**

(ii) $\dfrac{z}{w}$ **1**

(b) Given $t = 1 + i\sqrt{3}$ find:

(i) t in modulus–argument form, **2**

(ii) t^8 in Cartesian form. **1**

(c) Find:

(i) $\displaystyle\int \frac{1}{x^2 + 6x + 13}\,dx$ **2**

(ii) $\displaystyle\int x \sin x \, dx$ **2**

(d) Evaluate the following integral, expressing your answer in simplest exact form: **2**

$$\int_{10}^{17} \frac{dx}{\sqrt{x^2 - 64}}\,.$$

(e) (i) Find constants A, B and C such that **2**

$$\frac{-4x^2 + 5x + 1}{(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + C\,.$$

(ii) Hence find **2**

$$\int \frac{-4x^2 + 5x + 1}{(x-1)^2}\,dx\,.$$

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the hyperbola $9x^2 - 16y^2 = 144$.

(i) Find the eccentricity, foci, directrices and asymptotes of the hyperbola. **3**

(ii) Sketch the curve, locating the foci, directrices, y-intercepts and asymptotes. **2**

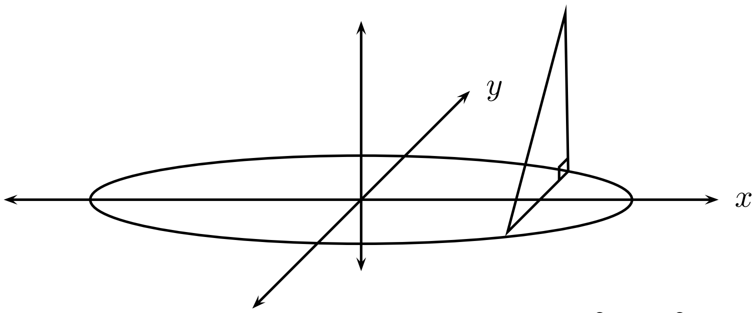
(iii) Use calculus to find the gradient of the tangent at $x = 5$ in quadrant one. **2**

(iv) Consider a tangent with point of contact in quadrant one. Explain geometrically why its gradient will always be greater than 0·75. **1**

(b) (i) Solve the equation $z^5 = -1$, leaving your answers in modulus–argument form. **1**

(ii) Hence factorise $z^5 + 1$ as a product of real linear and quadratic factors. **2**

(c)



A certain solid has a base which is the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$. Slices perpendicular to the base and parallel to the y-axis are right-angled triangles of height 3 units.

(i) Show that the cross-sectional area of a slice parallel to the y-axis is **2**

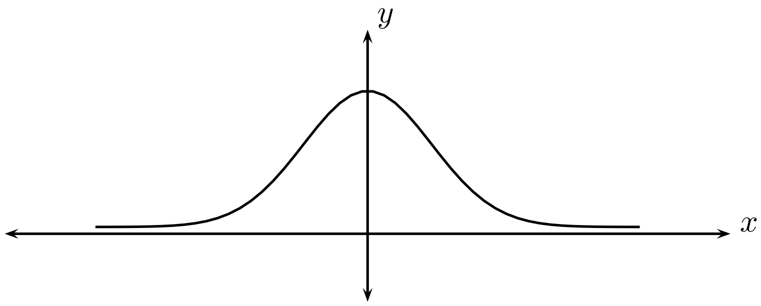
$$A(x) = \frac{3}{2}\sqrt{16 - x^2}$$

(ii) Hence find the volume of the solid. **2**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The cubic polynomial $P(x) = 2x^3 + 15x^2 + 24x + d$ is known to have a repeated real root and a distinct real root. The distinct root and repeated roots have opposite sign. Find the constant d. **3**

(b)



The curve $y = e^{-x^2}$ is shown above.

(i) Use the method of cylindrical shells to find the volume obtained when the region bounded by the axes, the curve and the line $x = 2$ is rotated about the y-axis. **3**

(ii) What is the limiting value as $N \to \infty$ of the volume obtained when the region bounded by the axes, the curve and the line $x = N$ is rotated about the y-axis? **1**

(c) A landing aeroplane of mass m kg is brought to rest by the action of two retarding forces: a force of $4m$ Newtons due to the reverse thrust of the engines; and a force due to the brakes of $\dfrac{mv^2}{40\,000}$ Newtons.

(i) Show that the aeroplane's equation of motion for its speed v at time t seconds after landing is **1**

$$\dot{v} = -\frac{v^2 + 400^2}{40\,000} .$$

(ii) Assuming the aeroplane lands at a speed of U m/s, find an expression for the time it takes to come to rest. **3**

(iii) Show that, given a sufficiently long runway, then no matter how fast its landing speed, it will always come to rest within approximately 2·6 minutes of landing. **1**

(d) Consider the locus of z such that $\left|z - \sqrt{2} - i\right| = 1$.

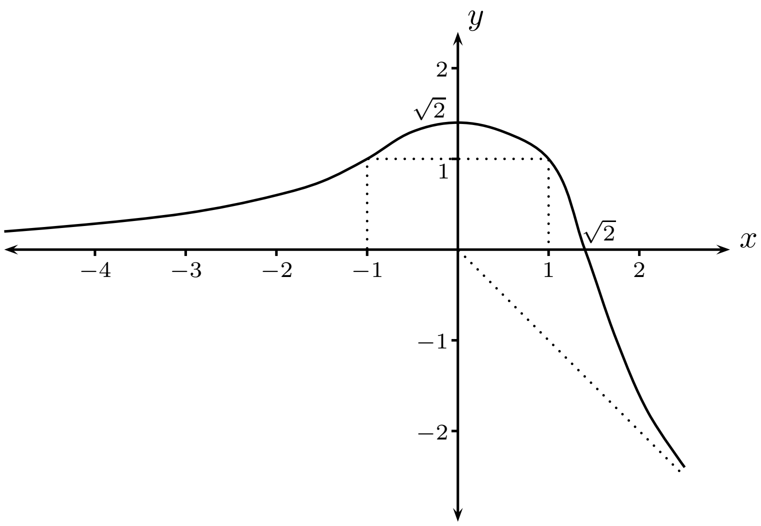
(i) Sketch the locus of z in the complex plane. **1**

(ii) Find the minimum value of $|z|$. **1**

(iii) Find the maximum value of $\arg(z)$, for $0° < \arg(z) < 90°$, correct to the nearest degree. **1**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



The curve $y = f(x)$, sketched above, has asymptotes $y = 0$ and $y = -x$.

