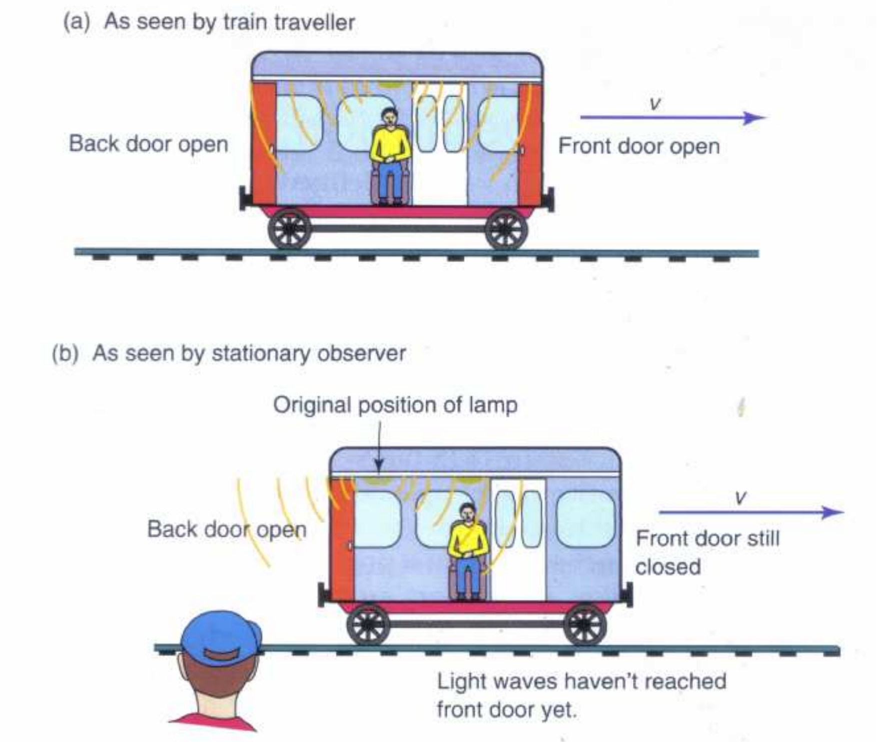
Derivations of Consequences of Special Theory of Relativity

1

An operator of a lamp rides in the middle of a rather special train carriage. The doors at either end of the carriage are light-operated. At an instant in time when the operator happens to be alongside an observer on the embankment (outside the moving train), the operator switches on the lamp which, in turn, opens the doors.



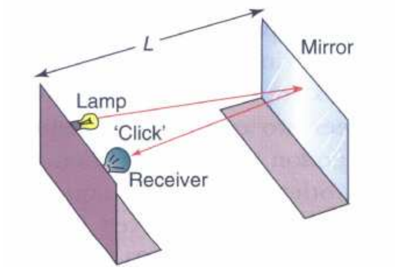
The operator of the lamp will see the two doors opening simultaneously. The distance of each door from the lamp is the same and light will travel at the same speed (c) both forward and backward so that each door receives the light at the same time and they open simultaneously.

The observer on the embankment, however, sees the situation differently. After the lamp is turned on, but before the light has reached the doors, the train has moved so that the front door is now further away and the back door is closer. He sees the light travelling both forward and backward at the same speed (c), but the forward journey is now longer than the backward journey, so that the back door is seen to open before the front door. They are most definitely not judged to be simultaneous events.

