MECHANICS

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Abstract. We shall begin discussing the fundamental concepts behind mechanics with Newton's Laws of motion which underpin all motions which are non-relativistic. We shall then explore motion one and two dimensional motions through space, resisted motion, circular motion and forces on a conical pendulum.

Part 1. Laws of Motion

In everyday language, a force is a push or a pull. A better definition of force would be an interaction between two bodies or between a body and its environment. We can categorise forces in contact and non-contact forces. Pushing someone over using your hands involves contact forces, but a ball falling from the top of a building experiences non-contact forces such as gravity and air resistance. Some common types of forces include:

- **Normal force**: When an object rests of pushes on a surface, the surface exerts a push on it that is directed perpendicular to the surface.
- **Friction force**: A surface may exert a frictional force on an object, directed parallel to the surface and opposes the motion.
- **Tension force**: A pulling force exerted on an object by a rope, cord etc
- **Weight force**: The pull of gravity on an object.

The Standard Internationale (SI) unit for force is Newtons (N). As with most things, force can be superimposed; any number of forces applied at a point on a body have the same effect as a single force equal to the vector sum of the forced.

Newton's First Law

Newton's first law describes an object when no net force acts on it.

> A body acted on by no net force moves with constant velocity or remains at rest, since no acceleration occurs.

Consider an ice hockey puck on ice. If the puck is hit, it continues to move across the ice until it is stopped by a barrier since the ice is an almost frictionless surface. The reason why it is so difficult to see this in everyday life is that frictional forces exist and act opposite to the motion of an object. Hence, a puck pushed across a table will not move across the table very far.

Newton's Second Law

Newton's second law describes what happens when a force is exerted on an object. From this law we see that the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass.

> The net force on an object is proportional to the acceleration of an object:
> $$\sum \overrightarrow{F} = m\overrightarrow{a}$$

Newton's Third Law

If you press against a corner of a book with your fingertip, the book pushes back and makes a small dent in your skin. If you push it even harder, the book does the same and the dent in your skin is a little larger. This simple activity shows the interactions between two objects and is a demonstration of Newton's Third law:

> If two objects interact with force $\overrightarrow{F_{12}}$exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force $\overrightarrow{F_{21}}$exerted by object 2 on object 1:
> $$\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$$

Part 2. Application of Newton's Law

In the application of Newton's Law, there are 2 cases to consider - when an object is in equilibrium and when it is under net force.

Particle in Equilibrium

If the acceleration of an object modelled as a particle is zero, it is in equilibrium because no net force acts on the object:

$$\sum \overrightarrow{F} = 0$$

Consider a lamp suspended from a light chain fastened to the ceiling, as shown in the free body diagram. No forces act horizontally so $\sum \overrightarrow{F_x} = 0$.The lamp is in equilibrium because the gravitational force downward is equal in magnitude but opposite in direction to the tension of the chain pulling the lamp upward so we get:

$$\sum \overrightarrow{F_y} = T - F_g = 0 \rightarrow T = F_g$$

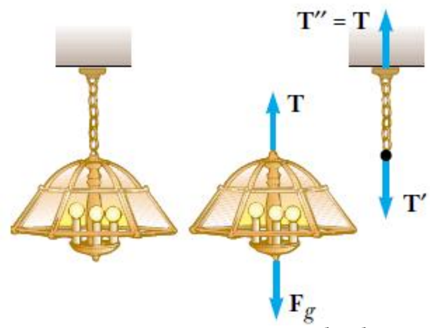


FIGURE 0.1. Freebody diagram of a lamp

Particle under a Net Force

If an object experiences acceleration, its motion can be analysed with the particle under a net force model. In most questions, we shall be using this model. Consider a person pulling a crate with a rope on a frictionless surface. We can analyse the x-y components of the forces acting on the crate.