Copy or trace the above graph onto three separate number planes. Use your diagrams to show sketches of the following graphs, showing all essential features clearly.

(i) $y = \big(f(x)\big)^2$ **2**

(ii) $|y| = f(x)$ **2**

(iii) $y = \ln f(x)$ **2**

(b) (i) Let $z = \operatorname{cis}\theta$. Use de Moivre's Theorem to prove that for any integer n, **1**

$$z^n - \frac{1}{z^n} = 2i \sin n\theta\,.$$

(ii) By considering $\left(z + \dfrac{1}{z}\right)^5$, show that $\sin^5\theta = \dfrac{1}{16}\big(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\big)$. **2**

(iii) Solve the following equation for $0 \le \theta \le 2\pi$: **2**

$$\sin 5\theta - 5\sin 3\theta + 9\sin\theta = 0\,.$$

(c) You may assume the equation for the chord of contact for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ from (x_0, y_0) is $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = 1$.

Show that that chord of contact from a point on a directrix is a focal chord. **2**

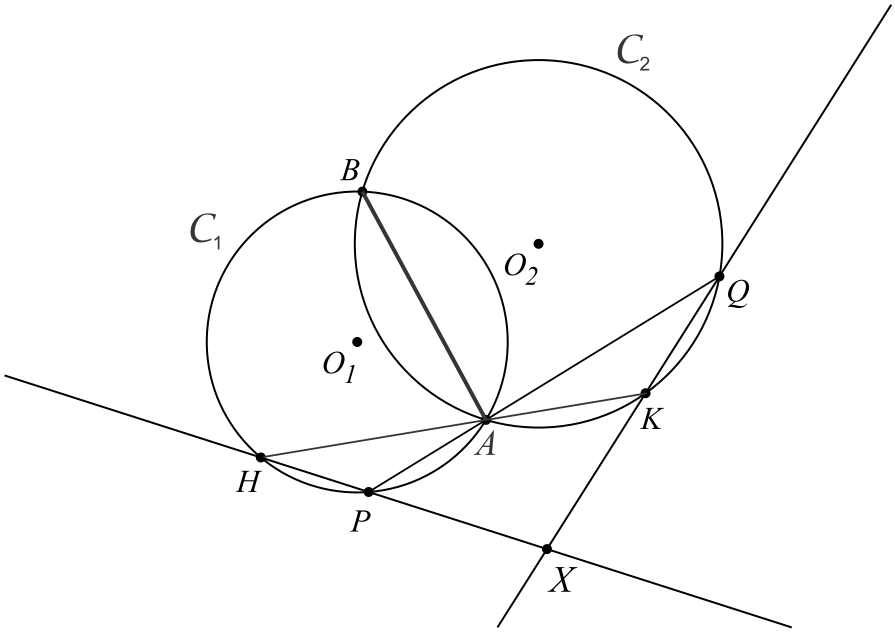
(d) Use the substitution $x = \dfrac{\pi}{2} - u$ to evaluate **2**

$$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx\,.$$

Exam continues overleaf ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



Two intersecting circles $\mathcal{C}_1$ and $\mathcal{C}_2$ share a common chord AB. Points P and H lie on circle $\mathcal{C}_1$ and points Q and K lie on circle $\mathcal{C}_2$, such that PAQ and HAK are straight. Lines HP and QK intersect at X.

Let $\angle BKQ = \theta$.

Copy or trace the diagram into your answer book.

(i) Find $\angle BAQ$ in terms of θ, giving a reason for your answer. **1**

(ii) Show that $BKXH$ is cyclic. **2**

(iii) Assuming that XAB is straight, show that XAB bisects angle PBK. **2**

(b) (i) List all 10 ways that 3 non-negative integers can add to 3. **1**

(ii) Use the identity $(1+x)^{3n} = \left((1+x)^n\right)^3$ to prove that **2**

$$^{3n}\mathrm{C}_3 = n^3 + 6n \times {}^{n}\mathrm{C}_2 + 3 \times {}^{n}\mathrm{C}_3$$

The question continues over the page

QUESTION FIFTEEN (Continued)

(c) As part of a test of a new capsule delivery system, a capsule of mass m is fired straight up at speed u m/s. Air resistance is negligible and the magnitude of the acceleration due to gravity is g.

The capsule subsequently deploys a parachute and falls back to earth, subject to gravity and to a resistive of force of magnitude mkv^2.

(i) Use calculus to show that the maximum height attained by the capsule is $H = \dfrac{u^2}{2g}$. **2**

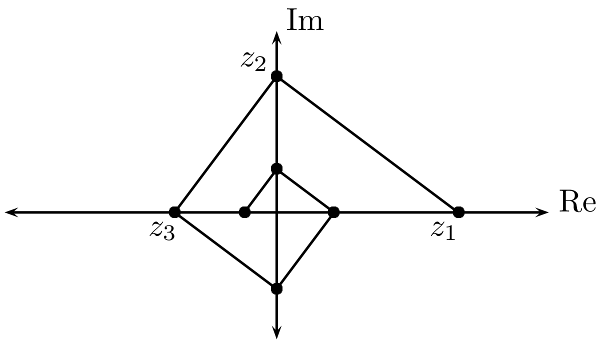
(ii) For the return trip, take the origin at the point it begins falling and assume down is positive. Show that the motion is determined by the equation $\ddot{x} = k(\alpha^2 - v^2)$, where $\alpha^2 = \dfrac{g}{k}$. **1**

(iii) Let U be the impact speed of the package. Find an expression for the square of the speed U in terms of H, k and α. **3**

(iv) Assume that the package is launched at speed $u = \alpha$. Find the impact speed as a percentage of the launch speed. **1**

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



An infinite sequence of complex numbers is defined by

$$z_1 = 1, z_{n+1} = \frac{3}{4} i z_n \,.$$

The path $z_1 z_2 z_3 \cdots$ defines a piecewise linear spiral in the Argand plane.

(i) Show that the n^{th} edge satisfies the relationship **1**

$$z_{n+1} - z_n = \left(\frac{3}{4} i - 1 \right) z_n \,.$$

(ii) Hence find a simplified expression for the length of the n^{th} edge. **1**

(iii) Find the length of the spiral, by considering the limiting sum of the lengths of its edges as $n \to \infty$. **1**

Exam continues overleaf ...