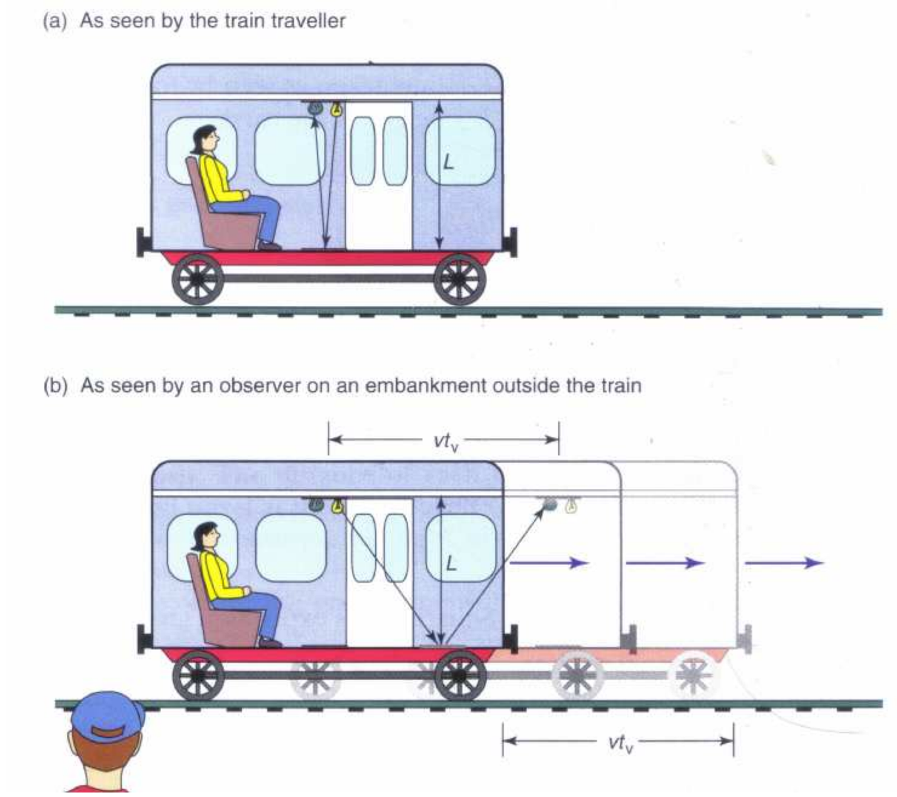
It is tempting to ask who is correct — the operator in the train or the observer on the embankment. The answer is that they both are. Both observers judged the situation correctly from their different frames of reference and this is a direct consequence of the constancy of the speed of light.

2

The following thought experiment uses a 'light clock'. As shown, a light pulse is released by a lamp, travels the length of the clock and is then reflected back to a sensor next to the lamp. When the sensor receives the pulse of light, it goes 'click'.



For this thought experiment we shall return to the train scenario favoured by Einstein. Imagine a traveller, seated in a speeding train. The light clock is arranged vertically, with the lamp at the ceiling and the mirror on the floor. An observer is watching from the embankment out- side the train. Our question now is this: when a light pulse is released, how long does it take to travel down to the mirror and return back to the ceiling, as seen by both the train traveller and the observer?



Let us first examine the situation as seen by the train traveller in the rest frame; that is, the frame within which the event occurs. If L is the height of the train carriage, for the total journey we can say that:

$$\text{distance} = 2L = ct_o$$

where

t_o = time taken as seen by traveller
L = height of the carriage

so that

$$t_o = \frac{2L}{c}.$$

Examine now the situation as seen by the observer on the embankment. Figure 5.9 (page 76) compares the way the situation is viewed by each person. From outside the train the observer sees the light travelling along a much longer journey, and its length can be determined using Pythagoras' theorem:

Total journey = $$ct_v = 2\sqrt{L^2 + \left(\frac{vt_v}{2}\right)^2}$$

$$= \sqrt{4L^2 + v^2 t_v^2}.$$

Squaring this expression gives: $$c^2 t_v^2 = 4L^2 + v^2 t_v^2.$$

Rearranging this leads to: $$t_v^2 = \frac{4L^2}{(c^2 - v^2)}.$$

but, from above, $t_o = \frac{2L}{c}$.

Substituting this into the expression gives: $t_v = \frac{t_o}{\sqrt{1 - \frac{v^2}{c^2}}}$

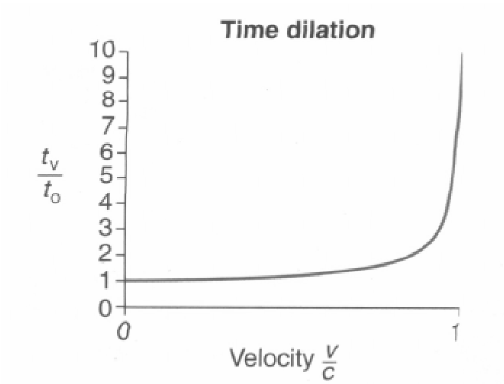
(the time dilation equation)

where

t_o = time taken in the rest frame of reference
= proper time
t_v = time taken as seen from the frame of reference in relative motion to the rest frame
v = velocity of the train
c = speed of light.