We notice that in the vertical direction, the crate is in equilibrium since it does not move up and down. So we use the equilibrium model:

$$\sum F_y = n - F_g = 0 \rightarrow n = F_g.$$

In the horizontal direction however there is only one force acting on the crate - tension towards the right so:

$$\sum F_y = T = ma_x \rightarrow a_x = \frac{T}{m}$$

Resolving Forces involving resistance

In one dimensional motion, all the forces lie in the same plane. To begin with consider a mass m falling from the sky. The forces acting on it include the weight force and air resistance (R), as shown. Air resistance always opposes the motion of an object, and hence by convention, it is mathematically represented by a negative value. Hence the y-component of the force is given by:

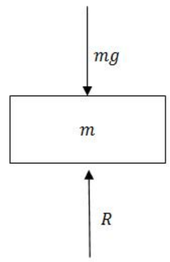


Figure 0.2. A mass falling from the sky

$$\sum F_y = mg - R$$

When motion occurs in n dimension, we can treat the motion as a superposition of motions in n dimensions. Take projectile motion for example. From experiments conducted by Galileo, we see that projectile motion consists of independent components of motion - an x component where no acceleration occurs and a y component that is subject to gravitational acceleration.

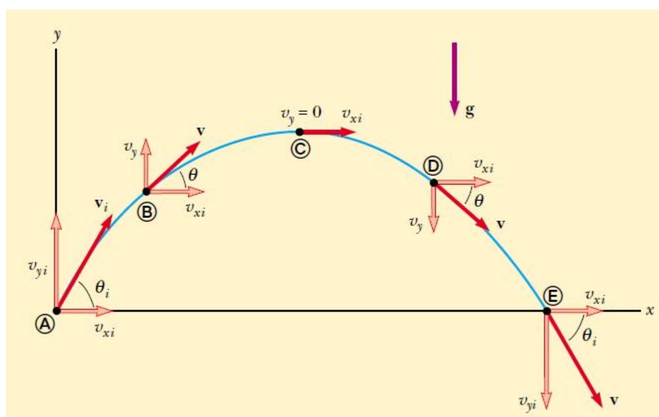


Figure 0.3. Projectile Motion

When you did projectile motion in 3 unit, more often than not, you would have drawn a diagram to represent what was happening in the situation. Similarly in the study of mechanics, a diagram is an invaluable tool. A diagram which shows what forces act on an object is called a free body diagram. By convention, a force is represented by a vector which represents the magnitude of the force and the direction it is acting on the object; so gravity would act downwards towards the ground.

Methodology

(1) Draw a free body diagram of the situation.
(2) Determine which components are in equilibrium and which are under a net force.
(3) Analyse the basic problem. Usually you are given the law governing the motion of a body, together with the information regarding the initial position and/or initial velocity. You are then required the velocity or displacement of an object at any time t by performing integration:

Part 3. Motion in One Dimension & Resisted Motion

The analysis of motion in one dimension involves only one component or plane of forces. Hence, we only need to consider whether the motion in that plane is in equilibrium (stationary or moving with constant velocity) or under a net force (thereby accelerating).

Example 0.1. A mass of m kilograms falls from a stationary balloon of height h metres above the ground. It experiences air resistance during its fall proportional to the product of its mass and square of its velocity. Let x be the distance in metres of the mass from the balloon, measured positively as it falls.

(1) Show that the motion of the mass is given by: $\ddot{x} = g - kv^2$
(2) Find v^2as a function of x
(3) Find the velocity of the mass as it hits the ground in terms of g,k,h.

Draw the freebody diagram:

i)

$$m\ddot{x} = mg - mkv^2 \rightarrow \ddot{x} = g - kv^2$$

ii)

$$v\frac{dv}{dx} = g - kv^2$$
$$\int \frac{v}{g-kv^2}dv = \int dx$$
$$-\frac{1}{2k}ln(g-kv^2) = x + C\text{,substituting x=0 v=0 then:}$$
$$v^2 = \frac{g}{k}(1 - e^{-2kx})$$

iii)

$$\text{substitute x=h to get } v = \sqrt{\frac{g}{k}(1 - e^{-2kh})}$$

Example 0.2. Experiments show that the acceleration due to gravity of an object at a distance x from te centre of the earth is proportional to the inverse square of the distance x, towards the centre of the earth. a) Show that if a particle is projected vertically upwards from the Earth's surface with initial speed U then its velocity at a distance x from the centre is given by $v^2 = U^2 - 2gR^2(\frac{1}{R} - \frac{1}{x})$. b) Calculate the maximum height of the particle above the Earth's surface. c) Deduce the escape velocity taking R=6400 km and g=9.8ms^{-2}.