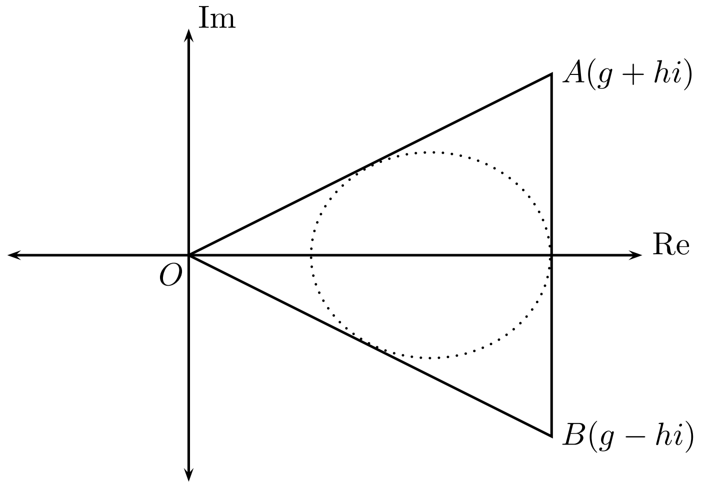
(b) (i) Use algebra to prove, for any integer $k \geq 0$, that **1**

$$\frac{2k+1}{2k+2} \leq \frac{\sqrt{2k+1}}{\sqrt{2k+3}}.$$

(ii) Prove, by induction on $n \geq 0$, that the central binomial coefficient $\binom{2n}{n}$ satisfies **3**

$$\binom{2n}{n} \leq \frac{4^n}{\sqrt{2n+1}}.$$

(c)



Consider a cubic polynomial $y = P(x)$ with real coefficients and roots 0, $g \pm hi$ where g and h are real and $h > 0$. In the diagram above, the roots form the vertices of an isosceles triangle OAB in the complex plane. The roots of $P'(x) = 0$ are the foci of the sketched ellipse which touches the triangle at $g + 0i$. We have sketched the case $g > 0$ and with major axis lying on the real axis. The centre of the ellipse is NOT the origin.

(i) Show that the cubic polynomial has equation $y = x^3 - 2gx^2 + (g^2 + h^2)x$. **1**

(ii) Show that the turning points of $y = P(x)$, and hence the foci of the ellipse, occur at **1**

$$x = \tfrac{2}{3}g \pm \tfrac{1}{3}\sqrt{g^2 - 3h^2}\,.$$

(iii) Find the condition on g and h and hence on $\angle AOB$ which ensures that the major axis of this ellipse lies on the real axis. You may assume this condition holds in parts (iv) and (v). **1**

(iv) Find the equation of the ellipse. **2**

(v) Show that the ellipse is tangential to the triangle at the midpoints of OA and OB. **2**

(vi) If the triangle is equilateral, describe the behaviour of the polynomial $P(x)$ at the centre of the ellipse, for real x. **1**

End of Section II

END OF EXAMINATION

SECTION I - Multiple Choice

QUESTION ONE

The discriminant $\Delta = (-8i)^2 - 4 \times 1 \times (-20) = 16 = (4)^2$. Hence the roots are

$$\frac{8i \pm 4}{2} = 4i \pm 2$$

Hence $\boxed{\text{A}}$.

QUESTION TWO

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2x \, dx \\ &= \left[-\frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{4}\big((-\cos \pi) + \cos 0\big) \\ &= \frac{1}{2}\end{aligned}$$

Hence $\boxed{\text{C}}$.

QUESTION THREE

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= 3\sec^2\theta \div 2\sec\theta\tan\theta \\ &= \frac{3}{2}\sec\theta \div \tan\theta \\ &= \frac{3}{2}\frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta} \\ &= \frac{3}{2}\operatorname{cosec}\theta\end{aligned}$$

Hence $\boxed{\text{D}}$.

QUESTION FOUR

Options A, C and D are even. Only option B is odd. Hence $\boxed{\text{B}}$.

QUESTION FIVE

Angle $XDY = \theta + 20$ (Exterior opposite angle in $\triangle AXD$)

Angle $BCD = \theta + 20 + 30$ (Exterior opposite angle in $\triangle DCY$)

Thus $\theta + (\theta + 50) = 180$ opposite angles of cyclic quad $ABCD$)

So $\theta = 65$.

Hence $\boxed{\text{D}}$.

QUESTION SIX

Option $\boxed{\text{C}}$ is incorrect.

QUESTION SEVEN

All statements are TRUE. Hence $\boxed{\text{D}}$.

QUESTION EIGHT

The linear equation $y = (6 - \frac{3}{4}z)$ satisfies the conditions $y = 6$ when $z = 0$ and $y = 3$ when $z = 4$. The area of the triangle is $\frac{1}{2}y^2 = \frac{1}{2}(6 - \frac{3}{4})^2$, hence the correct answer is $\boxed{\text{A}}$.

QUESTION NINE

The foci are $\pm 3i$, hence the distance between the two foci is $2ae = 6$. But the sum of the distances from a point on the ellipse to the foci is $2a = 12$. Combining these two equations, $e = \frac{1}{2}$.

Hence the correct answer is $\boxed{\text{B}}$.

QUESTION TEN

The correct answer is 2. Note that there must be an even number of complex non-real roots, because of the real coefficients, and two is a possible answer. This is easily seen by drawing a polynomial with zero when $x = 1$, stationary point of inflexion when $x = 2$ and a turning point at some larger x value.

SECTION II - Written Response

QUESTION ELEVEN

(a) (i) $w^2 = (1-2i)^2$

$= 1 - 4 - 4i$

$= -3 - 4i$

(ii) $\dfrac{z}{w} = \dfrac{3+4i}{1-2i}$

$= \dfrac{(3+4i)(1+2i)}{1+4}$

$= \dfrac{3-8+4i+6i}{5}$

$= -1 + 2i$

(b) (i) We have $|t| = 2$, $\arg(t) = \tan^{-1}\big(\sqrt{3}\,\big) = \frac{\pi}{3}$.

Hence $t = 2\operatorname{cis}\frac{\pi}{3}$.