Note that t_o is the time taken for the clock to go 'click' as observed by the train traveller, while t_v is the time taken as observed by the person on the embankment. Looking at the last expression above, we can see that

Note that t_o is the time taken for the clock to go 'click' as observed by the train traveller, while t_v is the time taken as observed by the person on the embankment. Looking at the last expression above, we can see that the term $\sqrt{1-\frac{v^2}{c^2}}$ is always less than one so that t_v is always greater than t_o.
This means that the clock takes longer to go 'click' as observed by the person on the embankment or, put another way, the outside observer hears the light clock clicking slower than does the train traveller. Time is passing more slowly on the train as observed by the person outside the train!



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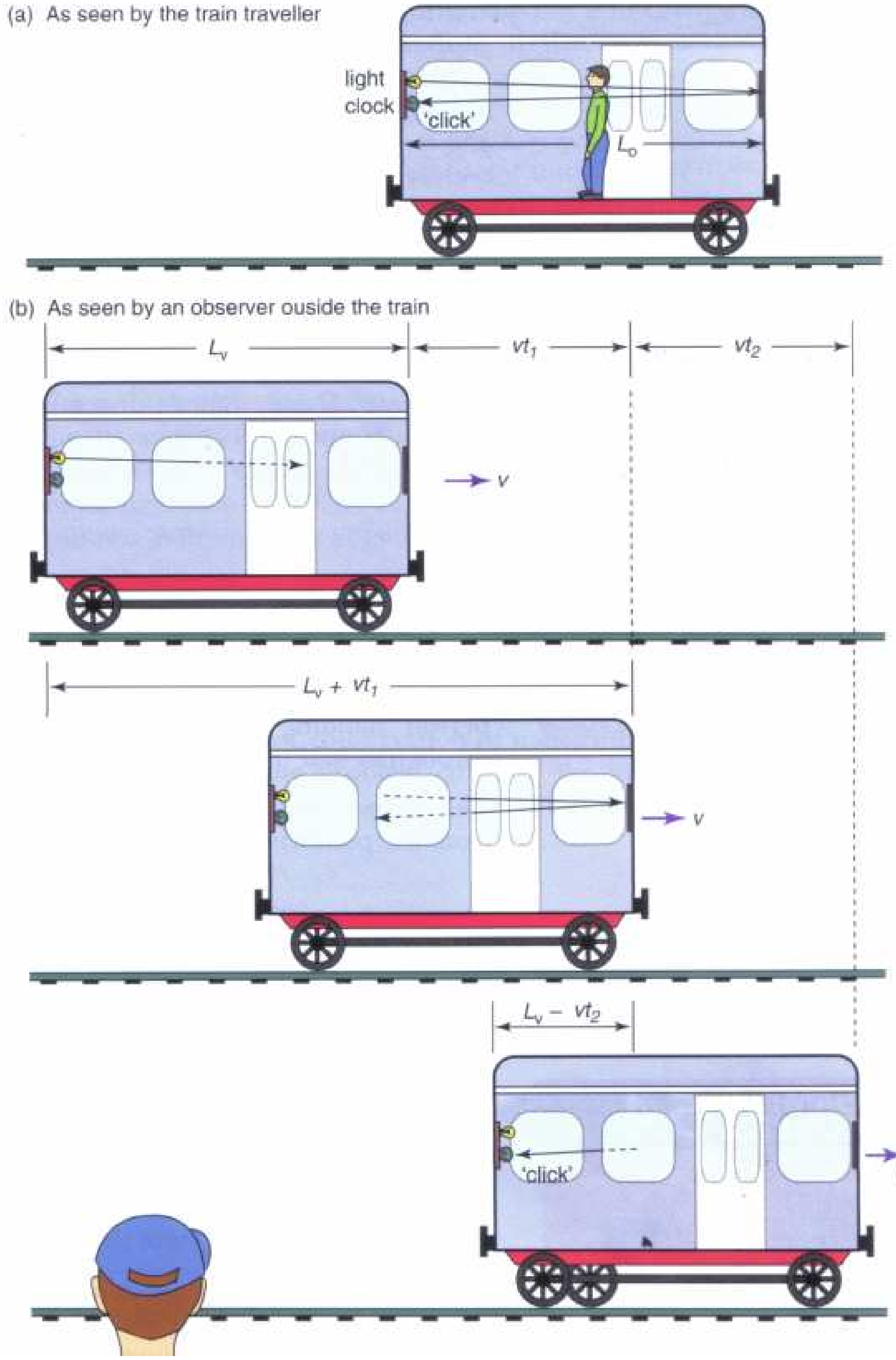
This time our train traveller has arranged the light clock so that it runs the length of the train, with the lamp and sensor located on the back wall and the mirror on the front wall. As the train passes the observer on the embankment, the light clock emits a light pulse which travels to the front wall and then returns to the back wall where it is picked up by the sensor which then goes 'click'. This journey is observed by both people, but what is the length of the journey that each perceives?
Let us start with the situation as seen by the train traveller. It is a simple situation:



(a) As seen by the train traveller
light clock
'click'
L_o
(b) As seen by an observer ouside the train
L_v
vt_1
vt_2
v
$L_v + vt_1$
v
$L_v - vt_2$
'click'
v

$$\text{length of light journey} = ct_o = 2L_o$$

where

L_o = length of train as perceived by train traveller
t_o = *time taken as perceived by train traveller.*

The situation seen by the observer at the side of the track is somewhat different because the train is moving at the same time, lengthening the forward leg of the light pulse's journey and shortening the return leg, as shown in figure 5.11(b).

If t_1 is the time taken for the forward part of the journey, then:

$$\text{length of forward journey} = ct_1 = L_v + vt_1$$

$$\text{and hence, } t_1 = \frac{L_v}{c - v}$$

where

L_v = length of the train as measured by the observer on the embankment.

Similarly, t_2 is the time taken for the return, so that:

$$\text{length of the return journey} = ct_2 = L_v + vt_2$$

$$\text{and hence, } t_2 = \frac{L_v}{c + v}$$

The time for the whole journey as seen by the observer on the embankment is:

$$\text{time for journey, } t_v = t_1 + t_2 = \frac{L_v}{c - v} + \frac{L_v}{c + v}$$

$$\text{which can be rearranged to give } t_v = \frac{2L_v}{c\left(1 - \frac{v^2}{c^2}\right)}.$$

It is now crucial to appreciate that each observer perceives time differently. To take that into account, we need to equate the time dilation equation to the one just derived:

$$\frac{t_o}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2L_v}{c\left(1 - \frac{v^2}{c^2}\right)} \text{ but we know that } t_o = \frac{2L_o}{c}$$

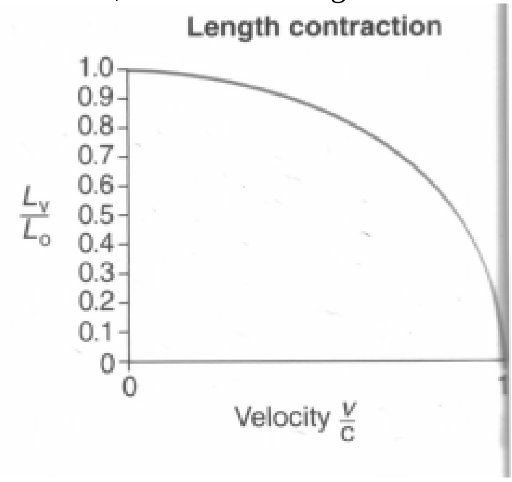
$$\text{so } \frac{2L_o}{c\sqrt{1-\frac{v^2}{c^2}}} = \frac{2L_v}{c\left(1-\frac{v^2}{c^2}\right)} \text{ which reduces down to give}$$

$$L_v = L_o\sqrt{1-\frac{v^2}{c^2}} \text{ (the length contraction equation)}$$

where

L_o = the length of an object measured from its rest frame
L_v = the length of an object measured from a different frame of reference
v = relative speed of the two frames of reference
c = speed of light.