Draw a freebody diagram

a)

$$a = \frac{dv}{dt} = -\frac{k}{x^2}$$

We can already solve for k since we know that on the Earth's surface, x=R and a=-g

$$-g = -\frac{k}{R^2} \rightarrow k = gR^2$$

$$\therefore \frac{dv}{dt} = -\frac{gR^2}{x^2}$$

Since the acceleration of the object is given in terms of x, we seek to express the acceleration in some form containing dx

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

This is a separable differential equation and hence we separate the v's to the left, and the x's to the right:

$$\int vdv = -\frac{gR^2}{x^2}dx$$

$$\frac{v^2}{2} = -gR^2\left[-\frac{1}{x}\right] + C$$

When x=R, v=U:

$$\frac{U^2}{2} = -gR^2(-\frac{1}{R}) + C \rightarrow C = \frac{U^2}{2} + gR$$

$$v^2 = U^2 - 2gR^2(\frac{1}{R} - \frac{1}{x})$$

b)

At the maximum height, the velocity will be zero hence:

$$U^2 = 2gR^2(\frac{1}{R} - \frac{1}{x})$$

$$x = \frac{2gR^2}{2gR - U^2}$$

c)

The escape velocity is the initial velocity required to escape Earth's gravitational attraction. This occurs when $x \rightarrow \infty$

$$2gR - U^2 \rightarrow 0$$

$$U = \sqrt{2gR} \approx 11.2kms^{-1}$$

Exercise. 1

(1) The acceleration of a particle moving in a straight line is given by $a = k(1-v^2), k > 0$, where b is its velocity at any time t. a) Find an expression for the velocity in terms of t, hence its terminal velocity. b) Find an expression for the position in terms of velocity, hence its limiting position.

(2) A particle moving in a striaght line experiences an acceleration of $-\frac{v^2}{10}ms^{-2}$. Initially the particle is projected from the origin with velocity Ums^{-1}. Show that its velocity and position after t seconds respectively are $v = \frac{10U}{10+Ut}\,ms^{-1}x = 10ln(1+\frac{Ut}{10})\,m.$

(3) A particle is projected vertically upwards from the ground with initial velocity U. It experiences an air resistance which is proportional to the square of its speed. a) Find the time required to reach the maximum height and the maximum height.

(4) A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity Q, Q>0. a) Show that v is related to the displacement x by the formula $x = \arctan(\frac{Q-v}{1+Qv})$. b) Show that the time t which has elapsed when the particle is travelling with velocity v is given by $t = \frac{1}{2}ln(\frac{Q^2(1+v^2)}{v^2(1+Q^2)}$. c) Find v^2 as a function of t. d) Find the limiting value of v and x as $t \to \infty$.

(5) A man with a parachute jumps from a plane flying horizontally. If he opens the parachute after 10 seconds, find his velocity at the end of a) 10 seconds b) his terminal velocity. You may assume that air resistance produces a retardation proportional to the velocity, with the constant of proportion being 0.1 for the first 10 seconds and 2 seconds thereafter.

Part 4. Circular Motion

An object that moves in circular motion experiences a centripetal force: a force that acts towards the centre of the circle. Hence, we can resolve the components of circular motion as:

$$F_x = Fcos\theta$$

$$F_y = Fsin\theta$$

Many terms in circular motion are analogous to kinematics. We see that:

- Angular speed is the change in angular displacement over time: $\omega = \frac{d\theta}{dt}$
- Tangential velocity is given by $v = r\omega$
- Tangential acceleration is given by: $a_t = \frac{d\omega}{dt} = \frac{d}{d\theta}\frac{1}{2}\omega^2 = \omega\frac{d\omega}{d\theta} = \ddot{\theta}$
- Centripetal, radial, normal acceleration is given by: $a_c = r\omega^2 = \frac{v^2}{r}$
- Centripetal force is given by: $F_c = m\frac{v^2}{r} = mr\omega$
- The period of motion is given by: $T = \frac{2\pi}{\omega}$

In this topic, a light string or rod is one with negligble mass; unless stated to be slack, the string is assumed taut & inextensible.

Example 0.3. A particle of mass 3 kg moves on a smooth table with a speed of 4 m/s, being attached to a fixed point on the table by a string of length 2 m. a) Find the tension in the string. b) If the maximum tension that the string can withstand is 147 N, find the greatest number of revolutions per second the particle can make without breaking the string.