(ii) $t^8 = 2^8 \operatorname{cis}\frac{8\pi}{3}$

$= 256\operatorname{cis}\frac{2\pi}{3}$

$= 256\big(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\big)$

$= 128(-1 + i\sqrt{3}\,)$

(c) (i) $\displaystyle\int \frac{1}{x^2+6x+13}\,dx = \int \frac{1}{(x+3)^2+2^2}\,dx$

$= \frac{1}{2}\tan^{-1}\dfrac{x+3}{2} + C$

(ii) $\displaystyle\int x\sin x\,dx = x(-\cos x) - \int 1 \times (-\cos x)\,dx$

$= -x\cos x + \sin x + C$

(d) $\displaystyle\int_{10}^{17} \frac{dx}{\sqrt{x^2-64}} = \Big[\ln\big(x + \sqrt{x^2-64}\,\big)\Big]_{10}^{17}$

$= \ln\big(17 + \sqrt{17^2-64}\,\big) - \ln\big(10 + \sqrt{10^2-64}\,\big)$

$= \ln 32 - \ln 16$

$= \ln 2$

(e)

(i) $\dfrac{-4x^2+5x+1}{(x-1)^2} = \dfrac{A}{(x-1)^2} + \dfrac{B}{x-1} + C$

$-4x^2+5x+1 = A + B(x-1) + C(x-1)^2$

Equating coefficients of x^2 tells us $C = -4$.

Substituting $x = 1$ tells us $A = 2$.

Substituting $x = 0$, tells us:

$A - B + C = 1$

$2 - B + 4 = 1$

$B = -3$

(ii) Hence $\displaystyle\int \frac{-4x^2 + 5x + 1}{(x-1)^2}\,dx = \int \frac{2}{(x-1)^2}\,dx + \int \frac{-3}{(x-1)}\,dx - \int 4\,dx$

$$= \frac{-2}{(x-1)} - 3\ln|x-1| - 4x + C$$

QUESTION TWELVE

(a)

(i) $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$

Hence $a = 4$ and $b = 3$.

Now $b^2 = a^2(e^2 - 1)$

$$e^2 = \frac{9}{16} + 1$$

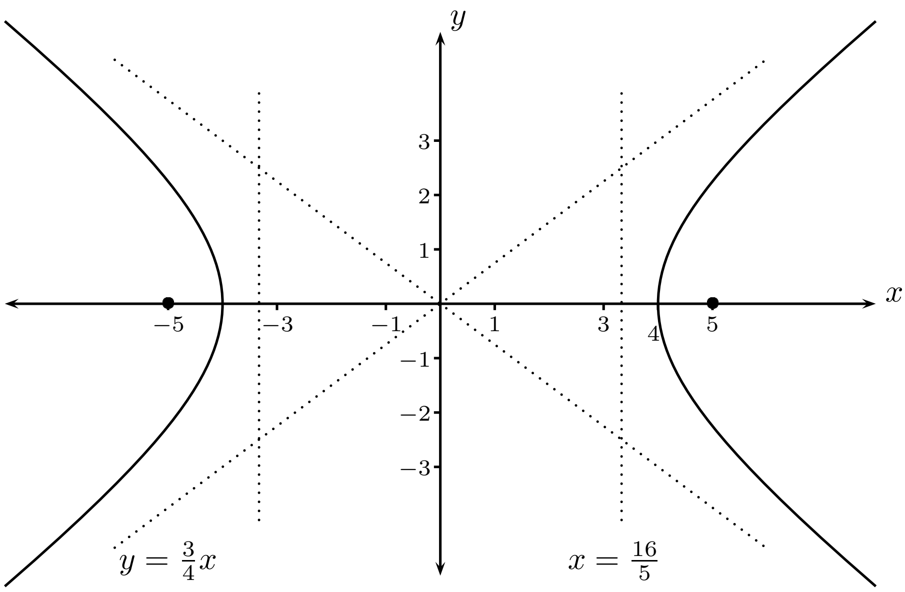
$$e^2 = \frac{25}{16}$$

$$e = \frac{5}{4}$$

The foci are $(\pm ae, 0) = (\pm 5, 0)$. The directrices are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$. Thus $x = \frac{16}{5}$ and $x = -\frac{16}{5}$.

The asymptotes are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. Thus $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$.

(ii)



(iii) By substituting in the equation for the hyperbola, when $x = 5$, we find $y = \frac{9}{4}$ in quadrant one. Differentiating with respect to x:

$$18x - 32y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{9x}{16y}$$
$$= \frac{45}{36} \qquad \text{when } x = 5$$
$$= 1{\cdot}25$$

(iv) The tangent in Quadrant One will be steeper than the asymptote, thus its gradient will never be less than that of the asymptote.

(b) (i) Let $z = \operatorname{cis}\theta$. Then

$$z^5 = -1$$
$$\operatorname{cis} 5\theta = \operatorname{cis}(\pi + 2k\pi) \quad \text{for any integer } k$$

Equating arguments gives:

$$5\theta = (2k+1)\pi$$
$$\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, -\frac{\pi}{5}, -\frac{3\pi}{5}$$

Hence (in conjugate pairs) the roots are:

$$z = -1, \qquad z = \operatorname{cis}\left(\pm\tfrac{1}{5}\pi\right), \qquad z = \operatorname{cis}\left(\pm\tfrac{3}{5}\pi\right)$$

(ii) Grouping the conjugate roots, we get:

$$z^5 + 1 = \left(z+1\right) \times \left(z - \operatorname{cis}\tfrac{1}{5}\pi\right)\left(z - \operatorname{cis}(-\tfrac{1}{5}\pi)\right) \times \left(z - \operatorname{cis}\tfrac{1}{5}\pi\right)\left(z - \operatorname{cis}(-\tfrac{1}{5}\pi)\right)$$
$$= \left(z+1\right)\left(z^2 - 2\cos(\tfrac{1}{5}\pi)z + 1\right)\left(z^2 - 2\cos(\tfrac{3}{5}\pi)z + 1\right)$$

(c) (i) The area of the triangle is:

$$\frac{1}{2}bh = \frac{1}{2}(2y) \times 3$$
$$= 3y$$
$$= 6 \times \sqrt{1 - \frac{x^2}{16}}$$
$$= \frac{6}{4} \times \sqrt{16 - x^2}$$
$$= \frac{3}{2}\sqrt{16 - x^2}$$

(ii) The volume is

$$V = 3 \times \int_0^4 \sqrt{16 - x^2}\,dx$$
$$= 3 \times \frac{1}{4}\pi 4^2$$

since the integral is the area of a quarter circle of radius 4.

Thus the volume is $V = 12\pi$.

(This integral can also be evaluated using the trig substitution $x = 4\sin u$.)