Returning to the thought experiment, this equation means that, since the term $\sqrt{1-\frac{v^2}{c^2}}$ is always less than one, the length of the train as observed by the person on the embankment is less than that observed by the person inside the train. The person outside the train has seen the train shorten, and the faster it goes the shorter it gets!



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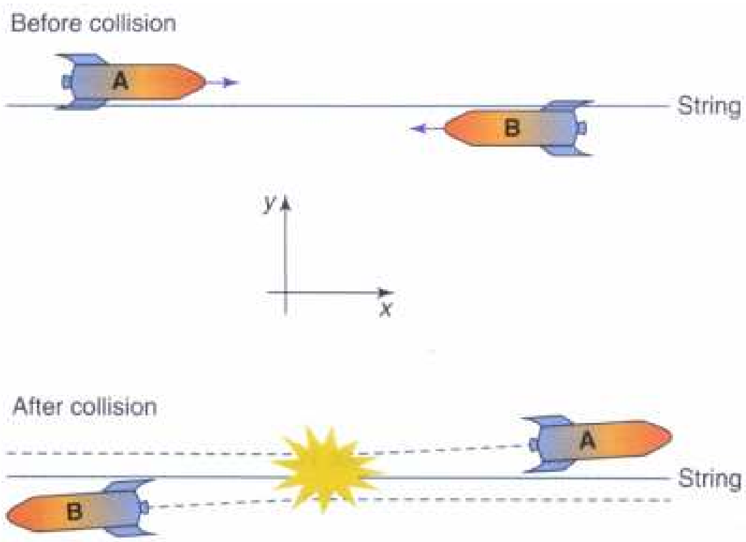


A long string is stretched through space. Two identical spacecraft travel toward each other on either side of the string, each with a velocity of 0.3c relative to the string. As they meet they collide with a glancing blow that marginally reduces their velocity along the string but also gives each a small velocity away from the string. The momentum prior to this collision is zero as the spacecraft have equal masses and equal but opposite velocities. Following the collision, the spacecraft are moving apart, however, their velocities in the x direction are still equal but opposite and almost unchanged. Due to the symmetry of the collision, the velocities of the spacecraft in the y direction, although very small, are also equal but opposite. As a result the total momentum in both the x and y directions is zero, hence momentum has been conserved.

The situation is different as seen from the point of view of one of the spacecraft, however, and the difference is due to time dilation and its effect on the y velocities.

As seen by a passenger of spacecraft A, prior to the collision space- craft B speeds toward it with a velocity of O.6c (actually 0.55c) , strikes it a sideways blow and then departs at a slight angle to its original direction. Due to the presence of the string, the passengers of spacecraft A can identify that they are now moving slowly away from the string — covering, say, 10 metres in one second,

giving a velocity of 10 m s^{-1}. Looking across, the passenger sees that the clocks in spacecraft B are running slow, so that it covers 10 metres in

$$\frac{1}{\sqrt{1-0.6^2}} = 1.25 \text{ s.}$$

The velocity of spacecraft B is thus $\frac{10 \text{ m}}{1.25 \text{ s}} = 8$ m s^{-1}. Therefore, the y velocities of the spacecraft are not identical and momentum is not conserved in the y direction.

Algebraically:

$$\begin{aligned} p_y \text{ before collision} &= 0 \\ p_y \text{ after collision} &= m_A v_A + m_B v_B \\ &= m(10) + m(-8) \\ &= 2m \text{ where } m = m_A = m_B \end{aligned}$$

Hence, momentum is not conserved.

Einstein believed very strongly that momentum must be conserved in all inertial frames of reference. In order to solve this dilemma he suggested that the mass of an object must increase, or dilate, at relativistic speeds by a factor that compensates for the effect of time dilation on speed measurement. We can use this idea to derive a formula for mass dilation.