Draw a freebody diagram of the situation.

a) The tension in the string is provided by the centripetal force so:

$$T = m\frac{v^2}{r} = (3)(\frac{16}{2}) = 8\ N$$

b) We are given the maximum tension so:

$$m\frac{v^2}{r} = 147 \rightarrow v = \sqrt{147\frac{r}{m}} \rightarrow= 7\sqrt{2}$$

$$v = r\omega = 7\sqrt{2} \rightarrow \omega = \frac{7\sqrt{2}}{2}\ rad/s$$

$$\omega = \frac{7\sqrt{2}}{4\pi}\ rev/s$$

Example 0.4. A car of mass 1 tonne passes over a bridge formed by the arc of a circle of radius 10 metres. a) Find the force exerted by the car on the road at the top of the bridge if the car is travelling at 8 m/s. b) What speed would cause the car to be on the point of leaving the bridge at its highest point?

Draw a freebody diagram

a) Resolving the forces radially and tangentially:
Since the horizontal motion is in equilibrium, the magnitude of the friction and tractive force of the car cancel out.

$$P - F = 0$$

In circular motion, the car is not in equilibrium so:

$$mg - N = \frac{mv^2}{r}$$

Remembering Newton's third law that states that the force exerted by the car on a surface is of the same magnitude as the force exerted by the surface on the car, si:

$$N = mg - \frac{mv^2}{r} = 3400\,N$$

b) When the car leaves the bridge, there is no contact between the wheels and the bridge, and hence the normal force can not exist ie. N=0:

$$mg = \frac{mv^2}{r} \rightarrow v^2 = gr$$

Example 0.5. The gravitational force between two objects of masses m and M places at a distance r apart is proportional to their masses and inversely proportional to the square of their distance apart. Determine the height and the linear velocity of a satelitte that, when placed in a circular orbit around earth will complete one revolution per day. Take g=9.8ms^{-2} R=6400km

$$F = k\frac{mM}{r^2}$$

At r=R, F=mg so:

$$mg = k\frac{mM}{R^2} \rightarrow k = \frac{gR^2}{M}$$

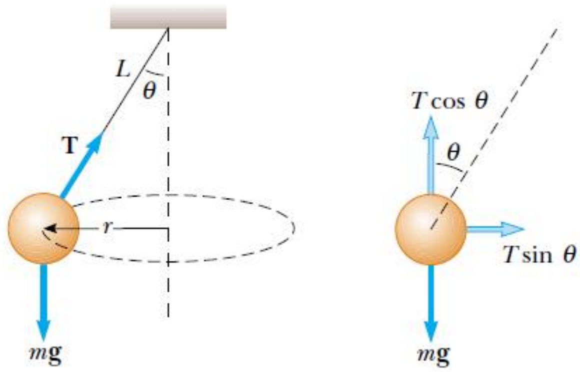
$$F = \frac{mgR^2}{r^2} = mr\omega^2$$

$$\omega = \frac{2\pi}{T} \rightarrow 7.3 \times 10^{-5}\,rad/s$$

$$r^3 = \frac{gR^2}{\omega^2} = \frac{(9.8)(6.4 \times 10^6)^2}{(7.3 \times 10^{-5})^2} = 7.53 \times 10^{22} \rightarrow r = 4.22 \times 10^7\,m$$

$$v = r\omega = (4.22 \times 10^7)(7.3 \times 10^{-5}) = 3000\,ms^{-1}$$

Part 5. Conical Pendulum



Consider a string of length l that is fixed at one end and attached to a particle on the other. This system is said to be a conical pendulum. We begin by resolving the forces horizontally and vertically:

$$Horizontally: Tsin\theta = mr\omega^2 \tag{0.1}$$

$$Vertically: Tcos\theta = mg \tag{0.2}$$

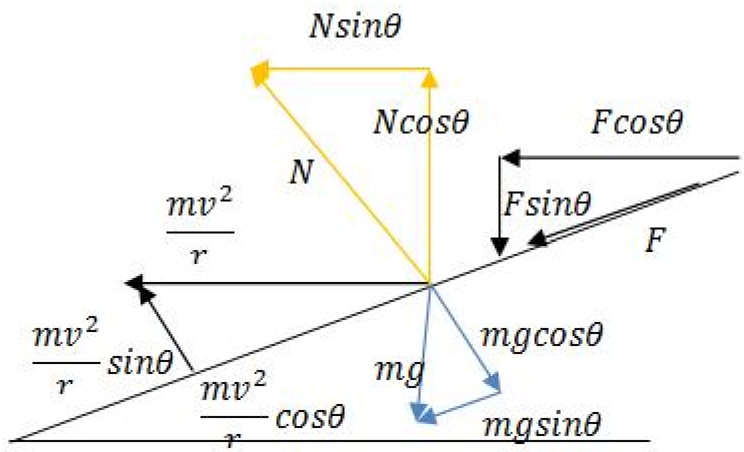
We can also use the physical dimensions of the conical pendulum:

- $$l^2 = r^2 + h^2 \tag{0.3}$$
- $$r = lsin\theta \tag{0.4}$$
- $$h = lcos\theta \tag{0.5}$$

From this we can attain:

- 0.4 in 0.1: $T = ml\omega^2$
- 0.1/0.2: $tam\theta = \frac{r\omega^2}{g} = \frac{v^2}{gr}$
- 0.4/0.5: $tan\theta = \frac{r}{h}$
- 0.6&0.7: $\omega = \sqrt{\frac{g}{h}}$

Part 6. Banked Tracks



Let us consider an object on a banked track, as shown in the diagram. Resolving the forces along the slope and perpendicular to the slope:

$$F + mgsin\theta = \frac{mv^2}{r}cos\theta$$

$$N - mgcos\theta = \frac{mv^2}{r}sin\theta$$

Alternatively, we can resolve the forces horizontally and vertically, noting that the friction and normal force establish the centripetal force:

$$\frac{mv^2}{r} = Fcos\theta + Nsin\theta$$

$$mg = Ncos\theta - Fsin\theta$$

We note that when there is no side slipe, there is no lateral force. Alternatively we can say that an object does no rely on friction to go around the curve without skidding. These conditions are met when:

$$F = 0$$

$$tan\theta = \frac{v^2}{gr}$$

Example 0.6. A civil engineer wishes to design a curved exit ramp for a highway in such a way that a car will not have to rely on friction to round the curve without skidding. In other words, a car moving at the designated speed can negotiate the curve even when the road is covered with ice. Such a ramp is usually banked; this means the roadway is tilted toward the inside of the curve. Suppose the designated speed for the ramp is to be 13.4 m/s (30.0 mi/h) and the radius of the curve is 50.0 m. At what angle should the curve be banked?

Resolving along the plane:

$$F + mgsin\theta = \frac{mv^2}{r}cos\theta$$

Since friction isn't being relied on and the car does not skid then F=0.

$$tan\theta = \frac{v^2}{gr} = \frac{13.4^2}{(9.8)(50)} \rightarrow \theta = 20.1^\circ$$

Exercise. 2

(1) A ring of mass m kg is threaded through a circular wire that lies in a horizontal plane. The wire has diameter 1 m. The ring is then given a velocity of 5m/s tangentially to the circular wire. Due to a constant friction F, the ring only describes a semi-circle before it comes to rest. a) Find the value of the tangential acceleration. b) Find the time taken before it stops.

(2) A light string OP is fixed at the end O and is attached at the other end P to a particle which is moving uniformly in a horizontaly circle whose centre C is vertically below and distant h from O. Prove that a) The period of motion is given by: $T = 2\pi\sqrt{\frac{h}{g}}$b) If the number of revolutions per second is increased from 1 to 2, find the rise in the level of the circle.

(3) A particle of mass m kg rests on a smooth horizontal disc being attached by two strings each of length l metres to the extremities of a diameter. If the disc rotates at a uniform rate with period T about a vertical axis through its centre, find the magnitude of the centripetal force.

(4) An airplane flying in a horizontal circle at a speed of 480 km/hr is tilted at 40degrees to the horizontal. Assuming the required force is provided entirely by the aerodynamic lift that is perpendicular to the wind's surface, show that the radius of the circle is 2.2km.

(5) A bowl of depth h is made from a spherical surface whose radius is a, and which rotates about its vertical axis. Find the maximum number of revolutioners per minute which it can make such that a particle can rest on its surface without being thrown off.

(6) A ball of mass 0.500 kg is attached to the end of a cord 1.50 m long. The ball is whirled in a horizontal circle as shown in Figure 6.1. If the cord can withstand a maximum tension of 50.0 N, what is the maximum speed at which the ball can be whirled before the cord breaks? Assume that the string remains horizontal during the motion.

(7) A 1 500-kg car moving on a flat, horizontal road negotiates a curve, as shown in Figure 6.5. If the radius of the curve is 35.0 m and the coefficient of static friction between the tires and dry pavement is 0.500, find the maximum speed the car can have and still make the turn successfully.