QUESTION THIRTEEN

(a) At a double root we have a zero of the derivative.

$$\begin{aligned} P'(x) &= 6x^2 + 30x + 24 \\ &= 6(x+4)(x+1) \end{aligned}$$

Hence the possibilities are $x = -1$ or $x = -4$. The other root must be positive and the product of roots must be negative, i.e. $d < 0$.

If $x = -1$ then $P(-1) = 2(-1)^3 + 15(-1)^2 + 24(-1) + d$

$$\begin{aligned} 0 &= -2 + 15 - 24 + d \\ d &= 11 \end{aligned}$$

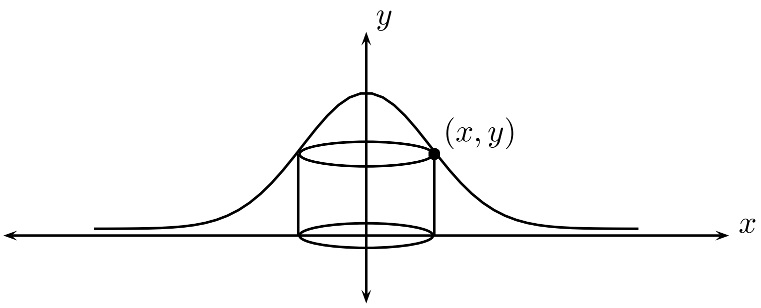
If $x = -4$ then $P(-4) = 2(-4)^3 + 15(-4)^2 + 24(-4) + d$

$$\begin{aligned} 0 &= -128 + 240 - 96 + d \\ d &= -16 \end{aligned}$$

Hence $d = -16$.

This question can also be solved using sum and product of roots methods.

(b) (i)



The volume of the cylindrical shell $dV = 2\pi xy dx$. Total volume is

$$\begin{aligned} V &= \int_0^2 2\pi xy\, dx \\ &= \pi \int_0^2 2xe^{-x^2}\, dx \\ &= \pi \times \left[e^{-x^2}\right]_0^2 \\ &= \pi \times \left(1 - e^{-4}\right) \end{aligned}$$

(ii)

$$\begin{aligned} V &= \pi \times \lim_{N\to\infty} \left(1 - e^{-N^2}\right) \\ &= \pi \end{aligned}$$

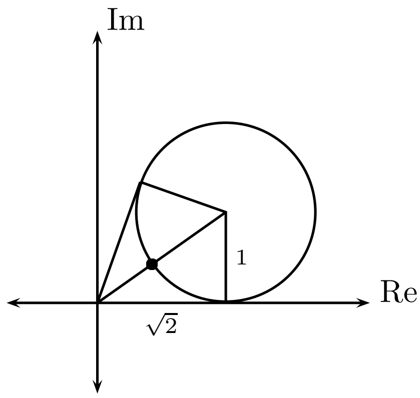
(c) (i)

$$\begin{aligned} m\dot{v} &= \frac{-mv^2}{40\,000} - 4m \\ \dot{v} &= -\frac{v^2}{40\,000} - 4 \\ \dot{v} &= -\frac{v^2 + 160000}{40\,000} \\ &= -\frac{v^2 + 400^2}{40\,000} \end{aligned}$$

(ii) $$\frac{dv}{dt} = -\frac{v^2 + 400^2}{40\,000}$$
$$dt = \frac{-40\,000\,dv}{v^2 + 400^2}$$
$$\int_0^T dt = -40\,000 \times \int_U^0 \frac{dv}{v^2 + 400^2}$$
$$T = 40\,000 \times \frac{1}{400}\tan^{-1}\frac{U}{400}$$
$$T = 100\tan^{-1}\frac{U}{400}$$

(iii) As $U \to \infty$, $T \to 100 \times \frac{\pi}{2}$ seconds, which is about 2·6 minutes.

(d) (i)

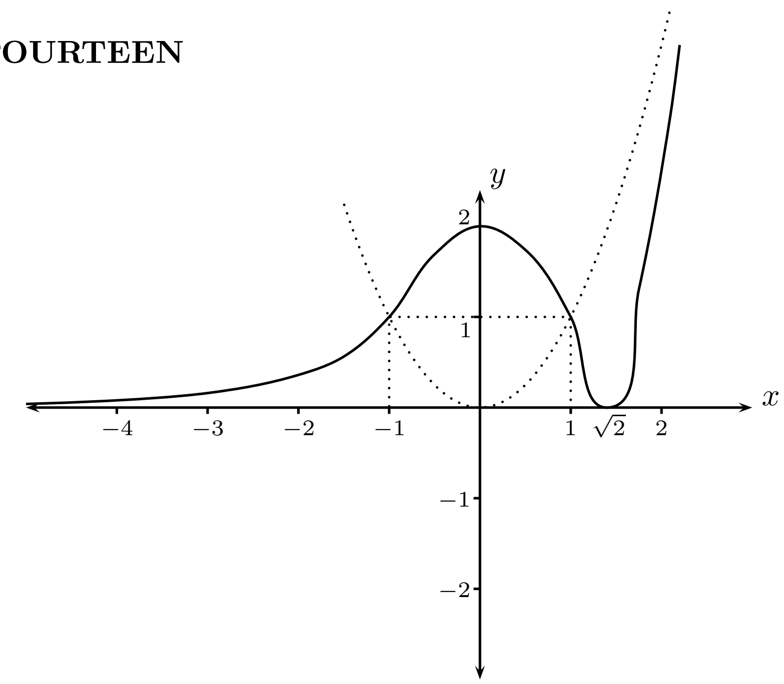


(ii) The point z of minimum modulus is the point on the circle closest to the origin. This distance is:
(distance from origin to centre of circle) − (radius) $= \sqrt{3} - 1$

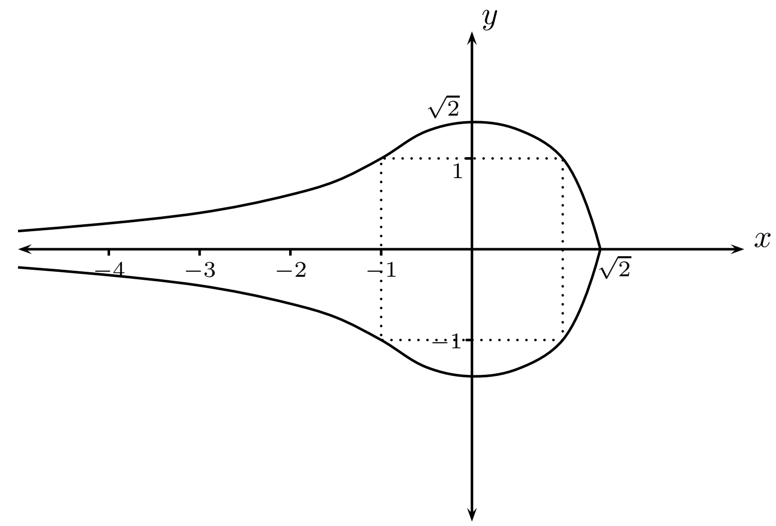
(iii) The point with maximum $\arg(z)$ on the circle is defined by the tangent to the circle. The argument is $2 \times \tan^{-1}\frac{1}{\sqrt{2}} \doteqdot 71^\circ$.