Referring back to the spacecraft problem, assume that total momentum is conserved in the y direction, as seen by the passenger of spacecraft A.

momentum before collision = momentum after collision

$$\begin{aligned} 0 &= m_A v_A + m_B v_B \text{ (as seen by A)} \\ &= m_A\left(\frac{r}{t_0}\right) - m_B\left(\frac{r}{t_V}\right) \end{aligned}$$

$$\text{hence } \frac{m_A}{t_0} = \frac{m_B}{t_V} \quad \text{and} \quad t_V = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\text{so that, as seen by A, } m_B = \frac{m_A}{\sqrt{1-\frac{v^2}{c^2}}}$$

This expression shows how the mass of spacecraft B increases with speed, as seen by the passenger of spacecraft A. By substituting this new expression back into the problem we can see how the momentum works out.

$$p_{By} \text{ after collision} = m_B v_B$$

$$= \left(\frac{m_A}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(\frac{r}{t_0\Big/\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$= \left(\frac{m_A}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(\frac{r\sqrt{1-\frac{v^2}{c^2}}}{t_0}\right)$$

$$= \frac{m_A r}{t_0}$$

Hence, we can now say:

$$p_y \text{ before collision} = 0$$

$$p_y \text{ after collision} = p_{Ay} + p_{By}$$

$$= m_A v_A + m_B v_B$$

$$= m_A\left(\frac{r}{t_0}\right) - \left(\frac{m_A r}{t_0}\right)$$

$$= 0$$

Hence, momentum is now conserved.

The masses of the two spacecraft were originally identical, so the expression relating the masses can be generalised to the form

$$m_v = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

where

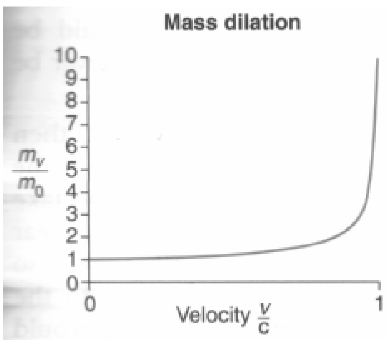
m_0 = mass measured in the rest frame of reference
= rest mass

m_v = mass as seen from the frame of reference in relative motion to the rest frame

v = velocity

c = speed of light

This effect is called mass dilation and can be generally stated as follows: The mass of an object within its own rest frame is called its rest mass m_o. Measurements of this mass m, made from any other inertial reference frame in relative motion to the first, are always greater. The degree of mass dilation varies with velocity The effect can be most simply stated as: moving objects gain mass.



Note in figure 5.14 that as the speed of an object approaches the speed of light c, its mass approaches an infinite value. It is this enormous increase in mass that prevents any object from exceeding the speed of light. This is because an applied force is required to create acceleration. Acceleration leads to higher velocities, which eventually leads to increased mass. This means that further accelerations will require ever-greater force. As mass becomes infinite, an infinite force would be required to achieve any acceleration at all. Sufficient force can never be supplied to accelerate beyond the speed of light.

But herein lies a problem — if a force is applied to an object, then work is done on it. Another way to say this is that energy is given to the object. In the sort of situation we are considering, this energy would take the form of increased kinetic energy as the object speeds up. But at near light speed the object does not speed up as we would normally expect, so where is the energy going? The applied force is giving energy to the object and the object does not acquire the kinetic energy we would expect. Instead, it acquires extra mass, as shown in figure 5.14. Einstein made an inference here and stated that the mass (or inertia) of the object contained the extra energy.

Relativity results in a new definition of energy as follows:

$$E = E_k + mc^2$$

where

E = total energy
E_k = kinetic energy
m = mass
c = speed of light.

Notice that when an object is stationary, so that it has no kinetic energy, it still has some energy due to its mass. This is called its mass energy or **rest energy** and is given by:

$$E = mc^2$$

where

E = rest energy (J)
m = mass (kg)
c = speed of light (3×10^8 m s^{-1}).