QUESTION FOURTEEN

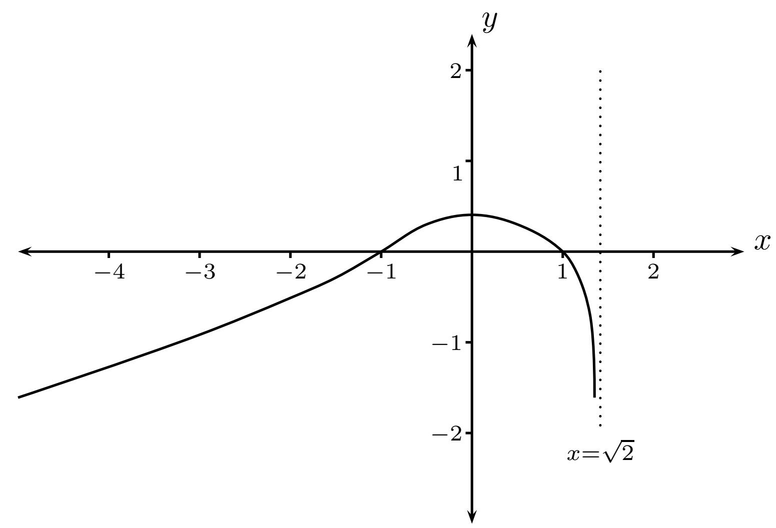
(a) (i)



(ii)



(iii)



(b) (i) Let $z = \operatorname{cis}\theta$. Then by de Moivre's Theorem,

$$\begin{aligned} z^n - \frac{1}{z^n} &= (\operatorname{cis}\theta)^n - (\operatorname{cis}\theta)^{-n} \\ &= \operatorname{cis}\big(n\theta\big) - \operatorname{cis}\big(-n\theta\big) \\ &= \cos\big(n\theta\big) + i\sin\big(n\theta\big) - \cos\big(-n\theta\big) - i\sin\big(-n\theta\big) \\ &= \cos\big(n\theta\big) + i\sin\big(n\theta\big) - \cos\big(n\theta\big) + i\sin\big(n\theta\big) \\ &= 2i\sin\big(n\theta\big) \end{aligned}$$

Where we have used the evenness of the cosine function and the oddness of the sine function.

(ii) $$\begin{aligned} \left(z - \frac{1}{z}\right)^5 &= z^5 - 5z^4\frac{1}{z} + 10z^3\frac{1}{z^2} - 10z^2\frac{1}{z^3} + 5z\frac{1}{z^4} - \frac{1}{z^5} \\ &= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) \\ &= 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta \end{aligned}$$

Now the LHS of this expression is $\big(2i\sin\theta\big)^5$, hence

$$\begin{aligned} 32i\sin^5\theta &= 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta \\ 16\sin^5\theta &= \sin 5\theta - 5\sin 3\theta + 10\sin\theta \\ \sin^5\theta &= \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right) \end{aligned}$$

(iii) From this equation $16\sin^5\theta - 10\sin\theta = \sin 5\theta - 5\sin 3\theta$.

Hence $\quad \sin 5\theta - 5\sin 3\theta + 9\sin\theta = 0$

Becomes $16\sin^5\theta - 10\sin\theta + 9\sin\theta = 0$

$$16\sin^5\theta - \sin\theta = 0$$

$$\sin\theta(16\sin^4\theta - 1) = 0.$$

So $\sin\theta = 0$ or $\sin\theta = \pm\frac{1}{2}$.

The solutions of these equations in the given domain are:

$$\theta = 0, \pi, 2\pi, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

(c) Let the point be $(x_0, y_0) = (\frac{a}{e}, y_0)$. The chord of contact is $\dfrac{x}{a^2}\left(\dfrac{a}{e}\right) - \dfrac{y_0 y}{b^2} = 1$.

Is $(ae, 0)$ on this chord?

$$\begin{aligned} LHS &= \frac{x}{a^2}\left(\frac{a}{e}\right) - \frac{y_0 y}{b^2} \\ &= 1 - 0 \\ &= RHS \end{aligned}$$

So yes, the chord passes through the focus.

(d) $$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx = \int_{\frac{\pi}{2}}^{0} \frac{\cos^3(\frac{\pi}{2}-u)}{\cos^3(\frac{\pi}{2}-u)+\sin^3(\frac{\pi}{2}-u)}\,(-dx)$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin^3 u}{\sin^3 u + \cos^3 u}\,du$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx \quad \text{(relabelling } u \text{ as } x\text{)}$$

Hence $$2 \times \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos^3 x + \sin^3 x}{\cos^3 x + \sin^3 x}\,dx$$
$$= \int_0^{\frac{\pi}{2}} 1\,dx$$
$$= \frac{\pi}{2}$$

Thus $$\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx = \frac{\pi}{4}.$$

QUESTION FIFTEEN

(a) (i) $\angle BAQ = \theta$ (angles at the circumference on arc BQ)

(ii) Hence $\angle BHP = \theta$ (interior opposite angle in cyclic quadrilateral $BHPA$)
Since $\angle BHP = \angle BHX = \theta$ and the exterior opposite angle $\angle BKQ = \theta$, we have that quadrilateral $BKXH$ is cyclic.

(iii) $\angle PBA = \angle PHA$ (angles at the circumference on arc PA in circle $\mathcal{C}_1$)
$= \angle XHK$ (same angle)
$= \angle XBK$ (angles at the circumference on arc XK in circle $BKXH$)
$= \angle ABK$ (same angle)

(b) (i) In any order, the ways three integers can add to 3 are:

$$3+0+0,\ 0+3+0,\ 0+0+3$$
$$1+2+0,\ 2+1+0,\ 0+2+1,\ 0+1+2,\ 1+0+2,\ 2+0+1$$
$$1+1+1$$

(ii) We look to be equating coefficients of x^3.
$LHS = (1+x)^{3n}$
$= {}^{3n}\mathrm{C}_0 x^0 + {}^{3n}\mathrm{C}_1 x^1 + \cdots$
... and the coefficient of x^3 is ${}^{3n}\mathrm{C}_3$.
The RHS is $(1+x)^n(1+x)^n(1+x)^n$.
We need to consider how we can get an x^3 term when we expand the brackets. Part (i) gives us a hint here, since the sum of the three indices (one from each bracket) must be 3.

Method 1 (list them all): The coefficient of x^3 is:

$$\begin{aligned}&{}^n\mathrm{C}_3{}^n\mathrm{C}_0{}^n\mathrm{C}_0 + {}^n\mathrm{C}_0{}^n\mathrm{C}_3{}^n\mathrm{C}_0 + {}^n\mathrm{C}_0{}^n\mathrm{C}_0{}^n\mathrm{C}_3\\ &\quad + {}^n\mathrm{C}_1{}^n\mathrm{C}_2{}^n\mathrm{C}_0 + {}^n\mathrm{C}_2{}^n\mathrm{C}_1{}^n\mathrm{C}_0 + {}^n\mathrm{C}_0{}^n\mathrm{C}_2{}^n\mathrm{C}_1 + {}^n\mathrm{C}_0{}^n\mathrm{C}_1{}^n\mathrm{C}_2 + {}^n\mathrm{C}_1{}^n\mathrm{C}_0{}^n\mathrm{C}_2\\ &\qquad\quad + {}^n\mathrm{C}_2{}^n\mathrm{C}_0{}^n\mathrm{C}_1\\ &\quad + {}^n\mathrm{C}_1{}^n\mathrm{C}_1{}^n\mathrm{C}_1\\ &= 3\times{}^n\mathrm{C}_3 + 6\times{}^n\mathrm{C}_1{}^n\mathrm{C}_2 + ({}^n\mathrm{C}_1)^3\end{aligned}$$

Method 2 (Avoid listing them all):

- There are 3 ways to get x^3 from one bracket, x^0 from each of the others. This gives a contribution $3\times{}^n\mathrm{C}_3x^3\times x^0\times x^0$
- There are 6 ways to get x^1 from one bracket, x^2 from a second bracket and x^0 from a third. This gives a contribution $6\times{}^n\mathrm{C}_1x^1\times{}^n\mathrm{C}_2x^2\times x^0$
- There is 1 way to get x^1 from all of the brackets This gives a contribution $1\times{}^n\mathrm{C}_1x\times{}^n\mathrm{C}_1x\times{}^n\mathrm{C}_1x$

Thus from the RHS the coefficient of x^3 is:
$3\times{}^n\mathrm{C}_3 + 6\times{}^n\mathrm{C}_1{}^n\mathrm{C}_2 + ({}^n\mathrm{C}_1)^3$

Thus either method gives us

$$\begin{aligned}&3\times{}^n\mathrm{C}_3 + 6\times{}^n\mathrm{C}_1{}^n\mathrm{C}_2 + ({}^n\mathrm{C}_1)^3\\ &\qquad= 3\times{}^n\mathrm{C}_3 + 6n\times{}^n\mathrm{C}_2 + n^3\end{aligned}$$

Equating coefficients of x^3 yields the required result.

(c) (i) Starting from the equation of motion $\ddot{x} = -g$, we get

$$\begin{aligned}\ddot{x} &= -g\\ v\frac{dv}{dx} &= -g\\ \int_u^0 v\,dv &= -\int_0^H g\,dx\\ \left[\tfrac{1}{2}v^2\right]_u^0 &= \left[-gx\right]_0^H\\ -gH &= -\tfrac{1}{2}u^2\\ H &= \frac{u^2}{2g}\end{aligned}$$

(ii) Working the equations with down as positive and including the resistive term;

$$\begin{aligned}m\ddot{x} &= mg - mkv^2\\ \ddot{x} &= g - kv^2\\ &= k\big(\frac{g}{k} - v^2\big)\\ &= k\big(\alpha^2 - v^2\big)\end{aligned}$$

where $\alpha^2 = \dfrac{g}{k}$.

(iii) We need to integrate the equation of motion:

$$v\frac{dv}{dx} = k(\alpha^2 - v^2)$$

$$-\frac{1}{2} \times \int_0^U \frac{-2v\,dv}{\alpha^2 - v^2} = k \times \int_0^H dx$$

$$\left[-\tfrac{1}{2}\ln\left(\alpha^2 - v^2\right)\right]_0^U = \left[kx\right]_0^H$$

$$-\tfrac{1}{2}\ln\frac{\alpha^2 - U^2}{\alpha^2} = kH$$

Hence $kH = -\frac{1}{2}\ln\dfrac{\alpha^2 - U^2}{\alpha^2}$. (Note that $U < \alpha$, the terminal velocity.)

Rearranging we get:

$$U^2 = \alpha^2\left(1 - e^{-2kH}\right)$$

(iv) We have:

$$\frac{(\text{impact speed})^2}{(\text{launch speed})^2} = \frac{\alpha^2\left(1 - e^{-2kH}\right)}{\alpha^2}$$
$$= 1 - e^{-2kH}$$

But $2kH = 2 \times \dfrac{g}{\alpha^2} \times \dfrac{\alpha^2}{2g} = 1$. Hence

$$\frac{(\text{impact speed})^2}{(\text{launch speed})^2} = 1 - e^{-1}$$
$$= \frac{e-1}{e}$$
$$\doteqdot 0{\cdot}632$$

So $\dfrac{\text{impact speed}}{\text{launch speed}} = 79{\cdot}5\%$

That is, the impact speed=79·5% of the launch speed.

QUESTION SIXTEEN

(a) (i) $z_{n+1} - z_n = \frac{3}{4}iz_n - z_n$
$$= (\tfrac{3}{4}i - 1)z_n$$

(ii) $\left|z_{n+1} - z_n\right| = \left|\frac{3}{4}i - 1\right| \times |z_n|$
$$= \tfrac{5}{4}|z_n|$$

But $|z_n| = (\frac{3}{4})^{n-1}$, so

(iii) $\left|z_{n+1} - z_n\right| = \frac{5}{4} \times (\frac{3}{4})^{n-1}$

$$\text{Total length} = \tfrac{5}{4}\left(1 + \tfrac{3}{4} + \tfrac{9}{16} + \cdots\right)$$
$$= \tfrac{5}{4} \times \frac{1}{1 - \frac{3}{4}}$$
$$= \tfrac{5}{4} \times 4$$
$$= 5$$

(b) Consider RHS/LHS. We want to show that this ratio is greater than 1.

$$\left(\frac{RHS}{LHS}\right)^2 = \frac{(2k+1)(2k+2)^2}{(2k+3)(2k+1)^2}$$
$$= \frac{(2k+2)^2}{(2k+3)(2k+1)}$$
$$= \frac{4k^2+8k+4}{4k^2+8k+3}$$
$$> 1$$

(Since the numerator is greater than the denominator.)

(i)

Step A: Let us check the result for $n = 0$. When $n = 0$:

$$LHS = \binom{0}{0} = 1 \qquad RHS = \frac{4^n}{\sqrt{2n+1}} = 1$$

Step B: Assume the results holds for $n = k$, that is assume

$$\binom{2k}{k} \le \frac{4^k}{\sqrt{2k+1}}$$

We need to show that the result holds for $n = k + 1$, that is to show that

$$\binom{2k+2}{k+1} \le \frac{4^{k+1}}{\sqrt{2k+3}}$$

$$LHS = \frac{(2k+2)!}{(k+1)!(k+1)!}$$
$$= \frac{(2k)!}{k!k!} \times \frac{(2k+1)(2k+2)}{(k+1)(k+1)}$$
$$\le \frac{4^k}{\sqrt{2k+1}} \times \frac{4(2k+1)}{(2k+2)}$$
$$\le \frac{4^{k+1}}{\sqrt{2k+1}} \times \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$$
$$\le \frac{4^{k+1}}{\sqrt{2k+3}}$$

as required.

Step C: Hence the result holds for all n by the Principle of Mathematical Induction.

(c) (i)
$$y = x\big(x-(g+hi)\big)\big(x-(g-hi)\big)$$
$$= x\big(x^2-2gx+(g^2+h^2)\big)$$
$$= x^3-2gx^2+x(g^2+h^2)$$

(ii) $y' = 3x^2 - 4gx + (g^2+h^2)$

So $y' = 0$ when

$$x = \frac{4g \pm \sqrt{16g^2 - 4\times 3\times (g^2+h^2)}}{6}$$
$$= \tfrac{2}{3}g \pm \tfrac{1}{3}\sqrt{g^2-3h^2}$$

(iii) We need $g^2 - 3h^2 \geq 0$, so that the foci are real.

Thus $\dfrac{h^2}{g^2} \leq \frac{1}{3}$

$\dfrac{h}{g} \leq \frac{1}{\sqrt{3}}$

But $\tan \frac{1}{2}\angle AOB = \dfrac{h}{g}$, so $\frac{1}{2}\angle AOB \leq 30°$. Thus $\angle AOB \leq 60°$.

(iv) The centre of the ellipse occurs at the midpoint of the foci, which are the stationary points of $P(x)$. Thus the centre is at $x = \frac{2}{3}g$.

The equation of the ellipse is

$$\frac{(x - \frac{2}{3}g)^2}{a^2} + \frac{y^2}{b^2} = 1$$

The endpoint of the ellipse is given to be $(g, 0)$, so that $a = g - \frac{2}{3}g = \frac{1}{3}g$. The distance between the foci is $2ae$, so using part (i)

$2ae = \frac{2}{3}\sqrt{g^2 - 3h^2}$

Thus $b^2 = a^2(1 - e^2)$

$= a^2 - (ae)^2$

$= \frac{1}{9}g^2 - \frac{1}{4} \times \frac{4}{9}(g^2 - 3h^2)$

$= \frac{1}{3}h^2$

Hence the equation is

$$\frac{(x - \frac{2}{3}g)^2}{\frac{1}{9}g^2} + \frac{y^2}{\frac{1}{3}h^2} = 1$$

(v) We need to show that $(\frac{1}{2}g, \frac{1}{2}h)$ lies on the ellipse, and that at this point the ellipse has gradient $\frac{h}{g}$.

Substitututing in the equation for the ellipse:

$LHS = \dfrac{(\frac{1}{2}g - \frac{2}{3}g)^2}{\frac{1}{9}g^2} + \dfrac{(\frac{1}{2}h)^2}{\frac{1}{3}h^2}$

$= \dfrac{\frac{1}{36}g^2}{\frac{1}{9}g^2} + \dfrac{3}{4}$

$= \dfrac{1}{4} + \dfrac{3}{4}$

$= 1$

Hence the point lies on the ellipse.

By implicit differentiation of the equation of the ellipse:

$$\frac{2(x - \frac{2}{3}g)}{\frac{1}{9}g^2} + \frac{2yy'}{\frac{1}{3}h^2} = 0$$

$$\begin{aligned}\text{Thus } y' &= \frac{-2(x - \frac{2}{3}g)}{\frac{1}{9}g^2} \times \frac{\frac{1}{3}h^2}{2y} \\ &= \frac{-2(\frac{1}{2}g - \frac{2}{3}g)}{\frac{1}{9}g^2} \times \frac{\frac{1}{3}h^2}{2\frac{1}{2}h} \\ &= \frac{-2(-\frac{1}{6}g)}{\frac{1}{9}g^2} \times \frac{1}{3}h \\ &= \frac{3}{g} \times \frac{1}{3}h \\ &= \frac{h}{g}\end{aligned}$$

Which is the gradient of OA, and hence the ellipse is tangential to the triangle at the midpoint of OA.

A similar proof (not required) would show that the ellipse is tangential to the triangle at the midpoint of OB also.

(vi) Angle $AOB = 60°$, hence $\frac{h}{g} = \tan 30° = \frac{1}{\sqrt{3}}$. Thus $g^2 - 3h^2 = 0$. The foci are both $x = \frac{2}{3}g$ and the ellipse is a circle, with centre $(\frac{2}{3}g, 0)$ and radius $\frac{1}{3}g$.

Note: At $x = \frac{2}{3}g$ the polynomial has a double root of $P'(x)$ and a point of inflexion – it defines a stationary point of inflexion.